STEINER TRIPLE SYSTEMS of order 3 and 9 are Kirkman triple systems with $n = 0$ and 1. Solution to KIRKMAN'S SCHOOLGIRL PROBLEM requires construction of a Kirkman triple system of order $n = 2$.

Ray-Chaudhuri and Wilson (1971) showed that there exists at least one Kirkman triple system for every NONNEGATIVE order n. Earlier editions of Ball and Coxeter (1987) gave constructions of Kirkman triple systems with $9 \leq v \leq 99$. For $n = 1$, there is a single unique (up to an isomorphism) solution, while there are 7 different systems for $n = 2$ (Mulder 1917, Cole 1922, Ball and Coxeter 1987).

see also STEINER TRIPLE SYSTEM

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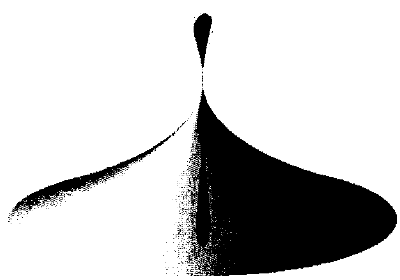
Kiss Surface



The QUINTIC SURFACE given by the equation

$$\tfrac{1}{2}x^5 + \tfrac{1}{2}x^4 - (y^2 + z^2) = 0.$$

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Kissing Circles Problem

see DESCARTES CIRCLE THEOREM, SODDY CIRCLES

Kissing Number

The number of equivalent HYPERSPHERES in n-D which can touch an equivalent HYPERSPHERE without any intersections, also sometimes called the NEWTON NUMBER, CONTACT NUMBER, COORDINATION NUMBER, or LIGANCY. Newton correctly believed that the kissing number in 3-D was 12, but the first proofs were not produced until the 19th century (Conway and Sloane 1993, p. 21) by Bender (1874), Hoppe (1874), and Günther (1875). More concise proofs were published by Schütte and van der Waerden (1953) and Leech (1956). Exact values for *lattice packings* are known for $n = 1$ to 9 and $n = 24$ (Conway and Sloane 1992, Sloane and Nebe). Odlyzko and Sloane (1979) found the exact value for 24-D.

The following table gives the largest known kissing numbers in DIMENSION D for lattice (L) and nonlattice (NL) packings (if a nonlattice packing with higher number exists). In nonlattice packings, the kissing number may vary from sphere to sphere, so the largest value is given below (Conway and Sloane 1993, p. 15). An more extensive and up-to-date tabulation is maintained by Sloane and Nebe.

D	L	NL	D	L	NL
1	2		13	≥ 918	$\geq 1,130$
2	6		14	$\geq 1,422$	$\geq 1,582$
3	12		15	$\geq 2,340$	
4	24		16	$\geq 4,320$	
5	40		17	$\geq 5,346$	
6	72		18	$\geq 7,398$	
7	126		19	$\geq 10,668$	
8	240		20	$\geq 17,400$	
9	272	≥ 306	21	$\geq 27,720$	
10	≥ 336	≥ 500	22	$\geq 49,896$	
11	≥ 438	≥ 582	23	$\geq 93,150$	
12	≥ 756	≥ 840	24	196,560	

The lattices having maximal packing numbers in 12- and 24-D have special names: the COXETER-TODD LATTICE and LEECH LATTICE, respectively. The general form of the lower bound of n-D lattice densities given by

$$\eta \geq \frac{\zeta(n)}{2^{n-1}},$$

where $\zeta(n)$ is the RIEMANN ZETA FUNCTION, is known as the MINKOWSKI-HLAWKA THEOREM.

see also COXETER-TODD LATTICE, HERMITE CONSTANTS, HYPERSPHERE PACKING, LEECH LATTICE, MINKOWSKI-HLAWKA THEOREM

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Kite

see DIAMOND, LOZENGE, PARALLELOGRAM, PENROSE TILES, QUADRILATERAL, RHOMBUS

Klarner-Rado Sequence

The thinnest sequence which contains 1, and whenever it contains x, also contains $2x$, $3x+2$, and $6x+3$: 1, 2, 4, 5, 8, 9, 10, 14, 15, 16, 17, ... (Sloane's A005658).

References

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Klarner's Theorem

An $a \times b$ RECTANGLE can be packed with $1 \times n$ strips IFF $n|a$ or $n|b$.

see also BOX-PACKING THEOREM, CONWAY PUZZLE, DE BRUIJN'S THEOREM, SLOTHOUBER-GRAATSMA PUZZLE

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Klein's Absolute Invariant

$$J(q) \equiv \frac{4}{27}\frac{[1-\lambda(q)+\lambda^2(q)]^3}{\lambda^2(q)[1-\lambda(q)]^2} = \frac{[E_4(q)]^3}{[E_4(q)]^3-[E_6(q)^2]}$$

(Cohn 1994), where $q \equiv e^{i\pi t}$ is the NOME, $\lambda(q)$ is the ELLIPTIC LAMBDA FUNCTION

$$\lambda(q) \equiv k^2(q) = \left[\frac{\vartheta_2(q)}{\vartheta_3(q)}\right]^4,$$

$\vartheta_i(q)$ is a THETA FUNCTION, and the $E_i(q)$ are RAMANUJAN-EISENSTEIN SERIES. $J(t)$ is GAMMA-MODULAR.

see also ELLIPTIC LAMBDA FUNCTION, j-FUNCTION, PI, RAMANUJAN-EISENSTEIN SERIES, THETA FUNCTION

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Klein-Beltrami Model

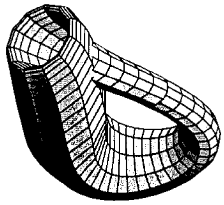
The Klein-Beltrami model of HYPERBOLIC GEOMETRY consists of an OPEN DISK in the Euclidean plane whose open chords correspond to hyperbolic lines. Two lines l and m are then considered parallel if their chords fail to intersect and are PERPENDICULAR under the following conditions,

1. If at least one of l and m is a diameter of the DISK, they are hyperbolically perpendicular IFF they are perpendicular in the Euclidean sense.
2. If neither is a diameter, l is perpendicular to m IFF the Euclidean line extending l passes through the pole of m (defined as the point of intersection of the tangents to the disk at the "endpoints" of m).

There is an isomorphism between the POINCARÉ HYPERBOLIC DISK model and the Klein-Beltrami model. Consider a Klein disk in Euclidean 3-space with a SPHERE of the same radius seated atop it, tangent at the ORIGIN. If we now project chords on the disk orthogonally upward onto the SPHERE's lower HEMISPHERE, they become arcs of CIRCLES orthogonal to the equator. If we then stereographically project the SPHERE's lower HEMISPHERE back onto the plane of the Klein disk from the north pole, the equator will map onto a disk somewhat larger than the Klein disk, and the chords of the original Klein disk will now be arcs of CIRCLES orthogonal to this larger disk. That is, they will be Poincaré lines. Now we can say that two Klein lines or angles are congruent iff their corresponding Poincaré lines and angles under this isomorphism are congruent in the sense of the Poincaré model.

see also HYPERBOLIC GEOMETRY, POINCARÉ HYPERBOLIC DISK

Klein Bottle



A closed NONORIENTABLE SURFACE of GENUS one having no inside or outside. It can be physically realized only in 4-D (since it must pass through itself without the presence of a HOLE). Its TOPOLOGY is equivalent to a pair of CROSS-CAPS with coinciding boundaries. It can be cut in half along its length to make two MÖBIUS STRIPS.

The above picture is an IMMERSION of the Klein bottle in $\mathbb{R}^3$ (3-space). There is also another possible IMMERSION called the "figure-8" IMMERSION (Geometry Center).

The equation for the usual IMMERSION is given by the implicit equation

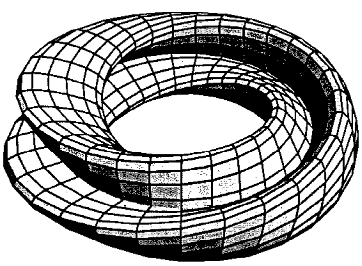
$$(x^2+y^2+z^2+2y-1)[(x^2+y^2+z^2-2y-1)^2-8z^2] +16xz(x^2+y^2+z^2-2y-1)=0 \quad (1)$$

(Stewart 1991). Nordstrand gives the parametric form

$$x=\cos u[\cos(\tfrac{1}{2}u)(\sqrt{2}+\cos v)+\sin(\tfrac{1}{2}u)\sin v\cos v] \quad (2)$$

$$y=\sin u[\cos(\tfrac{1}{2}u)(\sqrt{2}+\cos v)+\sin(\tfrac{1}{2}u)\sin v\cos v] \quad (3)$$

$$z=-\sin(\tfrac{1}{2}u)(\sqrt{2}+\cos v)+\cos(\tfrac{1}{2}u)\sin v\cos v. \quad (4)$$



The "figure-8" form of the Klein bottle is obtained by rotating a figure eight about an axis while placing a twist in it, and is given by parametric equations

$$x(u,v)=[a+\cos(\tfrac{1}{2}u)\sin(v)-\sin(\tfrac{1}{2}u)\sin(2v)]\cos(u) \quad (5)$$

$$y(u,v)=[a+\cos(\tfrac{1}{2}u)\sin(v)-\sin(\tfrac{1}{2}u)\sin(2v)]\sin(u) \quad (6)$$

$$z(u,v)=\sin(\tfrac{1}{2}u)\sin(v)+\cos(\tfrac{1}{2}u)\sin(2v) \quad (7)$$

for $u\in[0,2\pi)$, $v\in[0,2\pi)$, and $a>2$ (Gray 1993).

The image of the CROSS-CAP map of a TORUS centered at the ORIGIN is a Klein bottle (Gray 1993, p. 249).

Any set of regions on the Klein bottle can be colored using ss colors only (Franklin 1934, Saaty 1986).

see also CROSS-CAP, ETRUSCAN VENUS SURFACE, IDA SURFACE, MAP COLORING MÖBIUS STRIP

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Klein's Equation

If a REAL curve has no singularities except nodes and CUSPS, BITANGENTS, and INFLECTION POINTS, then

$$n+2\tau_2'+\iota'=m+2\delta_2'+\kappa',$$

where n is the order, τ' is the number of conjugate tangents, ι' is the number of REAL inflections, m is the class, δ' is the number of REAL conjugate points, and κ' is the number of REAL CUSPS. This is also called KLEIN'S THEOREM.

see also PLÜCKER'S EQUATION

References

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Klein Four-Group

see VIERGRUPPE

Klein-Gordon Equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2}-\mu^2\psi.$$

see also SINE-GORDON EQUATION, WAVE EQUATION

Klein Quartic

The 3-holed TORUS.

Klein's Theorem

see KLEIN'S EQUATION

Kleinian Group

A finitely generated discontinuous group of linear fractional transformation acting on a domain in the COMPLEX PLANE.

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Kloosterman's Sum

$$S(u,v,n) \equiv \sum_n \exp\left[\frac{2\pi i(uh+v\bar{h})}{n}\right], \qquad (1)$$

where h runs through a complete set of residues RELATIVELY PRIME to n, and $\bar{h}$ is defined by

$$h\bar{h} \equiv 1 \pmod{n}. \qquad (2)$$

If $(n,n')=1$ (if n and n' are RELATIVELY PRIME), then

$$S(u,v,n)S(u,v',n') = S(u, vn'^2 + v'n^2, nn'). \qquad (3)$$

Kloosterman's sum essentially solves the problem introduced by Ramanujan of representing sufficiently large numbers by QUADRATIC FORMS $a{x_1}^2 + b{x_2}^2 + c{x_3}^2 + d{x_4}^2$. Weil improved on Kloosterman's estimate for Ramanujan's problem with the best possible estimate

$$|S(u,u,n)| \leq 2\sqrt{n} \qquad (4)$$

(Duke 1997).

see also GAUSSIAN SUM

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Knapsack Problem

Given a SUM and a set of WEIGHTS, find the WEIGHTS which were used to generate the SUM. The values of the weights are then encrypted in the sum. The system relies on the existence of a class of knapsack problems which can be solved trivially (those in which the weights are separated such that they can be "peeled off" one at a time using a GREEDY-like algorithm), and transformations which convert the trivial problem to a difficult one and vice versa. Modular multiplication is used as the TRAPDOOR FUNCTION. The simple knapsack system was broken by Shamir in 1982, the Graham-Shamir system by Adleman, and the iterated knapsack by Ernie Brickell in 1984.

References

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Kneser-Sommerfeld Formula

Let J_ν be a BESSEL FUNCTION OF THE FIRST KIND, N_ν a NEUMANN FUNCTION, and $j_{\nu,n}$ the zeros of $z^{-\nu}J_\nu(z)$ in order of ascending REAL PART. Then for $0 < x < X < 1$ and $\Re[z] > 0$,

$$\frac{\pi J_\nu(xz)}{4J_\nu(z)}[J_\nu(z)N_\nu(Xz) - N_\nu(z)J_\nu(Xz)] = \sum_{n=1}^\infty \frac{J_\nu(j_{\nu,n}x)J_\nu(j_{\nu,n}X)}{(z^2-{j_{\nu,n}}^2){J'_{\nu,n}}^2(j_{\nu,n})}.$$

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Knights Problem

Kt		Kt		Kt		Kt	
	Kt		Kt		Kt		Kt
Kt		Kt		Kt		Kt	
	Kt		Kt		Kt		Kt
Kt		Kt		Kt		Kt	
	Kt		Kt		Kt		Kt
Kt		Kt		Kt		Kt	
	Kt		Kt		Kt		Kt

The problem of determining how many nonattacking knights $K(n)$ can be placed on an $n \times n$ CHESSBOARD. For $n = 8$, the solution is 32 (illustrated above). In general, the solutions are

$$K(n) = \begin{cases} \frac{1}{2}n^2 & n > 2 \text{ even} \\ \frac{1}{2}(n^2+1) & n > 1 \text{ odd,} \end{cases}$$

giving the sequence 1, 4, 5, 8, 13, 18, 25, ... (Sloane's A030978, Dudeney 1970, p. 96; Madachy 1979).

					Kt		
	Kt	Kt		Kt	Kt		
		Kt					
					Kt		
		Kt	Kt		Kt	Kt	
		Kt					

The minimal number of knights needed to occupy or attack every square on an $n \times n$ CHESSBOARD is given by 1, 4, 4, 4, 5, 8, 10, ... (Sloane's A006075). The number of such solutions are given by 1, 1, 2, 3, 8, 22, 3, ... (Sloane's A006076).

see also BISHOPS PROBLEM, CHESS, KINGS PROBLEM, KNIGHT'S TOUR, QUEENS PROBLEM, ROOKS PROBLEM

References

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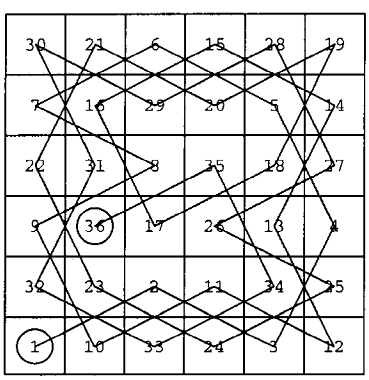
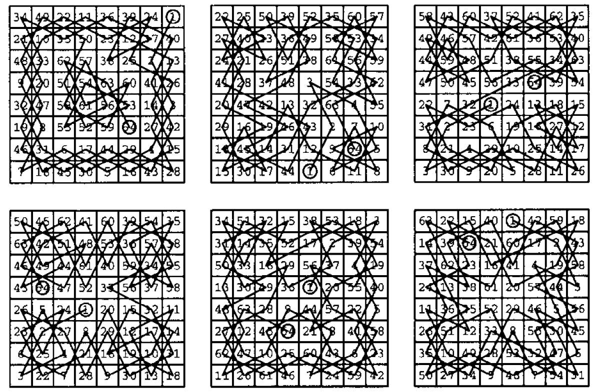
Knights of the Round Table

see NECKLACE

Knight's Tour





A knight's tour of a CHESSBOARD (or any other grid) is a sequence of moves by a knight CHESS piece (which may only make moves which simultaneously shift one square along one axis and two along the other) such that each square of the board is visited exactly once (i.e., a HAMILTONIAN CIRCUIT). If the final position is a knight's move away from the first position, the tour is called re-entrant. The first figure above shows a knight's tour on a 6×6 CHESSBOARD. The second set of figures shows six knight's tours on an 8×8 CHESSBOARD, all but the first of which are re-entrant. The final tour has the additional property that it is a SEMIMAGIC SQUARE with row and column sums of 260 and main diagonal sums of 348 and 168.

Löbbing and Wegener (1996) computed the number of cycles covering the directed knight's graph for an 8×8 CHESSBOARD. They obtained α^2, where $\alpha = 2,849,759,680$, i.e., 8,121,130,233,753,702,400. They also computed the number of undirected tours, obtaining an incorrect answer 33,439,123,484,294 (which is not divisible by 4 as it must be), and so are currently redoing the calculation.

The following results are given by Kraitchik (1942). The number of possible tours on a $4k \times 4k$ board for $k = 3$, 4, ... are 8, 0, 82, 744, 6378, 31088, 189688, 1213112, ... (Kraitchik 1942, p. 263). There are 14 tours on the 3×7 rectangle, two of which are symmetrical. There are 376 tours on the 3×8 rectangle, none of which is closed. There are 16 symmetric tours on the 3×9 rectangle and 8 closed tours on the 3×10 rectangle. There are 58 symmetric tours on the 3×11 rectangle and 28 closed tours on the 3×12 rectangle. There are five doubly symmetric tours on the 6×6 square. There are 1728 tours on the 5×5 square, 8 of which are symmetric. The longest "uncrossed" knight's tours on an $n \times n$ board for $n = 3$, 4, ... are 2, 5, 10, 17, 24, 35, ... (Sloane's A003192).

see also CHESS, KINGS PROBLEM, KNIGHTS PROBLEM, MAGIC TOUR, QUEENS PROBLEM, TOUR

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Knödel Numbers

For every $k \geq 1$, let C_k be the set of COMPOSITE numbers $n > k$ such that if $1 < a < n$, $\mathrm{GCD}(a,n) = 1$ (where GCD is the GREATEST COMMON DIVISOR), then $a^{n-k} \equiv 1 \pmod n$. C_1 is the set of CARMICHAEL NUMBERS. Makowski (1962/1963) proved that there are infinitely many members of C_k for $k \geq 2$.

see also CARMICHAEL NUMBER, D-NUMBER, GREATEST COMMON DIVISOR

References
Makowski, A. "Generalization of Morrow's D-Numbers." *Simon Stevin* **36**, 71, 1962/1963.
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, p. 101, 1989.

Knot

A knot is defined as a closed, non-self-intersecting curve embedded in 3-D. A knot is a single component LINK. Klein proved that knots cannot exist in an EVEN-numbered dimensional space ≥ 4. It has since been shown that a knot cannot exist in *any* dimension ≥ 4. Two distinct knots cannot have the same KNOT COMPLEMENT (Gordon and Luecke 1989), but two LINKS can! (Adams 1994, p. 261). The KNOT SUM of any number of knots cannot be the UNKNOT unless each knot in the sum is the UNKNOT.

Knots can be cataloged based on the minimum number of crossings present. Knots are usually further broken down into PRIME KNOTS. Knot theory was given its first impetus when Lord Kelvin proposed a theory that atoms were vortex loops, with different chemical elements consisting of different knotted configurations (Thompson 1867). P. G. Tait then cataloged possible knots by trial and error.

Thistlethwaite has used DOWKER NOTATION to enumerate the number of PRIME KNOTS of up to 13 crossings, and ALTERNATING KNOTS up to 14 crossings. In this compilation, MIRROR IMAGES are counted as a single knot type. The number of distinct PRIME KNOTS $N(n)$ for knots from $n = 3$ to 13 crossings are 1, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988 (Sloane's A002863). Combining PRIME KNOTS gives one additional type of knot each for knots six and seven crossings.

Let $C(n)$ be the number of distinct PRIME KNOTS of n crossings, *counting* CHIRAL *versions of the same knot separately.* Then

$$\tfrac{1}{3}(2^{n-2} - 1) \leq N(n) \lesssim e^n$$

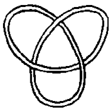
(Ernst and Summers 1987). Welsh has shown that the number of knots is bounded by an exponential in n.

A pictorial enumeration of PRIME KNOTS of up to 10 crossings appears in Rolfsen (1976, Appendix C). Note, however, that in this table, the PERKO PAIR 10_{161} and 10_{162} are actually identical, and the uppermost crossing in 10_{144} should be changed (Jones 1987). The kth knot having n crossings in this (arbitrary) ordering of knots is given the symbol n_k. Another possible representation for knots uses the BRAID GROUP. A knot with $n+1$ crossings is a member of the BRAID GROUP n. There is no general method known for deciding whether two given knots are equivalent or interlocked. There is no general ALGORITHM to determine if a tangled curve is a knot. Haken (1961) has given an ALGORITHM, but it is too complex to apply to even simple cases.

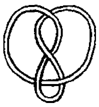
If a knot is AMPHICHIRAL, the "amphichirality" is $A = 1$, otherwise $A = 0$ (Jones 1987). ARF INVARIANTS are designated a. BRAID WORDS are denoted b (Jones 1987). CONWAY'S KNOT NOTATION C for knots up to 10 crossings is given by Rolfsen (1976). Hyperbolic volumes are given (Adams, Hildebrand, and Weeks 1991; Adams 1994). The BRAID INDEX i is given by Jones (1987). ALEXANDER POLYNOMIALS Δ are given in Rolfsen (1976), but with the POLYNOMIALS for 10_{083} and 10_{086} reversed (Jones 1987). The ALEXANDER POLYNOMIALS are normalized according to Conway, and given in abbreviated form $[a_1, a_2, \ldots$ for $a_1 + a_2(x^{-1} + x) + \ldots$.

The JONES POLYNOMIALS W for knots of up to 10 crossings are given by Jones (1987), and the JONES POLYNOMIALS V can be either computed from these, or taken from Adams (1994) for knots of up to 9 crossings (although most POLYNOMIALS are associated with the wrong knot in the first printing). The JONES POLYNOMIALS are listed in the abbreviated form $\{n\}\ a_0\ a_1\ \ldots$ for $t^{-n}(a_0 + a_1 t + \ldots)$, and correspond either to the knot depicted by Rolfsen or its MIRROR IMAGE, whichever has the lower POWER of t^{-1}. The HOMFLY POLYNOMIAL $P(\ell, m)$ and KAUFFMAN POLYNOMIAL $F(a, x)$ are given in Lickorish and Millett (1988) for knots of up to 7 crossings.

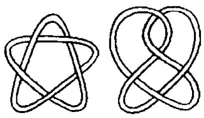
M. B. Thistlethwaite has tabulated the HOMFLY POLYNOMIAL and KAUFFMAN POLYNOMIAL F for KNOTS of up to 13 crossings.



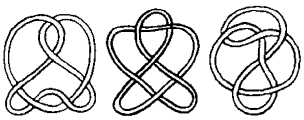
3_1



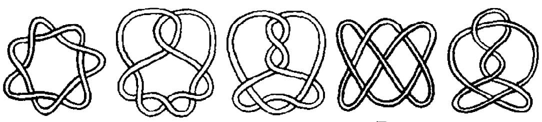
4_1



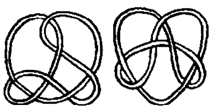
5_1 5_2



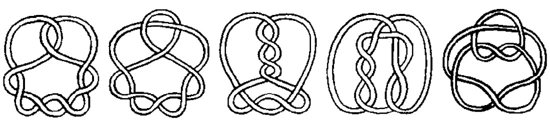
6_1 6_2 6_3



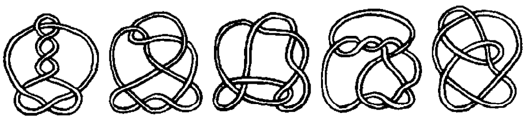
7_1 7_2 7_3 7_4 7_5



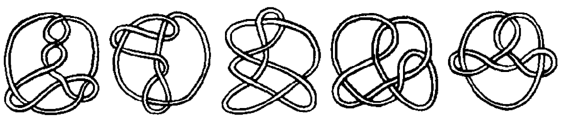
7_6 7_7



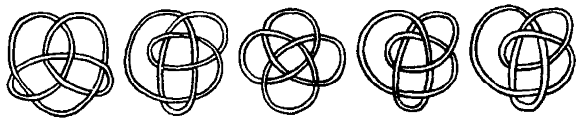
8_1 8_2 8_3 8_4 8_5



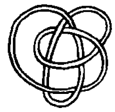
8_6 8_7 8_8 8_9 8_{10}



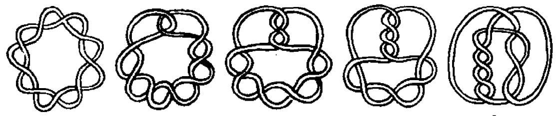
8_{11} 8_{12} 8_{13} 8_{14} 8_{15}



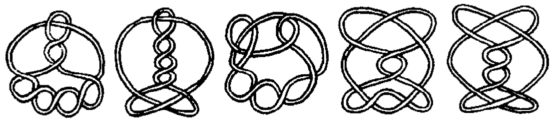
8_{16} 8_{17} 8_{18} 8_{19} 8_{20}



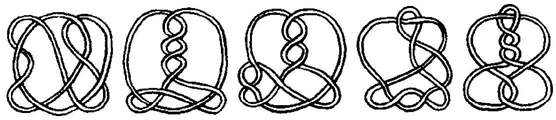
8_{21}



9_1 9_2 9_3 9_4 9_5



9_6 9_7 9_8 9_9 9_{10}



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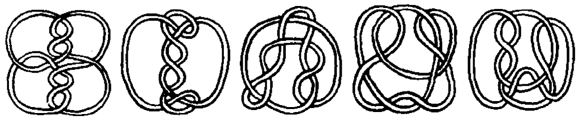

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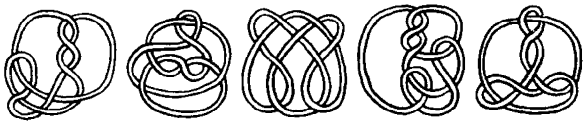

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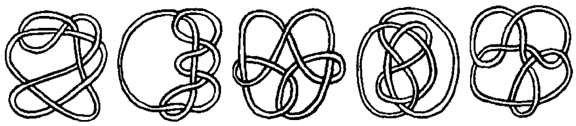
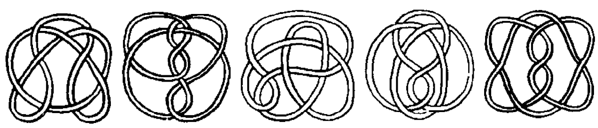


9_{26} 9_{27} 9_{28} 9_{29} 9_{30}



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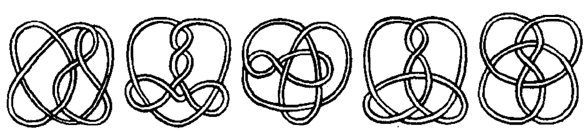

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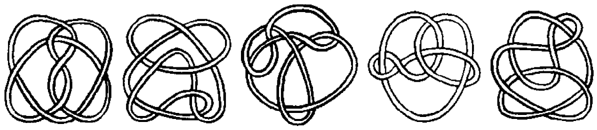
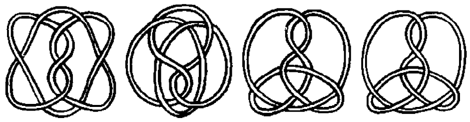


9_{41} 9_{42} 9_{43} 9_{44} 9_{45}



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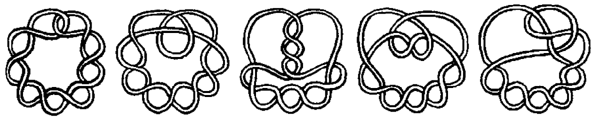

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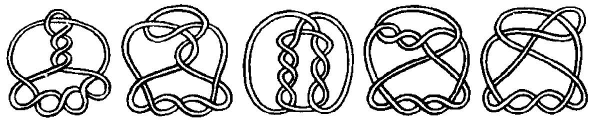

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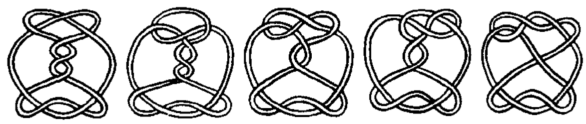

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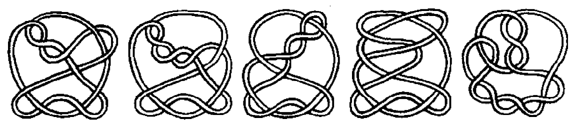

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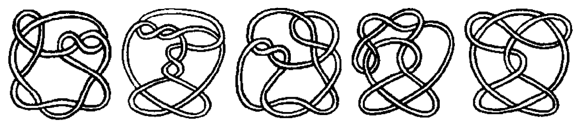

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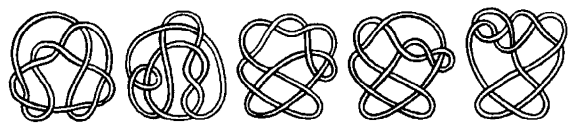

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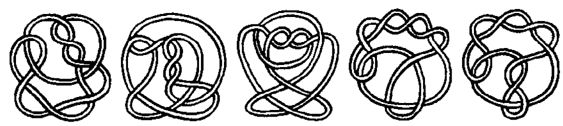

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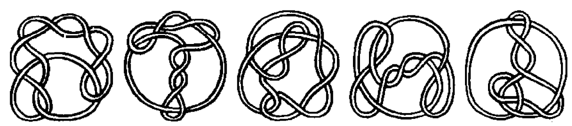

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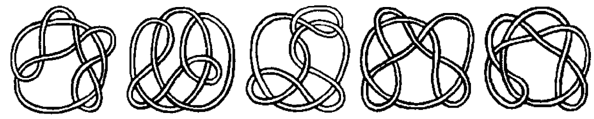

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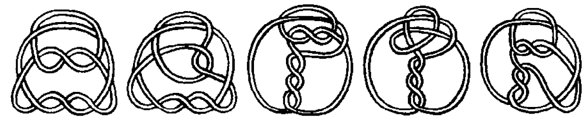

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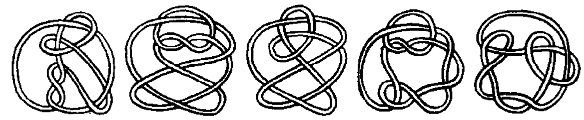

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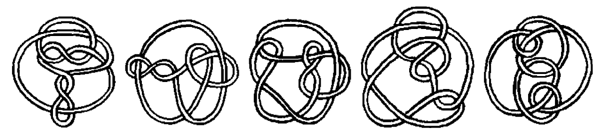

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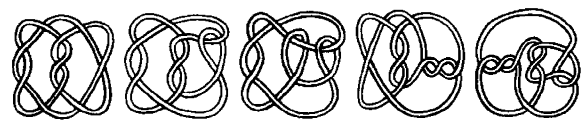

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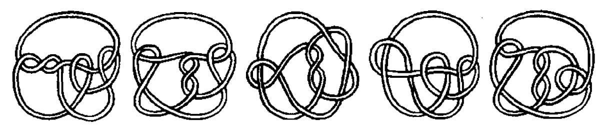

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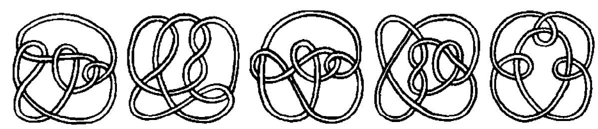

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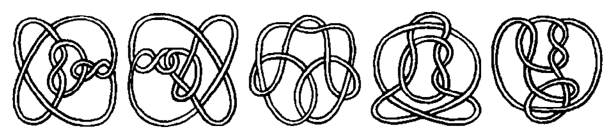

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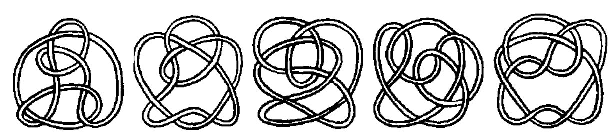

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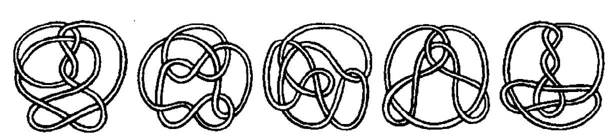

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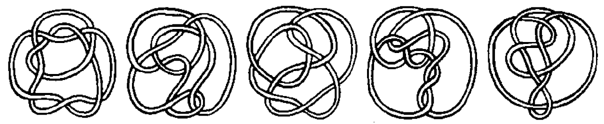

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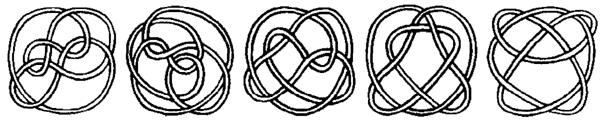

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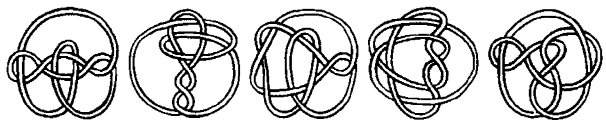

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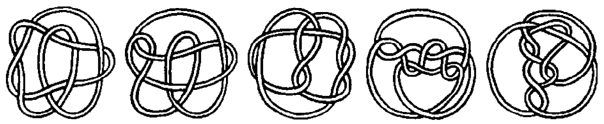

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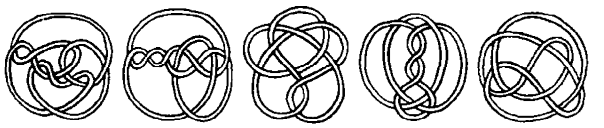

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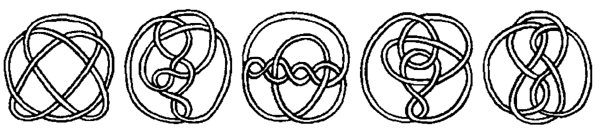

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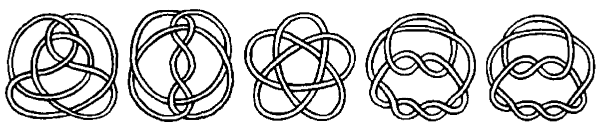

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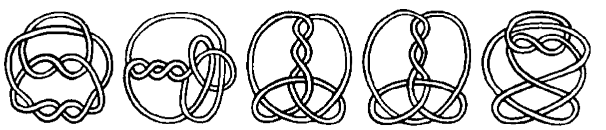

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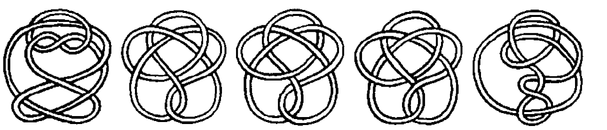

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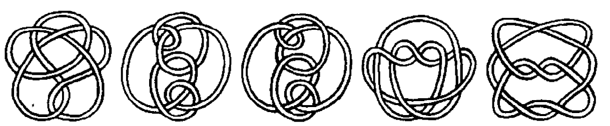

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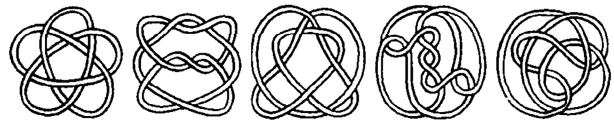

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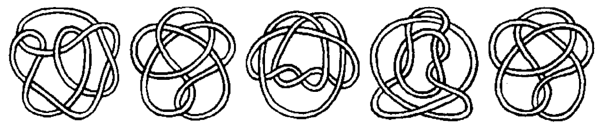

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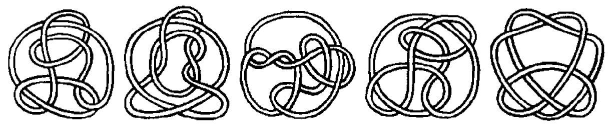

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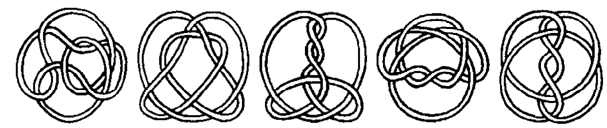

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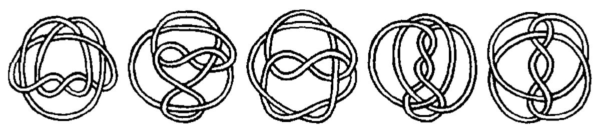
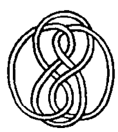


10_{161} 10_{162} 10_{163} 10_{164} 10_{165}



10_{166}

see also ALEXANDER POLYNOMIAL, ALEXANDER'S HORNED SPHERE, AMBIENT ISOTOPY, AMPHICHIRAL, ANTOINE'S NECKLACE, BEND (KNOT), BENNEQUIN'S CONJECTURE, BORROMEAN RINGS, BRAID GROUP, BRUNNIAN LINK, BURAU REPRESENTATION, CHEFALO KNOT, CLOVE HITCH, COLORABLE, CONWAY'S KNOT, CROOKEDNESS, DEHN'S LEMMA, DOWKER NOTATION, FIGURE-OF-EIGHT KNOT, GRANNY KNOT, HITCH, INVERTIBLE KNOT, JONES POLYNOMIAL, KINOSHITA-TERASAKA KNOT, KNOT POLYNOMIAL, KNOT SUM, LINKING NUMBER, LOOP (KNOT), MARKOV'S THEOREM, MENASCO'S THEOREM, MILNOR'S CONJECTURE, NASTY KNOT, PRETZEL KNOT, PRIME KNOT, REIDEMEISTER MOVES, RIBBON KNOT, RUNNING KNOT, SCHÖNFLIES THEOREM, SHORTENING, SIGNATURE (KNOT), SKEIN RELATIONSHIP, SLICE KNOT, SLIP KNOT, SMITH CONJECTURE, SOLOMON'S SEAL KNOT, SPAN (LINK), SPLITTING, SQUARE KNOT, STEVEDORE'S KNOT, STICK NUMBER, STOPPER KNOT, TAIT'S KNOT CONJECTURES, TAME KNOT, TANGLE, TORSION NUMBER, TREFOIL KNOT, UNKNOT, UNKNOTTING NUMBER, VASSILIEV POLYNOMIAL, WHITEHEAD LINK

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Knot Complement

Two distinct knots cannot have the same KNOT COMPLEMENT (Gordon and Luecke 1989).

References

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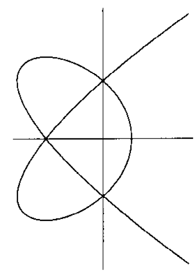
Knot Curve



$$(x^2-1)^2 = y^2(3+2y).$$

References

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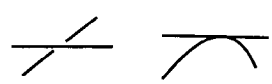
Knot Determinant

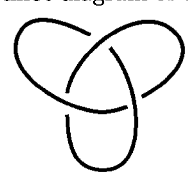
The determinant of a knot is $|\Delta(-1)|$, where $\Delta(z)$ is the ALEXANDER POLYNOMIAL.

Knot Diagram

A picture of a projection of a KNOT onto a PLANE. Usually, only double points are allowed (no more than two points are allowed to be superposed), and the double or crossing points must be "genuine crossings" which transverse in the plane. This means that double points must look like the below diagram on the left, and not the one on the right.



Also, it is usually demanded that a knot diagram contain the information if the crossings are overcrossings or undercrossings so that the original knot can be reconstructed. Here is a knot diagram of the TREFOIL KNOT,



KNOT POLYNOMIALS can be computed from knot diagrams. Such POLYNOMIALS often (but not always) allow the knots corresponding to given diagrams to be uniquely identified.

Knot Exterior

The COMPLEMENT of an open solid TORUS knotted at the KNOT. The removed open solid TORUS is called a tubular NEIGHBORHOOD.

Knot Linking

In general, it is possible to link two n-D HYPERSPHERES in $(n+2)$-D space in an infinite number of inequivalent ways. In dimensions greater than $n+2$ in the piecewise linear category, it is true that these spheres are themselves unknotted. However, they may still form nontrivial links. In this way, they are something like higher dimensional analogs of two 1-spheres in 3-D. The following table gives the number of nontrivial ways that two n-D HYPERSPHERES can be linked in k-D.

D of spheres	D of space	Distinct Linkings
23	40	239
31	48	959
102	181	3
102	182	10438319
102	183	3

Two 10-D HYPERSPHERES link up in 12, 13, 14, 15, and 16-D, then unlink in 17-D, link up again in 18, 19, 20, and 21-D. The proof of these results consists of an "easy part" (Zeeman 1962) and a "hard part" (Ravenel 1986). The hard part is related to the calculation of the (stable and unstable) HOMOTOPY GROUPS of SPHERES.

References

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Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 7, 1976.

Zeeman. "Isotopies and Knots in Manifolds." In *Topology of 3-Manifolds and Related Topics* (Ed. M. K. Fort). Englewood Cliffs, NJ: Prentice-Hall, 1962.

Knot Polynomial

A knot invariant in the form of a POLYNOMIAL such as the ALEXANDER POLYNOMIAL, BLM/HO POLYNOMIAL, BRACKET POLYNOMIAL, CONWAY POLYNOMIAL, JONES POLYNOMIAL, KAUFFMAN POLYNOMIAL F, KAUFFMAN POLYNOMIAL X, and VASSILIEV POLYNOMIAL.

References

Lickorish, W. B. R. and Millett, K. C. "The New Polynomial Invariants of Knots and Links." *Math. Mag.* **61**, 3–23, 1988.

Knot Problem

The problem of deciding if two KNOTS in 3-space are equivalent such that one can be continuously deformed into another.

Knot Shadow

A LINK DIAGRAM which does not specify whether crossings are under- or overcrossings.

Knot Sum

Two oriented knots (or links) can be summed by placing them side by side and joining them by straight bars so that orientation is preserved in the sum. This operation is denoted #, so the knot sum of knots K_1 and K_2 is written

$$K_1 \# K_2 = K_2 \# K_1.$$

see also CONNECTED SUM

Knot Theory

The mathematical study of KNOTS. Knot theory considers questions such as the following:

1. Given a tangled loop of string, is it really knotted or can it, with enough ingenuity and/or luck, be untangled without having to cut it?
2. More generally, given two tangled loops of string, when are they deformable into each other?
3. Is there an effective algorithm (or any algorithm to speak of) to make these determinations?

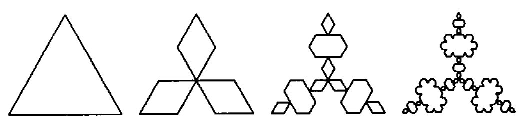
Although there has been almost explosive growth in the number of important results proved since the discovery of the JONES POLYNOMIAL, there are still many "knotty" problems and conjectures whose answers remain unknown.

see also KNOT, LINK

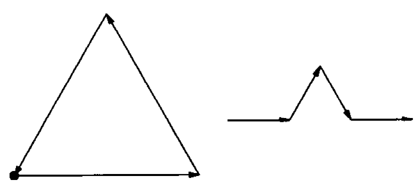
Knot Vector

see B-SPLINE

Koch Antisnowflake



A FRACTAL derived from the KOCH SNOWFLAKE. The base curve and motif for the fractal are illustrated below.



The AREA after the nth iteration is

$$A_n = A_{n-1} - \frac{1}{3}\frac{\ell_{n-1}}{a}\frac{\Delta}{3^n},$$

where Δ is the area of the original EQUILATERAL TRIANGLE, so from the derivation for the KOCH SNOWFLAKE,

$$A \equiv \lim_{n\to\infty} A_n = (1 - \tfrac{3}{5})\Delta = \tfrac{2}{5}\Delta.$$

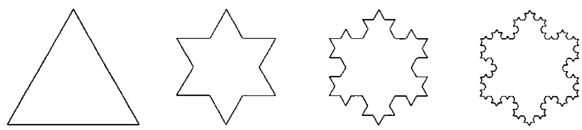
see also EXTERIOR SNOWFLAKE, FLOWSNAKE FRACTAL, KOCH SNOWFLAKE, PENTAFLAKE, SIERPIŃSKI CURVE

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 66–67, 1989.
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 36–37, 1991.
Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

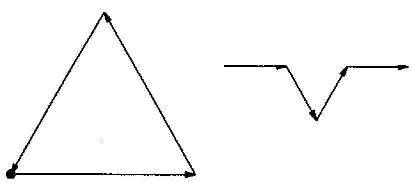
Koch Island

see KOCH SNOWFLAKE

Koch Snowflake



A FRACTAL, also known as the KOCH ISLAND, which was first described by Helge von Koch in 1904. It is built by starting with an EQUILATERAL TRIANGLE, removing the inner third of each side, building another EQUILATERAL TRIANGLE at the location where the side was removed, and then repeating the process indefinitely. The Koch snowflake can be simply encoded as a LINDENMAYER SYSTEM with initial string `"F--F--F"`, STRING REWRITING rule `"F" -> "F+F--F+F"`, and angle 60°. The zeroth through third iterations of the construction are shown above. The fractal can also be constructed using a base curve and motif, illustrated below.



Let N_n be the number of sides, L_n be the length of a single side, ℓ_n be the length of the PERIMETER, and A_n the snowflake's AREA after the nth iteration. Further, denote the AREA of the initial $n=0$ TRIANGLE Δ, and the length of an initial $n=0$ side 1. Then

$$N_n = 3\cdot 4^n \tag{1}$$

$$L_n = (\tfrac{1}{3})^n = 3^{-n} \tag{2}$$

$$\ell_n \equiv N_n L_n = 3(\tfrac{4}{3})^n \tag{3}$$

$$\begin{aligned} A_n &= A_{n-1} + \tfrac{1}{4} N_n {L_n}^2 \Delta = A_{n-1} + \frac{3\cdot 4^n}{4}\left(\frac{1}{3}\right)^{2n}\Delta \\ &= A_{n-1} + \frac{3\cdot 4^{n-1}}{9^n}\Delta = A_{n-1} + \frac{3\cdot 4^{4-1}}{9\cdot 9^{n-1}}\Delta \\ &= A_{n-1} + \tfrac{1}{3}(\tfrac{4}{9})^{n-1}\Delta. \end{aligned} \tag{4}$$

The CAPACITY DIMENSION is then

$$\begin{aligned} d_{\text{cap}} &= -\lim_{n\to\infty} \frac{\ln N_n}{\ln L_n} = -\lim_{n\to\infty} \frac{\ln(3\cdot 4^n)}{\ln(3^{-n})} \\ &= \lim_{n\to\infty} \frac{\ln 3 + n\ln 4}{n \ln 3} \\ &= \frac{\ln 4}{\ln 3} = \frac{2\ln 2}{\ln 3} = 1.261859507\ldots. \end{aligned} \tag{5}$$

Now compute the AREA explicitly,

$$A_0 = \Delta \tag{6}$$

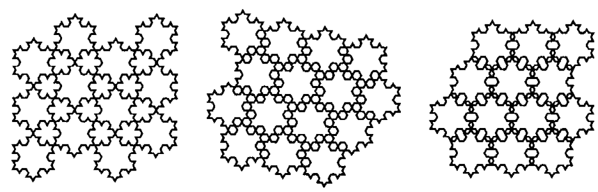
$$A_1 = A_0 + \frac{1}{3}\left(\frac{4}{9}\right)^0 \Delta = \Delta\left\{1 + \frac{1}{3}\left(\frac{4}{9}\right)^0\right\} \tag{7}$$

$$A_2 = A_1 + \frac{1}{3}\left(\frac{4}{9}\right)^1 \Delta = \Delta\left\{1 + \frac{1}{3}\left[\left(\frac{4}{9}\right)^0 + \left(\frac{4}{9}\right)^1\right]\right\} \tag{8}$$

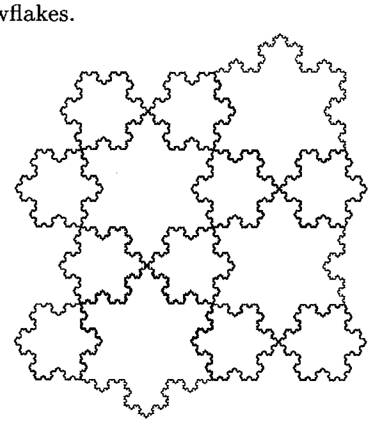
$$A_n = \left[1 + \frac{1}{3}\sum_{k=0}^{n}\left(\frac{4}{9}\right)^k\right]\Delta, \tag{9}$$

so as $n\to\infty$,

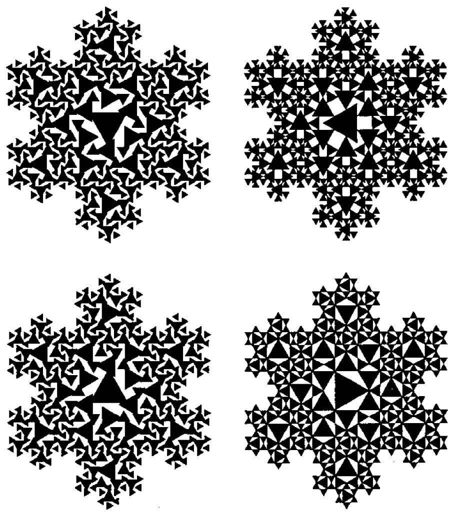
$$\begin{aligned} A \equiv A_\infty &= \left[1 + \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k\right] = \left(1 + \tfrac{1}{3}\frac{1}{1-\frac{4}{9}}\right)\Delta \\ &= \tfrac{8}{5}\Delta. \end{aligned} \tag{10}$$



Some beautiful TILINGS, a few examples of which are illustrated above, can be made with iterations toward Koch snowflakes.



In addition, two sizes of Koch snowflakes in AREA ratio 1:3 TILE the PLANE, as shown above (Gosper).



Another beautiful modification of the Koch snowflake involves inscribing the constituent triangles with filled-in triangles, possibly rotated at some angle. Some sample results are illustrated above for 3 and 4 iterations.

see also CESÀRO FRACTAL, EXTERIOR SNOWFLAKE, GOSPER ISLAND, KOCH ANTISNOWFLAKE, PEANO-GOSPER CURVE, PENTAFLAKE, SIERPIŃSKI SIEVE

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Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Kochansky's Approximation

The approximation for PI,

$$\pi \approx \sqrt{\frac{40}{3} - \sqrt{12}} = 3.141533\ldots.$$

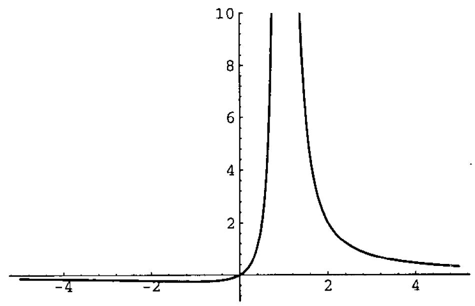
Koebe's Constant

A CONSTANT equal to one QUARTER, 1/4.

see also QUARTER

References
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 24, 1983.

Koebe Function



The function

$$f(z) \equiv \frac{z}{(1-z)^2}.$$

It has a MINIMUM at $z = -1$, where

$$f'(z) = -\frac{1+z}{(z-1)^3} = 0,$$

and an INFLECTION POINT at $z = -2$, where

$$f''(z) = \frac{2(2+z)}{(z-1)^4} = 0.$$

References
Stewart, I. *From Here to Infinity: A Guide to Today's Mathematics.* Oxford, England: Oxford University Press, pp. 164–165, 1996.

Kollros' Theorem

For every ring containing p SPHERES, there exists a ring of q SPHERES, each touching each of the p SPHERES, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{3}.$$

The HEXLET is a special case with $p = 3$.

see also HEXLET, SPHERE

References
Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., p. 50, 1976.

Kolmogorov-Arnold-Moser Theorem

A theorem outlined in 1954 by Kolmogorov which was subsequently proved in the 1960s by Arnold and Moser (Tabor 1989, p. 105). It gives conditions under which CHAOS is restricted in extent. Moser's 1962 proof was valid for TWIST MAPS

$$\theta' = \theta + 2\pi f(I) + g(\theta, I) \quad (1)$$

$$I' = I + f(\theta, I). \quad (2)$$

In 1963, Arnold produced a proof for Hamiltonian systems

$$H = H_0(\mathbf{I}) + \epsilon H_1(\mathbf{I}). \quad (3)$$

The original theorem required perturbations $\epsilon \sim 10^{-48}$, although this has since been significantly increased. Arnold's proof required C^∞, and Moser's original proof required C^{333}. Subsequently, Moser's version has been reduced to C^6, then $C^{2+\epsilon}$, although counterexamples are known for C^2. Conditions for applicability of the KAM theorem are:

1. small perturbations,
2. smooth perturbations, and
3. sufficiently irrational WINDING NUMBER.

Moser considered an integrable Hamiltonian function H_0 with a TORUS T_0 and set of frequencies ω having an incommensurate frequency vector $\boldsymbol{\omega}^*$ (i.e., $\boldsymbol{\omega}\cdot\mathbf{k}\neq 0$ for all INTEGERS k_i). Let H_0 be perturbed by some periodic function H_1. The KAM theorem states that, if H_1 is small enough, then for almost every ω^* there exists an invariant TORUS $T(\omega^*)$ of the perturbed system such that $T(\omega^*)$ is "close to" $T_0(\omega^*)$. Moreover, the TORI $T(\omega^*)$ form a set of POSITIVE measures whose complement has a measure which tends to zero as $|H_1|\to 0$. A useful paraphrase of the KAM theorem is, "For sufficiently small perturbation, almost all TORI (excluding those with rational frequency vectors) are preserved." The theorem thus explicitly excludes TORI with rationally related frequencies, that is, $n-1$ conditions of the form

$$\boldsymbol{\omega}\cdot\mathbf{k}=0. \tag{4}$$

These TORI are destroyed by the perturbation. For a system with two DEGREES OF FREEDOM, the condition of closed orbits is

$$\sigma=\frac{\omega_1}{\omega_2}=\frac{r}{s}. \tag{5}$$

For a QUASIPERIODIC ORBIT, σ is IRRATIONAL. KAM shows that the preserved TORI satisfy the irrationality condition

$$\left|\frac{\omega_1}{\omega_2}-\frac{r}{s}\right|>\frac{K(\epsilon)}{s^{2.5}} \tag{6}$$

for all r and s, although not much is known about $K(\epsilon)$.

The KAM theorem broke the deadlock of the small divisor problem in classical perturbation theory, and provides the starting point for an understanding of the appearance of CHAOS. For a HAMILTONIAN SYSTEM, the ISOENERGETIC NONDEGENERACY condition

$$\left|\frac{\partial^2 H_0}{\partial I_j \partial I_j}\right|\neq 0 \tag{7}$$

guarantees preservation of most invariant TORI under small perturbations $\epsilon\ll 1$. The Arnold version states that

$$\left|\sum_{k=1}^n m_k\omega_k\right|>K(\epsilon)\left(\sum_{k=1}^n |m_k|\right)^{-n-1} \tag{8}$$

for all $m_k\in\mathbb{Z}$. This condition is less restrictive than Moser's, so fewer points are excluded.

see also CHAOS, HAMILTONIAN SYSTEM, QUASIPERIODIC FUNCTION, TORUS

References

Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, 1989.

Kolmogorov Complexity

The complexity of a pattern parameterized as the shortest ALGORITHM required to reproduce it. Also known as ALGORITHMIC COMPLEXITY.

References

Goetz, P. "Phil's Good Enough Complexity Dictionary." http://www.cs.buffalo.edu/~goetz/dict.html.

Kolmogorov Constant

The exponent 5/3 in the spectrum of homogeneous turbulence, $k^{-5/3}$.

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 41, 1983.

Kolmogorov Entropy

Also known as METRIC ENTROPY. Divide PHASE SPACE into D-dimensional HYPERCUBES of CONTENT ϵ^D. Let $P_{i_0,\ldots,i_n}$ be the probability that a trajectory is in HYPERCUBE i_0 at $t=0$, i_1 at $t=T$, i_2 at $t=2T$, etc. Then define

$$K_n=h_K=-\sum_{i_0,\ldots,i_n} P_{i_0,\ldots,i_n}\ln P_{i_0,\ldots,i_n}, \tag{1}$$

where $K_{N+1}-K_N$ is the information needed to predict which HYPERCUBE the trajectory will be in at $(n+1)T$ given trajectories up to nT. The Kolmogorov entropy is then defined by

$$K\equiv\lim_{T\to 0}\lim_{\epsilon\to 0^+}\lim_{N\to\infty}\frac{1}{NT}\sum_{n=0}^{N-1}(K_{n+1}-K_n). \tag{2}$$

The Kolmogorov entropy is related to LYAPUNOV CHARACTERISTIC EXPONENTS by

$$h_K=\int_P\sum_{\sigma_i>0}\sigma_i\,d\mu. \tag{3}$$

see also HYPERCUBE, LYAPUNOV CHARACTERISTIC EXPONENT

References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 138, 1993.
Schuster, H. G. *Deterministic Chaos: An Introduction, 3rd ed.* New York: Wiley, p. 112, 1995.

Kolmogorov-Sinai Entropy

see KOLMOGOROV ENTROPY, METRIC ENTROPY

Kolmogorov-Smirnov Test

A goodness-of-fit test for any DISTRIBUTION. The test relies on the fact that the value of the sample cumulative density function is asymptotically normally distributed.

To apply the Kolmogorov-Smirnov test, calculate the cumulative frequency (normalized by the sample size) of the observations as a function of class. Then calculate the cumulative frequency for a true distribution (most commonly, the NORMAL DISTRIBUTION). Find the greatest discrepancy between the observed and expected cumulative frequencies, which is called the "D-STATISTIC." Compare this against the critical D-STATISTIC for that sample size. If the calculated D-STATISTIC is greater than the critical one, then reject the NULL HYPOTHESIS that the distribution is of the expected form. The test is an R-ESTIMATE.

see also ANDERSON-DARLING STATISTIC, D-STATISTIC, KUIPER STATISTIC, NORMAL DISTRIBUTION, R-ESTIMATE

References

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Knuth, D. E. §3.3.1B in *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms, 2nd ed.* Reading, MA: Addison-Wesley, pp. 45–52, 1981.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Kolmogorov-Smirnov Test." In *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 617–620, 1992.

König-Egeváry Theorem

A theorem on BIPARTITE GRAPHS.

see also BIPARTITE GRAPH, FROBENIUS-KÖNIG THEOREM

König's Theorem

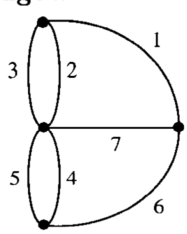
If an ANALYTIC FUNCTION has a single simple POLE at the RADIUS OF CONVERGENCE of its POWER SERIES, then the ratio of the coefficients of its POWER SERIES converges to that POLE.

see also POLE

References

König, J. "Über eine Eigenschaft der Potenzreihen." *Math. Ann.* **23**, 447–449, 1884.

Königsberg Bridge Problem



The Königsberg bridges cannot all be traversed in a single trip without doubling back. This problem was solved by Euler, and represented the beginning of GRAPH THEORY.

see also EULERIAN CIRCUIT, GRAPH THEORY

References

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Newman, J. "Leonhard Euler and the Königsberg Bridges." *Sci. Amer.* **189**, 66–70, 1953.

Pappas, T. "Königsberg Bridge Problem & Topology." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 124–125, 1989.

Korselt's Criterion

n DIVIDES $a^n - a$ for all INTEGERS a IFF n is SQUAREFREE and $(p-1)|n/p - 1$ for all PRIME DIVISORS p of n. CARMICHAEL NUMBERS satisfy this CRITERION.

References

Borwein, D.; Borwein, J. M.; Borwein, P. B.; and Girgensohn, R. "Giuga's Conjecture on Primality." *Amer. Math. Monthly* **103**, 40–50, 1996.

Kovalevskaya Exponent

see LEADING ORDER ANALYSIS

Kozyrev-Grinberg Theory

A theory of HAMILTONIAN CIRCUITS.

see also GRINBERG FORMULA, HAMILTONIAN CIRCUIT

Kramers Rate

The characteristic escape rate from a stable state of a potential in the absence of signal.

see also STOCHASTIC RESONANCE

References

Bulsara, A. R. and Gammaitoni, L. "Tuning in to Noise." *Phys. Today* **49**, 39–45, March 1996.

Krawtchouk Polynomial

Let $\alpha(x)$ be a STEP FUNCTION with the JUMP

$$j(x) = \binom{N}{x} p^x q^{N-x} \tag{1}$$

at $x = 0, 1, \ldots, N$, where $p > 0, q > 0$, and $p + q = 1$. Then

$$k_n^{(p)}(x) = \left[\binom{N}{n}\right]^{-1/2} (pq)^{-n/2} \times \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{N-x}{n-\nu} \binom{x}{\nu} p^{n-\nu} q^{\nu}, \tag{2}$$

for $n = 0, 1, \ldots, N$. It has WEIGHT FUNCTION

$$w = \frac{N! p^x q^{N-x}}{\Gamma(1+x)\Gamma(N+1-x)}, \tag{3}$$

where $\Gamma(x)$ is the GAMMA FUNCTION, RECURRENCE RELATION

$$(n+1)k_{n+1}^{(p)}(x) + pq(N-n+1)k_{n-1}^{(p)}(x) = [x - n - (N-2)]k_n^{(p)}(x), \tag{4}$$

and squared norm

$$\frac{N!}{n!(N-n)!}(pq)^n. \tag{5}$$

It has the limit

$$\lim_{n\to\infty}\left(\frac{2}{Npq}\right)^{n/2} n! k_n^{(p)}(Np + \sqrt{2Npqs}) = H_n(s), \tag{6}$$

where $H_n(x)$ is a HERMITE POLYNOMIAL, and is related to the HYPERGEOMETRIC FUNCTION by

$$\begin{aligned} k_n^{(p)}(x,N) &= k_n^{(p)}(x,N) \\ &= (-1)^n \binom{N}{n} p^n {}_2F_1(-n,-x;-N;1/p) \\ &\quad \frac{(-1)^n p^n}{n!} \frac{\Gamma(N-x+1)}{\Gamma(N-x-n+1)} \\ &\quad \times {}_2F_1(-n,-x;N-x-n+1;-q/p). \end{aligned} \tag{7}$$

see also ORTHOGONAL POLYNOMIALS

References

Nikiforov, A. F.; Uvarov, V. B.; and Suslov, S. S. *Classical Orthogonal Polynomials of a Discrete Variable.* New York: Springer-Verlag, 1992.

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 35–37, 1975.

Zelenkov, V. "Krawtchouk Polynomial Home Page." `http://www.isir.minsk.by/~zelenkov/physmath/kr_polyn/`.

Kreisel Conjecture

A CONJECTURE in DECIDABILITY theory which postulates that, if there is a uniform bound to the lengths of shortest proofs of instances of $S(n)$, then the universal generalization is necessarily provable in PEANO ARITHMETIC. The CONJECTURE was proven true by M. Baaz in 1988 (Baaz and Pudlák 1993).

see also DECIDABLE

References

Baaz, M. and Pudlák P. "Kreisel's Conjecture for $L\exists_1$. In *Arithmetic, Proof Theory, and Computational Complexity, Papers from the Conference Held in Prague, July 2–5, 1991* (Ed. P. Clote and J. Krajiček). New York: Oxford University Press, pp. 30–60, 1993.

Dawson, J. "The Gödel Incompleteness Theorem from a Length of Proof Perspective." *Amer. Math. Monthly* **86**, 740–747, 1979.

Kreisel, G. "On the Interpretation of Nonfinitistic Proofs, II." *J. Symbolic Logic* **17**, 43–58, 1952.

Kronecker Decomposition Theorem

Every FINITE ABELIAN GROUP can be written as a DIRECT PRODUCT of CYCLIC GROUPS of PRIME POWER ORDERS. In fact, the number of nonisomorphic ABELIAN FINITE GROUPS $a(n)$ of any given ORDER n is given by writing n as

$$n = \prod_i p_i^{\alpha_i},$$

where the p_i are distinct PRIME FACTORS, then

$$a(n) = \prod_i P(\alpha_i),$$

where P is the PARTITION FUNCTION. This gives 1, 1, 1, 2, 1, 1, 1, 3, 2, ... (Sloane's A000688).

see also ABELIAN GROUP, FINITE GROUP, ORDER (GROUP), PARTITION FUNCTION P

References

Sloane, N. J. A. Sequence A000688/M0064 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Kronecker Delta

The simplest interpretation of the Kronecker delta is as the discrete version of the DELTA FUNCTION defined by

$$\delta_{ij} \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases} \tag{1}$$

It has the COMPLEX GENERATING FUNCTION

$$\delta_{mn} = \frac{1}{2\pi i}\int z^{m-n-1}\,dz, \tag{2}$$

where m and n are INTEGERS. In 3-space, the Kronecker delta satisfies the identities

$$\delta_{ii} = 3 \tag{3}$$

$$\delta_{ij}\epsilon_{ijk} = 0 \tag{4}$$

$$\epsilon_{ipq}\epsilon_{jpq} = 2\delta_{ij} \tag{5}$$

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \tag{6}$$

where EINSTEIN SUMMATION is implicitly assumed, $i, j = 1, 2, 3$, and ϵ is the PERMUTATION SYMBOL.

Technically, the Kronecker delta is a TENSOR defined by the relationship

$$\delta_l^k \frac{\partial x_i'}{\partial x_k}\frac{\partial x_l}{\partial x_j'} = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_k}{\partial x_j'} = \frac{\partial x_i'}{\partial x_j'}. \tag{7}$$

Since, by definition, the coordinates x_i and x_j are independent for $i \neq j$,

$$\frac{\partial x_i'}{\partial x_j'} = \delta'^i_j, \tag{8}$$

so
$$\delta'^{i}_{j} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x_l}{\partial x'_j}\delta^k_l, \tag{9}$$
and δ^i_j is really a mixed second RANK TENSOR. It satisfies
$$\delta_{ab}{}^{jk} = \epsilon_{abi}\epsilon^{jki} = \delta^j_a\delta^k_b - \delta^k_a\delta^j_b \tag{10}$$
$$\delta_{abjk} = g_{aj}g_{bk} - g_{ak}g_{bj} \tag{11}$$
$$\epsilon_{aij}\epsilon^{bij} = \delta_{ai}{}^{bi} = 2\delta^b_a. \tag{12}$$

see also DELTA FUNCTION, PERMUTATION SYMBOL

Kronecker's Polynomial Theorem

An algebraically soluble equation of ODD PRIME degree which is irreducible in the natural FIELD possesses either

1. Only a single REAL ROOT, or
2. All REAL ROOTS.

see also ABEL'S IRREDUCIBILITY THEOREM, ABEL'S LEMMA, SCHOENEMANN'S THEOREM

References
Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover p. 127, 1965.

Kronecker Product

see DIRECT PRODUCT (MATRIX)

Kronecker Symbol

An extension of the JACOBI SYMBOL (n/m) to all INTEGERS. It can be computed using the normal rules for the JACOBI SYMBOL
$$\left(\frac{ab}{cd}\right) = \left(\frac{a}{cd}\right)\left(\frac{b}{cd}\right) = \left(\frac{ab}{c}\right)\left(\frac{ab}{d}\right)$$
$$= \left(\frac{a}{c}\right)\left(\frac{b}{c}\right)\left(\frac{a}{d}\right)\left(\frac{b}{d}\right)$$
plus additional rules for $m = -1$,
$$(n/-1) = \begin{cases} -1 & \text{for } n < 0 \\ 1 & \text{for } n > 0, \end{cases}$$
and $m = 2$. The definition for $(n/2)$ is variously written as
$$(n/2) \equiv \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd, } n \equiv \pm 1 \pmod 8 \\ -1 & \text{for } n \text{ odd, } n \equiv \pm 3 \pmod 8 \end{cases}$$
or
$$(n/2) \equiv \begin{cases} 0 & \text{for } 4|n \\ 1 & \text{for } n \equiv 1 \pmod 8 \\ -1 & \text{for } n \equiv 5 \pmod 8 \\ \text{undefined} & \text{otherwise} \end{cases}$$
(Cohn 1980). Cohn's form "undefines" $(n/2)$ for SINGLY EVEN NUMBERS $n \equiv 4 \pmod 2$ and $n \equiv -1, 3 \pmod 8$, probably because no other values are needed in applications of the symbol involving the DISCRIMINANTS d of QUADRATIC FIELDS, where $m > 0$ and d always satisfies $d \equiv 0, 1 \pmod 4$.

The KRONECKER SYMBOL is a REAL CHARACTER modulo n, and is, in fact, essentially the only type of REAL primitive character (Ayoub 1963).

see also CHARACTER (NUMBER THEORY), CLASS NUMBER, DIRICHLET L-SERIES, JACOBI SYMBOL, LEGENDRE SYMBOL

References
Ayoub, R. G. *An Introduction to the Analytic Theory of Numbers.* Providence, RI: Amer. Math. Soc., 1963.
Cohn, H. *Advanced Number Theory.* New York: Dover, p. 35, 1980.

Krull Dimension

If R is a RING (commutative with 1), the height of a PRIME IDEAL p is defined as the SUPREMUM of all n so that there is a chain $p_0 \subset \cdots p_{n-1} \subset p_n = p$ where all p_i are distinct PRIME IDEALS. Then, the Krull dimension of R is defined as the SUPREMUM of all the heights of all its PRIME IDEALS.

see also PRIME IDEAL

References
Eisenbud, D. *Commutative Algebra with a View Toward Algebraic Geometry.* New York: Springer-Verlag, 1995.
Macdonald, I. G. and Atiyah, M. F. *Introduction to Commutative Algebra.* Reading, MA: Addison-Wesley, 1969.

Kruskal's Algorithm

An ALGORITHM for finding a GRAPH's spanning TREE of minimum length.

see also KRUSKAL'S TREE THEOREM

References
Gardner, M. *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions and Other Mathematical Sleight-of-Mind from Scientific American.* New York: Vintage, pp. 248–249, 1978.

Kruskal's Tree Theorem

A theorem which plays a fundamental role in computer science because it is one of the main tools for showing that certain orderings on TREES are well-founded. These orderings play a crucial role in proving the termination of rewriting rules and the correctness of the Knuth-Bendix equational completion procedures.

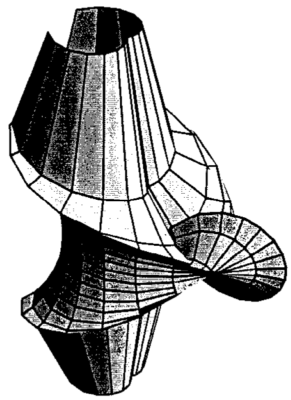
see also KRUSKAL'S ALGORITHM, NATURAL INDEPENDENCE PHENOMENON, TREE

References
Gallier, J. "What's so Special about Kruskal's Theorem and the Ordinal Gamma[0]? A Survey of Some Results in Proof Theory." *Ann. Pure and Appl. Logic* **53**, 199–260, 1991.

KS Entropy

see METRIC ENTROPY

Kuen Surface



A special case of ENNEPER'S SURFACES which can be given parametrically by

$$x = \frac{2(\cos u + u \sin u)\sin v}{1 + u^2 \sin^2 v} \quad (1)$$
$$= \frac{2\sqrt{1+u^2}\,\cos(u - \tan^{-1} u)\sin v}{1 + u^2 \sin^2 v} \quad (2)$$
$$y = \frac{2(\sin u - u \cos u)\sin v}{1 + u^2 \sin^2 v} \quad (3)$$
$$= \frac{2\sqrt{1+u^2}\,\sin(u - \tan^{-1} u)\sin v}{1 + u^2 \sin^2 v} \quad (4)$$
$$z = \ln[\tan(\tfrac{1}{2}v)] + \frac{2\cos v}{1 + u^2 \sin^2 v} \quad (5)$$

for $v \in [0, \pi)$, $u \in [0, 2\pi)$ (Reckziegel *et al.* 1986). The Kuen surface has constant NEGATIVE GAUSSIAN CURVATURE of $K = -1$. The PRINCIPAL CURVATURES are given by

$$\kappa_1 = -\frac{u \cos(\frac{1}{2}v)[-2 - u^2 + u^2 \cos(2v)]^4 \sin(\frac{1}{2}v)}{2[2 - u^2 + u^2 \cos(2v)](1 + u^2 \sin^2 v)^4} \quad (6)$$
$$\kappa_2 = \frac{[-2 - u^2 + u^2 \cos(2v)]^4 [2 - u^2 + u^2 \cos(2v)] \csc(v)}{64u(1 + u^2 \sin^2 v)^4}. \quad (7)$$

see also ENNEPER'S SURFACES, REMBS' SURFACES, SIEVERT'S SURFACE

References

Fischer, G. (Ed.). Plate 86 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 82, 1986.

Gray, A. "Kuen's Surface." §19.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 384–386, 1993.

Kuen, T. "Ueber Flächen von constantem Krümmungsmaass." *Sitzungsber. d. königl. Bayer. Akad. Wiss. Math.-phys. Classe,* Heft II, 193–206, 1884.

Nordstrand, T. "Kuen's Surface." `http://www.uib.no/people/nfytn/kuentxt.htm`.

Reckziegel, H. "Kuen's Surface." §3.4.4.2 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 38, 1986.

Kuhn-Tucker Theorem

A theorem in nonlinear programming which states that if a regularity condition holds and f and the functions h_j are convex, then a solution x^0 which satisfies the conditions h_j for a VECTOR of multipliers λ is a GLOBAL MINIMUM. The Kuhn-Tucker theorem is a generalization of LAGRANGE MULTIPLIERS. FARKAS'S LEMMA is key in proving this theorem.

see also FARKAS'S LEMMA, LAGRANGE MULTIPLIER

Kuiper Statistic

A statistic defined to improve the KOLMOGOROV-SMIRNOV TEST in the TAILS.

see also ANDERSON-DARLING STATISTIC

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 621, 1992.

Kulikowski's Theorem

For every POSITIVE INTEGER n, there exists a SPHERE which has exactly n LATTICE POINTS on its surface. The SPHERE is given by the equation

$$(x-a)^2 + (y-b)^2 + (z - \sqrt{2})^2 = c^2 + 2,$$

where a and b are the coordinates of the center of the so-called SCHINZEL CIRCLE

$$\begin{cases} (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}5^{k-1} & \text{for } n = 2k \text{ even} \\ (x - \frac{1}{3})^2 + y^2 = \frac{1}{9}5^{2k} & \text{for } n = 2k+1 \text{ odd} \end{cases}$$

and c is its RADIUS.

see also CIRCLE LATTICE POINTS, LATTICE POINT, SCHINZEL'S THEOREM

References

Honsberger, R. "Circles, Squares, and Lattice Points." Ch. 11 in *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., pp. 117–127, 1973.

Kulikowski, T. "Sur l'existence d'une sphère passant par un nombre donné aux coordonnées entières." *L'Enseignement Math. Ser. 2* **5**, 89–90, 1959.

Schinzel, A. "Sur l'existence d'un cercle passant par un nombre donné de points aux coordonnées entières." *L'Enseignement Math. Ser. 2* **4**, 71–72, 1958.

Sierpiński, W. "Sur quelques problèmes concernant les points aux coordonnées entières." *L'Enseignement Math. Ser. 2* **4**, 25–31, 1958.

Sierpiński, W. "Sur un problème de H. Steinhaus concernant les ensembles de points sur le plan." *Fund. Math.* **46**, 191–194, 1959.

Sierpiński, W. *A Selection of Problems in the Theory of Numbers.* New York: Pergamon Press, 1964.

Kummer's Conjecture

A conjecture concerning PRIMES.

Kummer's Differential Equation

see CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION

Kummer's Formulas

Kummer's first formula is

$$ {}_2F_1(\tfrac{1}{2}+m-k,-n;2m+1;1) = \frac{\Gamma(2m+1)\Gamma(m+\frac{1}{2}+k+n)}{\Gamma(m+\frac{1}{2}+k)\Gamma(2m+1+n)}, \quad (1) $$

where ${}_2F_1(a,b;c;z)$ is the HYPERGEOMETRIC FUNCTION with $m \neq -1/2, -1, -3/2, \ldots$, and $\Gamma(z)$ is the GAMMA FUNCTION. The identity can be written in the more symmetrical form as

$$ {}_2F_1(a,b;c;-1) = \frac{\Gamma(\frac{1}{2}b+1)\Gamma(b-a+1)}{\Gamma(b+1)\Gamma(\frac{1}{2}b-a+1)}, \quad (2) $$

where $a-b+c=1$ and b is a positive integer. If b is a negative integer, the identity takes the form

$$ {}_2F_1(a,b;c;-1) = 2\cos(\tfrac{1}{2}\pi b)\frac{\Gamma(|b|)\Gamma(b-a+1)}{\Gamma(\frac{1}{2}b-a+1)} \quad (3) $$

(Petkovšek *et al.* 1996).

Kummer's second formula is

$$ {}_1F_1(\tfrac{1}{2}+m;2m+1;z) = M_{0,m}(z) = z^{m+1/2}\left[1+\sum_{p=1}^{\infty}\frac{z^{2p}}{2^{4p}p!(m+1)(m+2)\cdots(m+p)}\right], \quad (4) $$

where ${}_1F_1(a;b;z)$ is the CONFLUENT HYPERGEOMETRIC FUNCTION and $m \neq -1/2, -1, -3/2, \ldots$.

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B*. Wellesley, MA: A. K. Peters, pp. 42–43 and 126, 1996.

Kummer's Function

see CONFLUENT HYPERGEOMETRIC FUNCTION

Kummer Group

A GROUP of LINEAR FRACTIONAL TRANSFORMATIONS which transform the arguments of Kummer solutions to the HYPERGEOMETRIC DIFFERENTIAL EQUATION into each other. Define

$$ A(z) = 1-z $$
$$ B(z) = 1/z, $$

then the elements of the group are $\{I, A, B, AB, BA, ABA = BAB\}$.

Kummer's Quadratic Transformation

A transformation of a HYPERGEOMETRIC FUNCTION,

$$ {}_2F_1\left(\alpha,\beta;2\beta;\frac{4z}{(1+z)^2}\right) = (1+z)^{2\alpha}{}_2F_1(\alpha,\alpha+\tfrac{1}{2}-\beta;\beta+\tfrac{1}{2};z^2). $$

Kummer's Relation

An identity which relates HYPERGEOMETRIC FUNCTIONS,

$$ {}_2F_1(2a,2b;a+b+\tfrac{1}{2};x) = {}_2F_1(a,b;a+b+\tfrac{1}{2},4x(1-x)). $$

Kummer's Series

see HYPERGEOMETRIC FUNCTION

Kummer's Series Transformation

Let $\sum_{k=0}^{\infty} a_k = a$ and $\sum_{k=0}^{\infty} c_k = c$ be convergent series such that

$$ \lim_{k\to\infty}\frac{a_k}{c_k} = \lambda \neq 0. $$

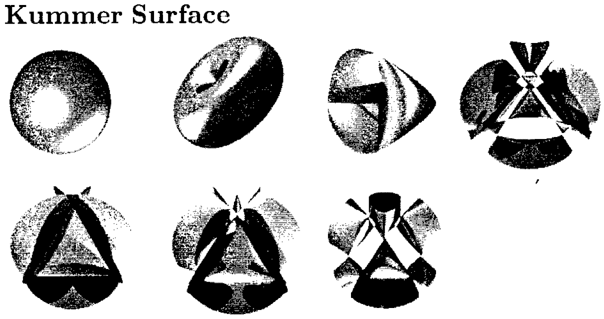
Then

$$ a = \lambda c + \sum_{k=0}^{\infty}\left(1-\lambda\frac{c_k}{a_k}\right)a_k. $$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Kummer Surface



The Kummer surfaces are a family of QUARTIC SURFACES given by the algebraic equation

$$ (x^2+y^2+z^2-\mu^2w^2)^2 - \lambda pqrs = 0, \quad (1) $$

where

$$ \lambda \equiv \frac{3\mu^2-1}{3-\mu^2}, \quad (2) $$

p, q, r, and s are the TETRAHEDRAL COORDINATES

$$ p = w - z - \sqrt{2}\,x \quad (3) $$
$$ q = w - z + \sqrt{2}\,x \quad (4) $$
$$ r = w + z + \sqrt{2}\,y \quad (5) $$
$$ s = w + z - \sqrt{2}\,y, \quad (6) $$

and w is a parameter which, in the above plots, is set to $w = 1$. The above plots correspond to $\mu^2 = 1/3$

$$(3x^2 + 3y^2 + 3z^2 + 1)^2 = 0,$$

(double sphere), 2/3, 1

$$x^4 - 2x^2y^2 + y^4 + 4x^2z + 4y^2z + 4x^2z^2 + 4y^2z^2 = 0 \quad (7)$$

(ROMAN SURFACE), $\sqrt{2}$, $\sqrt{3}$

$$[(z-1)^2 - 2x^2][y^2 - (z+1)^2] = 0 \quad (8)$$

(four planes), 2, and 5. The case $0 \leq \mu^2 \leq 1/3$ corresponds to four real points.

The following table gives the number of ORDINARY DOUBLE POINTS for various ranges of μ^2, corresponding to the preceding illustrations.

Range	Real Nodes	Complex Nodes
$0 \leq \mu^2 \leq \frac{1}{3}$	4	12
$\mu^2 = \frac{1}{3}$		
$\frac{1}{3} \leq \mu^2 < 1$	4	12
$\mu^2 = 1$		
$1 < \mu^2 < 3$	16	0
$\mu^2 = 3$		
$\mu^2 > 3$	16	0

The Kummer surfaces can be represented parametrically by hyperelliptic THETA FUNCTIONS. Most of the Kummer surfaces admit 16 ORDINARY DOUBLE POINTS, the maximum possible for a QUARTIC SURFACE. A special case of a Kummer surface is the TETRAHEDROID.

Nordstrand gives the implicit equations as

$$x^4+y^4+z^4-x^2-y^2-z^2-x^2y^2-x^2z^2-y^2z^2+1 = 0 \quad (9)$$

or

$$x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2) + b(x^2y^2 + x^2z^2 + y^2z^2) + cxyz - 1 = 0. \quad (10)$$

see also QUARTIC SURFACE, ROMAN SURFACE, TETRAHEDROID

References

Endraß, S. "Flächen mit vielen Doppelpunkten." *DMV-Mitteilungen* **4**, 17–20, Apr. 1995.

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Kummer's Test

Given a SERIES of POSITIVE terms u_i and a sequence of finite POSITIVE constants a_i, let

$$\rho \equiv \lim_{n\to\infty} \left(a_n \frac{u_n}{u_{n+1}} - a_{n+1} \right).$$

1. If $\rho > 0$, the series converges.
2. If $\rho < 0$, the series diverges.
3. If $\rho = 0$, the series may converge or diverge.

The test is a general case of BERTRAND'S TEST, the ROOT TEST, GAUSS'S TEST, and RAABE'S TEST. With $a_n = n$ and $a_{n+1} = n + 1$, the test becomes RAABE'S TEST.

see also CONVERGENCE TESTS, RAABE'S TEST

References

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Kummer's Theorem

$${}_2F_1(x, -x; x+n+1; -1) = \frac{\Gamma(x+n+1)\Gamma(\frac{1}{2}n+1)}{\Gamma(x+\frac{1}{2}n+1)\Gamma(n+1)}$$

$${}_2F_1(\alpha, \beta; 1+\alpha-\beta; -1) = \frac{\Gamma(1+\alpha-\beta)\Gamma(1+\frac{1}{2}\alpha)}{\Gamma(1+\alpha)\Gamma(1+\frac{1}{2}\alpha-\beta)},$$

where ${}_2F_1$ is a HYPERGEOMETRIC FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION.

Kuratowski's Closure-Component Problem

Let X be an arbitrary TOPOLOGICAL SPACE. Denote the CLOSURE of a SUBSET A of X by A^- and the complement of A by A'. Then at most 14 different SETS can be derived from A by repeated application of closure and complementation (Berman and Jordan 1975, Fife 1991). The problem was first proved by Kuratowski (1922) and popularized by Kelley (1955).

see also KURATOWSKI REDUCTION THEOREM

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Kuratowski Reduction Theorem

Every nonplanar graph is a SUPERGRAPH of an expansion of the UTILITY GRAPH $UG = K_{3,3}$ or the COMPLETE GRAPH K_5. This theorem was also proven earlier by Pontryagin (1927–1928), and later by Frink and Smith (1930). Kennedy *et al.* (1985) give a detailed history of the theorem, and there exists a generalization known as the ROBERTSON-SEYMOUR THEOREM.

see also COMPLETE GRAPH, PLANAR GRAPH, ROBERTSON-SEYMOUR THEOREM, UTILITY GRAPH

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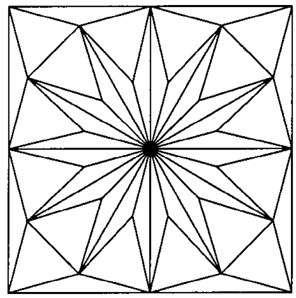
Kuratowski's Theorem

see KURATOWSKI REDUCTION THEOREM

Kürschák's Tile



An attractive tiling of the SQUARE composed of two types of triangular tiles.

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Weisstein, E. W. "Kürschák's Tile." `http://www.astro.virginia.edu/~eww6n/math/notebooks/KurschaksTile.m.`

Kurtosis

The degree of peakedness of a distribution, also called the EXCESS or EXCESS COEFFICIENT. Kurtosis is denoted γ_2 (or b_2) or β_2 and computed by taking the fourth MOMENT of a distribution. A distribution with a high peak ($\gamma_2 > 0$) is called LEPTOKURTIC, a flat-topped curve ($\gamma_2 < 0$) is called PLATYKURTIC, and the normal distribution ($\gamma_2 = 0$) is called MESOKURTIC. Let μ_i denote the ith MOMENT $\langle x^i \rangle$. The FISHER KURTOSIS is defined by

$$\gamma_2 \equiv b_2 \equiv \frac{\mu_4}{{\mu_2}^2} - 3 = \frac{\mu_4}{\sigma^4} - 3, \tag{1}$$

and the PEARSON KURTOSIS is defined by

$$\beta_2 \equiv \alpha_4 \equiv \frac{\mu_4}{\sigma^4}. \tag{2}$$

An ESTIMATOR for the γ_2 FISHER KURTOSIS is given by

$$g_2 = \frac{k_4}{{k_2}^2}, \tag{3}$$

where the ks are k-STATISTICS. The STANDARD DEVIATION of the estimator is

$$\sigma_{g_2}{}^2 \approx \frac{24}{N}. \tag{4}$$

see also FISHER KURTOSIS, MEAN, PEARSON KURTOSIS, SKEWNESS, STANDARD DEVIATION

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L

L_1-Norm

A VECTOR NORM defined for a VECTOR

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

with COMPLEX entries by

$$||\mathbf{x}||_1 = \sum_{r=1}^{n} |x_r|.$$

see also L_2-NORM, L_∞-NORM, VECTOR NORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1114–1125, 1979.

L_2-Norm

A VECTOR NORM defined for a VECTOR

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

with COMPLEX entries by

$$||\mathbf{x}||_2 = \sqrt{\sum_{r=1}^{n} |x_r|^2}.$$

The L_2-norm is also called the EUCLIDEAN NORM. The L_2-norm is defined for a function $\phi(x)$ by

$$||\phi(x)|| \equiv \phi(x) \cdot \phi(x) \equiv \left\langle [\phi(x)]^2 \right\rangle \equiv \int_a^b [\phi(x)]^2 \, dx.$$

see also L_1-NORM, L_2-SPACE, L_∞-NORM, PARALLELOGRAM LAW, VECTOR NORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1114–1125, 1979.

L_2-Space

A HILBERT SPACE in which a BRACKET PRODUCT is defined by

$$\langle \phi | \psi \rangle \equiv \int \psi^* \phi \, dx \tag{1}$$

and which satisfies the following conditions

$$\langle \phi | \psi \rangle^* = \langle \psi | \phi \rangle \, e \tag{2}$$

$$\langle \phi | \lambda_1 \psi_1 + \lambda_2 \psi_2 \rangle = \lambda_1 \langle \phi | \psi_1 \rangle + \lambda_2 \langle \phi | \psi_2 \rangle \tag{3}$$

$$\langle \lambda_1 \phi_1 + \lambda_2 \phi_2 | \psi \rangle = \lambda_1{}^* \langle \phi_1 | \psi \rangle + \lambda_2{}^* \langle \phi_2 | \psi \rangle \tag{4}$$

$$\langle \psi | \psi \rangle \in \mathbb{R} \geq 0 \tag{5}$$

$$| \langle \psi_1 | \psi_2 \rangle |^2 \leq \langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle . \tag{6}$$

The last of these is SCHWARZ'S INEQUALITY.

see also BRACKET PRODUCT, HILBERT SPACE, L_2-NORM, RIESZ-FISCHER THEOREM, SCHWARZ'S INEQUALITY

L_∞-Norm

A VECTOR NORM defined for a VECTOR

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

with COMPLEX entries by

$$||\mathbf{x}||_\infty = \max_i |x_i|.$$

see also L_1-NORM, L_2-NORM, VECTOR NORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1114–1125, 1979.

$L_{p'}$-Balance Theorem

If every component L of $X/O_{p'}(X)$ satisfies the "Schreler property," then

$$L_{p'}(Y) \leq L_{p'}(X)$$

for every p-local SUBGROUP Y of X, where $L_{p'}$ is the p-LAYER.

see also p-LAYER, SUBGROUP

L-Estimate

A ROBUST ESTIMATION based on linear combinations of ORDER STATISTICS. Examples include the MEDIAN and TUKEY'S TRIMEAN.

see also M-ESTIMATE, R-ESTIMATE

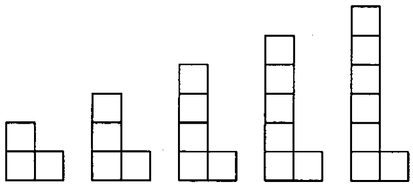
References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Robust Estimation." §15.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 694–700, 1992.

L-Function

see ARTIN L-FUNCTION, DIRICHLET L-SERIES, EULER L-FUNCTION, HECKE L-FUNCTION

L-Polyomino



The order $n \geq 2$ L-polyomino consists of a vertical line of n SQUARES with a single additional SQUARE attached at the bottom.

see also L-POLYOMINO, SKEW POLYOMINO, SQUARE, SQUARE POLYOMINO, STRAIGHT POLYOMINO

L-Series

see DIRICHLET L-SERIES

L-System

see LINDENMAYER SYSTEM

L'Hospital's Cubic

see TSCHIRNHAUSEN CUBIC

L'Hospital's Rule

Let lim stand for the LIMIT $\lim_{x\to c}$, $\lim_{x\to c^-}$, $\lim_{x\to c^+}$, $\lim_{x\to\infty}$, or $\lim_{x\to-\infty}$, and suppose that $\lim f(x)$ and $\lim g(x)$ are both ZERO or are both $\pm\infty$. If

$$\lim \frac{f'(x)}{g'(x)}$$

has a finite value or if the LIMIT is $\pm\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

L'Hospital's rule occasionally fails to yield useful results, as in the case of the function $\lim_{u\to\infty} u(u^2+1)^{-1/2}$. Repeatedly applying the rule in this case gives expressions which oscillate and never converge,

$$\begin{aligned}\lim_{u\to\infty} \frac{u}{(u^2+1)^{1/2}} &= \lim_{u\to\infty} \frac{1}{u(u^2+1)^{-1/2}} \\ &= \lim_{u\to\infty} \frac{(u^2+1)^{1/2}}{u} = \lim_{u\to\infty} \frac{u(u^2+1)^{-1/2}}{1} \\ &= \lim_{u\to\infty} \frac{u}{(u^2+1)^{1/2}}.\end{aligned}$$

(The actual LIMIT is 1.)

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 13, 1972.

L'Hospital, G. de *L'analyse des infiniment petits pour l'intelligence des lignes courbes.* 1696.

L'Huilier's Theorem

Let a SPHERICAL TRIANGLE have sides of length a, b, and c, and SEMIPERIMETER s. Then the SPHERICAL EXCESS Δ is given by

$$\begin{aligned}&\tan(\tfrac{1}{4}\Delta) \\ &\quad= \sqrt{\tan(\tfrac{1}{2}s)\tan[\tfrac{1}{2}(s-a)]\tan[\tfrac{1}{2}(s-b)]\tan[\tfrac{1}{2}(s-c)]}.\end{aligned}$$

see also GIRARD'S SPHERICAL EXCESS FORMULA, SPHERICAL EXCESS,SPHERICAL TRIANGLE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 148, 1987.

Labelled Graph

A labelled graph $G = (V, E)$ is a finite series of VERTICES V with a set of EDGES E of 2-SUBSETS of V. Given a VERTEX set $V_n = \{1, 2, \ldots, n\}$, the number of labelled graphs is given by $2^{n(n-1)/2}$. Two graphs G and H with VERTICES $V_n = \{1, 2, \ldots, n\}$ are said to be ISOMORPHIC if there is a PERMUTATION p of V_n such that $\{u, v\}$ is in the set of EDGES $E(G)$ IFF $\{p(u), p(v)\}$ is in the set of EDGES $E(H)$.

see also CONNECTED GRAPH, GRACEFUL GRAPH, GRAPH (GRAPH THEORY), HARMONIOUS GRAPH, MAGIC GRAPH, TAYLOR'S CONDITION, WEIGHTED TREE

References

Cahit, I. "Homepage for the Graph Labelling Problems and New Results." `http://193.140.42.134/~cahit/CORDIAL.html`.

Gallian, J. A. "Graph Labelling." *Elec. J. Combin.* DS6, 1–43, Mar. 5, 1998. `http://www.combinatorics.org/Surveys/`.

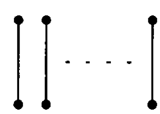
Lacunarity

Quantifies deviation from translational invariance by describing the distribution of gaps within a set at multiple scales. The more lacunar a set, the more heterogeneous the spatial arrangement of gaps.

Ladder

see ASTROID, CROSSED LADDERS PROBLEM, LADDER GRAPH

Ladder Graph



A GRAPH consisting of two rows of paired nodes each connected by an EDGE. Its complement is the COCKTAIL PARTY GRAPH.

see also COCKTAIL PARTY GRAPH

Lagrange Bracket

Let F and G be infinitely differentiable functions of x, u, and p. Then the Lagrange bracket is defined by

$$[F,G]=\sum_{\nu=1}^{n}\left[\frac{\partial F}{\partial p_\nu}\left(\frac{\partial G}{\partial x_p}+p_\nu\frac{\partial G}{\partial u}\right)-\frac{\partial G}{\partial p_\nu}\left(\frac{\partial F}{\partial x_\nu}+p_\nu\frac{\partial F}{\partial u}\right)\right]. \quad (1)$$

The Lagrange bracket satisfies

$$[F,G]=-[G,F] \quad (2)$$

$$[[F,G],H]+[[G,H],F]+[[H,F],G]=\frac{\partial F}{\partial u}[G,H]+\frac{\partial G}{\partial u}[H,F]+\frac{\partial H}{\partial u}[F,G]. \quad (3)$$

If F and G are functions of x and p only, then the Lagrange bracket $[F,G]$ collapses the POISSON BRACKET (F,G).

see also LIE BRACKET, POISSON BRACKET

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1004, 1980.

Lagrange-Bürmann Theorem

see LAGRANGE INVERSION THEOREM

Lagrangian Coefficient

COEFFICIENTS which appear in LAGRANGE INTERPOLATING POLYNOMIALS where the points are equally spaced along the ABSCISSA.

Lagrange's Continued Fraction Theorem

The REAL ROOTS of quadratic expressions with integral COEFFICIENTS have periodic CONTINUED FRACTIONS, as first proved by Lagrange.

Lagrangian Derivative

see CONVECTIVE DERIVATIVE

Lagrange's Equation

The PARTIAL DIFFERENTIAL EQUATION

$$(1+{f_y}^2)f_{xx}+2f_xf_yf_{xy}+(1+{f_x}^2)f_{yy}=0,$$

whose solutions are called MINIMAL SURFACES.

see also MINIMAL SURFACE

References

do Carmo, M. P. "Minimal Surfaces." §3.5 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 41–43, 1986.

Lagrange Expansion

Let $y=f(x)$ and $y_0=f(x_0)$ where $f'(x_0)\neq 0$, then

$$x=x_0+\sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}\left\{\frac{d^{k-1}}{dx^{k-1}}\left[\frac{x-x_0}{f(x)-y_0}\right]^k\right\}_{x=x_0}$$

$$g(x)=g(x_0)+\sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}\left\{\frac{d^{k-1}}{dx^{k-1}}\left[g'(x)\left(\frac{x-x_0}{f(x)-y_0}\right)^k\right]\right\}_{x=x_0}$$

see also MACLAURIN SERIES, TAYLOR SERIES

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

Lagrange's Four-Square Theorem

A theorem also known as BACHET'S CONJECTURE which was stated but not proven by Diophantus. It states that every POSITIVE INTEGER can be written as the SUM of at most four SQUARES. Although the theorem was proved by Fermat using infinite descent, the proof was suppressed. Euler was unable to prove the theorem. The first published proof was given by Lagrange in 1770 and made use of the EULER FOUR-SQUARE IDENTITY.

see also EULER FOUR-SQUARE IDENTITY, FERMAT'S POLYGONAL NUMBER THEOREM, FIFTEEN THEOREM, VINOGRADOV'S THEOREM, WARING'S PROBLEM

Lagrange's Group Theorem

Also known as LAGRANGE'S LEMMA. If A is an ELEMENT of a FINITE GROUP of order n, then $A^n=1$. This implies that $e|n$ where e is the smallest exponent such that $A^e=1$. Stated another way, the ORDER of a SUBGROUP divides the ORDER of the GROUP. The converse of Lagrange's theorem is not, in general, true (Gallian 1993, 1994).

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Birkhoff, G. and Mac Lane, S. *A Brief Survey of Modern Algebra, 2nd ed.* New York: Macmillan, p. 111, 1965.
Gallian, J. A. "On the Converse of Lagrange's Theorem." *Math. Mag.* **63**, 23, 1993.
Gallian, J. A. *Contemporary Abstract Algebra, 3rd ed.* Lexington, MA: D. C. Heath, 1994.
Herstein, I. N. *Abstract Algebra, 2nd ed.* New York: Macmillan, p. 66, 1990.
Hogan, G. T. "More on the Converse of Lagrange's Theorem." *Math. Mag.* **69**, 375–376, 1996.
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 86, 1993.

Lagrange's Identity

The vector identity

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}). \quad (1)$$

This identity can be generalized to n-D,

$$(\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-1}) \cdot (\mathbf{b}_1 \times \cdots \times \mathbf{b}_{n-1}) = \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n-1} \cdot \mathbf{b}_1 & \cdots & \mathbf{a}_{n-1} \cdot \mathbf{b}_{n-1} \end{vmatrix}, \quad (2)$$

where $|\mathsf{A}|$ is the DETERMINANT of A, or

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right) - \sum_{1 \leq k \leq j \leq n} (a_k b_j - a_j b_k)^2. \quad (3)$$

see also VECTOR TRIPLE PRODUCT, VECTOR QUADRUPLE PRODUCT

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1093, 1979.

Lagrange's Interpolating Fundamental Polynomial

Let $l(x)$ be an nth degree POLYNOMIAL with zeros at $x_1, \ldots, x_m$. Then the fundamental POLYNOMIALS are

$$l_\nu(x) = \frac{l(x)}{l'(x_\nu)(x - x_\nu)}. \quad (1)$$

They have the property

$$l_\nu(x) = \delta_{\nu\mu}, \quad (2)$$

where $\delta_{\nu\mu}$ is the KRONECKER DELTA. Now let $f_1, \ldots, f_n$ be values. Then the expansion

$$L_n(x) = \sum_{\nu=1}^{n} f_\nu l_\nu(x) \quad (3)$$

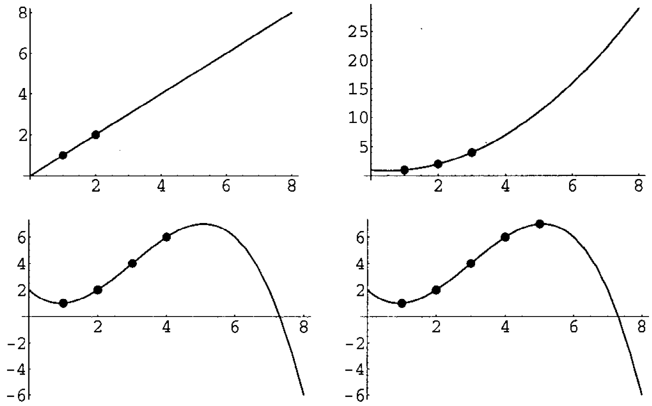
gives the unique LAGRANGE INTERPOLATING POLYNOMIAL assuming the values f_ν at x_ν. Let $d\alpha(x)$ be an arbitrary distribution on the interval $[a, b]$, $\{p_n(x)\}$ the associated ORTHOGONAL POLYNOMIALS, and $l_1(x), \ldots, l_n(x)$ the fundamental POLYNOMIALS corresponding to the set of zeros of $p_n(x)$. Then

$$\int_a^b l_\nu(x) l_\mu(x)\, d\alpha(x) = \lambda_\mu \delta_{\nu\mu} \quad (4)$$

for $\nu, \mu = 1, 2, \ldots, n$, where λ_ν are CHRISTOFFEL NUMBERS.

References
Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 329 and 332, 1975.

Lagrange Interpolating Polynomial



The Lagrange interpolating polynomial is the POLYNOMIAL of degree $n-1$ which passes through the n points $y_1 = f(x_1)$, $y_2 = f(x_2)$, ..., $y_n = f(x_n)$. It is given by

$$P(x) = \sum_{j=1}^{n} P_j(x), \quad (1)$$

where

$$P_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k} y_j. \quad (2)$$

Written explicitly,

$$P(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} y_2 + \cdots + \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} y_n. \quad (3)$$

For $n = 3$ points,

$$P(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3 \quad (4)$$

$$P'(x) = \frac{2x-x_2-x_3}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{2x-x_1-x_3}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{2x-x_1-x_2}{(x_3-x_1)(x_3-x_2)} y_3. \quad (5)$$

Note that the function $P(x)$ passes through the points (x_i, y_i), as can be seen for the case $n = 3$,

$$P(x_1) = \frac{(x_1-x_2)(x_1-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x_1-x_1)(x_1-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x_1-x_1)(x_1-x_2)}{(x_3-x_1)(x_3-x_2)} y_3 = y_1 \quad (6)$$

$$P(x_2) = \frac{(x_2-x_2)(x_2-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x_2-x_1)(x_2-x_3)}{(x_2-x_1)(x_2-x_3)} y_2$$

$$+\frac{(x_2-x_1)(x_2-x_2)}{(x_3-x_1)(x_3-x_2)}y_3 = y_2 \tag{7}$$

$$P(x_3) = \frac{(x_3-x_2)(x_3-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x_3-x_1)(x_3-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x_3-x_1)(x_3-x_2)}{(x_3-x_1)(x_3-x_2)}y_3 = y_3. \tag{8}$$

Generalizing to arbitrary n,

$$P(x_j) = \sum_{k=1}^{n} P_k(x_j) = \sum_{k=1}^{n} \delta_{jk} y_k = y_j. \tag{9}$$

The Lagrange interpolating polynomials can also be written using

$$\pi(x) \equiv \prod_{k=1}^{n}(x-x_k), \tag{10}$$

$$\pi(x_j) = \prod_{k=1}^{n}(x_j-x_k), \tag{11}$$

$$\pi'(x_j) = \left[\frac{d\pi}{dx}\right]_{x=x_j} = \prod_{\substack{k=1\\k\neq j}}^{n}(x_j-x_k), \tag{12}$$

so

$$P(x) = \sum_{k=1}^{n} \frac{\pi(x)}{(x-x_k)\pi'(x_k)} y_k. \tag{13}$$

Lagrange interpolating polynomials give no error estimate. A more conceptually straightforward method for calculating them is NEVILLE'S ALGORITHM.

see also AITKEN INTERPOLATION, LEBESGUE CONSTANTS (LAGRANGE INTERPOLATION), NEVILLE'S ALGORITHM, NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

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Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 878–879 and 883, 1972.

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 439, 1987.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Polynomial Interpolation and Extrapolation" and "Coefficients of the Interpolating Polynomial." §3.1 and 3.5 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 102–104 and 113–116, 1992.

Lagrange Inversion Theorem

Let z be defined as a function of w in terms of a parameter α by

$$z = w + \alpha\phi(z).$$

Then any function of z can be expressed as a POWER SERIES in α which converges for sufficiently small α and has the form

$$F(z) = F(w) + \frac{\alpha}{1}\phi(w)F'(w) + \frac{\alpha^2}{1\cdot 2}\frac{\partial}{\partial w}\{[\phi(w)]^2 F'(w)\} + \ldots + \frac{\alpha^{n+1}}{(n+1)!}\frac{\partial^n}{\partial w^n}\{[\phi(w)]^{n+1}F'(w)\} + \ldots.$$

References

Goursat, E. *Functions of a Complex Variable, Vol. 2, Pt. 1.* New York: Dover, 1959.

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Williamson, B. "Remainder in Lagrange's Series." §119 in *An Elementary Treatise on the Differential Calculus, 9th ed.* London: Longmans, pp. 158–159, 1895.

Lagrange's Lemma

see LAGRANGE'S FOUR-SQUARE THEOREM

Lagrange Multiplier

Used to find the EXTREMUM of $f(x_1, x_2, \ldots, x_n)$ subject to the constraint $g(x_1, x_2, \ldots, x_n) = C$, where f and g are functions with continuous first PARTIAL DERIVATIVES on the OPEN SET containing the curve $g(x_1, x_2, \ldots, x_n) = 0$, and $\nabla g \neq \mathbf{0}$ at any point on the curve (where ∇ is the GRADIENT). For an EXTREMUM to exist,

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n = 0. \tag{1}$$

But we also have

$$dg = \frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + \ldots + \frac{\partial g}{\partial x_n}dx_n = 0. \tag{2}$$

Now multiply (2) by the as yet undetermined parameter λ and add to (1),

$$\left(\frac{\partial f}{\partial x_1} + \lambda\frac{\partial g}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2} + \lambda\frac{\partial g}{\partial x_2}\right)dx_2 + \ldots + \left(\frac{\partial f}{\partial x_n} + \lambda\frac{\partial g}{\partial x_n}\right)dx_n = 0. \tag{3}$$

Note that the differentials are all independent, so we can set any combination equal to 0, and the remainder must still give zero. This requires that

$$\frac{\partial f}{\partial x_k} + \lambda\frac{\partial g}{\partial x_k} = 0 \tag{4}$$

for all $k = 1, \ldots, n$. The constant λ is called the Lagrange multiplier. For multiple constraints, $g_1 = 0$, $g_2 = 0, \ldots$,

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \ldots. \tag{5}$$

see also KUHN-TUCKER THEOREM

References

Arfken, G. "Lagrange Multipliers." §17.6 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 945–950, 1985.

Lagrange Number (Diophantine Equation)

Given a FERMAT DIFFERENCE EQUATION (a quadratic DIOPHANTINE EQUATION)

$$x^2 - r^2y^2 = 4$$

with r a QUADRATIC SURD, assign to each solution $x|y$ the Lagrange number

$$z \equiv \tfrac{1}{2}(x + yr).$$

The product and quotient of two Lagrange numbers are also Lagrange numbers. Furthermore, every Lagrange number is a POWER of the smallest Lagrange number with an integral exponent.

see also PELL EQUATION

References

Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 94–95, 1965.

Lagrange Number (Rational Approximation)

HURWITZ'S IRRATIONAL NUMBER THEOREM gives the best rational approximation possible for an arbitrary irrational number α as

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{L_n q^2}.$$

The L_n are called Lagrange numbers and get steadily larger for each "bad" set of irrational numbers which is excluded.

n	Exclude	L_n
1	none	$\sqrt{5}$
2	ϕ	$\sqrt{8}$
3	$\sqrt{2}$	$\frac{\sqrt{221}}{5}$

Lagrange numbers are of the form

$$\sqrt{9 - \frac{4}{m^2}},$$

where m is a MARKOV NUMBER. The Lagrange numbers form a SPECTRUM called the LAGRANGE SPECTRUM.

see also HURWITZ'S IRRATIONAL NUMBER THEOREM, LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, LIOUVILLE-ROTH CONSTANT, MARKOV NUMBER, ROTH'S THEOREM, SPECTRUM SEQUENCE, THUE-SIEGEL-ROTH THEOREM

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 187–189, 1996.

Lagrange Polynomial

see LAGRANGE INTERPOLATING POLYNOMIAL

Lagrange Remainder

Given a TAYLOR SERIES, the error after n terms is bounded by

$$R_n = \frac{f^{(n)}(\xi)}{n!}(x-a)^n$$

for some $\xi \in (a, x)$.

see also CAUCHY REMAINDER FORM, TAYLOR SERIES

Lagrange Resolvent

A quantity involving primitive cube roots of unity which can be used to solve the CUBIC EQUATION.

References

Faucette, W. M. "A Geometric Interpretation of the Solution of the General Quartic Polynomial." *Amer. Math. Monthly* **103**, 51–57, 1996.

Lagrange Spectrum

A SPECTRUM formed by the LAGRANGE NUMBERS. The only ones less than three are the LAGRANGE NUMBERS, but the last gaps end at FREIMAN'S CONSTANT. REAL NUMBERS larger than FREIMAN'S CONSTANT are in the MARKOV SPECTRUM.

see also FREIMAN'S CONSTANT, LAGRANGE NUMBER (RATIONAL APPROXIMATION), MARKOV SPECTRUM, SPECTRUM SEQUENCE

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 187–189, 1996.

Laguerre Differential Equation

$$xy'' + (1-x)y' + \lambda y = 0. \tag{1}$$

The Laguerre differential equation is a special case of the more general "associated Laguerre differential equation"

$$xy'' + (\nu + 1 - x)y' + \lambda y = 0 \tag{2}$$

with $\nu = 0$. Note that if $\lambda = 0$, then the solution to the associated Laguerre differential equation is of the form

$$y''(x) + P(x)y'(x) = 0, \tag{3}$$

and the solution can be found using an INTEGRATING FACTOR

$$\begin{aligned}\mu &= \exp\left(\int P(x)\,dx\right) = \exp\left(\int \frac{\nu+1-x}{x}\,dx\right)\\ &= \exp[(\nu+1)\ln x - x] = x^{\nu+1}e^{-x},\end{aligned} \tag{4}$$

so

$$y = C_1 \int \frac{dx}{\mu} + C_2 = C_1 \int \frac{e^x}{x^{\nu+1}}\,dx + C_2. \tag{5}$$

The associated Laguerre differential equation has a REGULAR SINGULAR POINT at 0 and an IRREGULAR

SINGULARITY at ∞. It can be solved using a series expansion,

$$x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (\nu+1)\sum_{n=1}^{\infty} n a_n x^{n-1} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n = 0 \quad (6)$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + (\nu+1)\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0 \quad (7)$$

$$\sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n + (\nu+1)\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0 \quad (8)$$

$$[(n+1)a_1 + \lambda a_0] + \sum_{n=1}^{\infty} \{[(n+1)n + (\nu+1)(n+1)]a_{n+1} - n a_n + \lambda a_n\} x^n = 0 \quad (9)$$

$$[(n+1)a_1 + \lambda a_0] + \sum_{n=1}^{\infty} [(n+1)(n+\nu+1)a_{n+1} + (\lambda - n)a_n] x^n = 0. \quad (10)$$

This requires

$$a_1 = -\frac{\lambda}{\nu+1} a_0 \quad (11)$$

$$a_{n+1} = \frac{n-\lambda}{(n+1)(n+\nu+1)} a_n \quad (12)$$

for $n > 1$. Therefore,

$$a_{n+1} = \frac{n-\lambda}{(n+1)(n+\nu+1)} a_n \quad (13)$$

for $n = 1, 2, \ldots$, so

$$y = a_0 \left[1 - \frac{\lambda}{\nu+1}x - \frac{\lambda(1-\lambda)}{2(\nu+1)(\nu+2)}x^2 - \frac{\lambda(1-\lambda)(2-\lambda)}{2\cdot 3(\nu+1)(\nu+2)(\nu+3)} + \cdots\right]. \quad (14)$$

If λ is a POSITIVE INTEGER, then the series terminates and the solution is a POLYNOMIAL, known as an associated LAGUERRE POLYNOMIAL (or, if $\nu = 0$, simply a LAGUERRE POLYNOMIAL).

see also LAGUERRE POLYNOMIAL

Laguerre-Gauss Quadrature

Also called GAUSS-LAGUERRE QUADRATURE or LAGUERRE QUADRATURE. A GAUSSIAN QUADRATURE over the interval $[0, \infty)$ with WEIGHTING FUNCTION $W(x) = e^{-x}$. The ABSCISSAS for quadrature order n are given by the ROOTS of the LAGUERRE POLYNOMIALS $L_n(x)$. The weights are

$$w_i = -\frac{A_{n+1}\gamma_n}{A_n L_n'(x_i) L_{n+1}(x_i)} = \frac{A_n}{A_{n-1}} \frac{\gamma_{n-1}}{L_{n-1}(x_i) L_n'(x_i)}, \quad (1)$$

where A_n is the COEFFICIENT of x^n in $L_n(x)$. For LAGUERRE POLYNOMIALS,

$$A_n = (-1)^n n!, \quad (2)$$

where $n!$ is a FACTORIAL, so

$$\frac{A_{n+1}}{A_n} = -(n+1). \quad (3)$$

Additionally,

$$\gamma_n = 1, \quad (4)$$

so

$$w_i = \frac{n+1}{L_{n+1}(x_i) L_n'(x_i)} = -\frac{n}{L_{n-1}(x_i) L_n'(x_i)}. \quad (5)$$

(Note that the normalization used here is different than that in Hildebrand 1956.) Using the recurrence relation

$$x L_n'(x) = n L_n(x) - n L_{n-1}(x) = (x - n - 1) L_n(x) + (n+1) L_{n+1}(x) \quad (6)$$

which implies

$$x_i L_n'(x_i) = -n L_{n-1}(x_i) = (n+1) L_{n+1}(x_i) \quad (7)$$

gives

$$w_i = \frac{1}{x_i [L_n'(x_i)]^2} = \frac{x_i}{(n+1)^2 [L_{n+1}(x_i)]^2}. \quad (8)$$

The error term is

$$E = \frac{(n!)^2}{(2n)!} f^{(2n)}(\xi). \quad (9)$$

Beyer (1987) gives a table of ABSCISSAS and weights up to $n = 6$.

n	x_i	w_i
2	0.585786	0.853553
	3.41421	0.146447
3	0.415775	0.711093
	2.29428	0.278518
	6.28995	0.0103893
4	0.322548	0.603154
	1.74576	0.357419
	4.53662	0.0388879
	9.39507	0.000539295
5	0.26356	0.521756
	1.4134	0.398667
	3.59643	0.0759424
	7.08581	0.00361176
	12.6408	0.00002337

The ABSCISSAS and weights can be computed analytically for small n.

n	x_i	w_i
2	$2-\sqrt{2}$	$\frac{1}{4}(2+\sqrt{2})$
	$2+\sqrt{2}$	$\frac{1}{4}(2-\sqrt{2})$

For the associated Laguerre polynomial $L_n^\beta(x)$ with WEIGHTING FUNCTION $w(x)=x^\beta e^{-x}$,

$$A_n=(-1)^n \tag{10}$$

and

$$\gamma_n=n!\int_0^\infty x^{\beta+n}e^{-x}\,dx=n!\Gamma(n+\beta+1). \tag{11}$$

The weights are

$$w_i=\frac{n!\Gamma(n+\beta+1)}{x_i[L_m^{\beta\,\prime}(x_i)]^2}=\frac{n!\Gamma(n+\beta+1)x_i}{[L_{n+1}^\beta(x_i)]^2}, \tag{12}$$

where $\Gamma(z)$ is the GAMMA FUNCTION, and the error term is

$$E_n=\frac{n!\Gamma(n+\beta+1)}{(2n)!}f^{(2n)}(\xi). \tag{13}$$

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 463, 1987.

Chandrasekhar, S. *Radiative Transfer.* New York: Dover, pp. 64–65, 1960.

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 325–327, 1956.

Laguerre's Method

A ROOT-finding algorithm which converges to a COMPLEX ROOT from any starting position.

$$P_n(x)=(x-x_1)(x-x_2)\cdots(x-x_n) \tag{1}$$

$$\ln|P_n(x)|=\ln|x-x_1|+\ln|x-x_2|+\ldots+\ln|x-x_n| \tag{2}$$

$$\begin{aligned}P_n'(x)&=(x-x_2)\cdots(x-x_n)+(x-x_1)\cdots(x-x_n)+\ldots\\&=P_n(x)\left(\frac{1}{x-x_1}+\ldots+\frac{1}{x-x_n}\right)\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{d\ln|P_n(x)|}{dx}&=\frac{1}{x-x_1}+\frac{1}{x-x_2}+\ldots+\frac{1}{x-x_n}\\&=\frac{P_n'(x)}{P_n(x)}\equiv G(x)\end{aligned} \tag{4}$$

$$\begin{aligned}&-\frac{d^2\ln|P_n(x)|}{dx^2}\\&=\frac{1}{(x-x_1)^2}+\frac{1}{(x-x_2)^2}+\ldots+\frac{1}{(x-x_n)^2}\\&=\left[\frac{P_n'(x)}{P_n(x)}\right]^2-\frac{P_n''(x)}{P_n(x)}\equiv H(x).\end{aligned} \tag{5}$$

Now let $a\equiv x-x_1$ and $b\equiv x-x_1$. Then

$$G\equiv\frac{1}{a}+\frac{n-1}{b} \tag{6}$$

$$H\equiv\frac{1}{a^2}+\frac{n-1}{b^2}, \tag{7}$$

so

$$a=\frac{n}{\max\left[G\pm\sqrt{(n-1)(nH-G^2)}\right]}. \tag{8}$$

Setting $n=2$ gives HALLEY'S IRRATIONAL FORMULA.

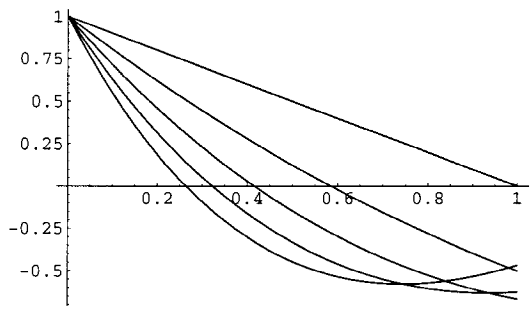
see also HALLEY'S IRRATIONAL FORMULA, HALLEY'S METHOD, NEWTON'S METHOD, ROOT

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 365–366, 1992.

Ralston, A. and Rabinowitz, P. §8.9–8.13 in *A First Course in Numerical Analysis, 2nd ed.* New York: McGraw-Hill, 1978.

Laguerre Polynomial



Solutions to the LAGUERRE DIFFERENTIAL EQUATION with $\nu=0$ are called Laguerre polynomials. The Laguerre polynomials $L_n(x)$ are illustrated above for $x\in[0,1]$ and $n=1, 2, \ldots, 5$.

The Rodrigues formula for the Laguerre polynomials is

$$L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}(x^ne^{-x}) \tag{1}$$

and the GENERATING FUNCTION for Laguerre polynomials is

$$\begin{aligned}g(x,z)&=\frac{\exp\left(-\frac{xz}{1-z}\right)}{1-z}=1+(-x+1)z\\&+(\tfrac{1}{2}x^2-2x+1)z^2+(-\tfrac{1}{6}x^3+\tfrac{3}{2}x^2-3x+1)z^3+\ldots.\end{aligned} \tag{2}$$

A CONTOUR INTEGRAL is given by

$$L_n(x)=\frac{1}{2\pi i}\int\frac{e^{-xz/(1-z)}}{(1-z)z^{n+1}}\,dz. \tag{3}$$

The Laguerre polynomials satisfy the RECURRENCE RELATIONS

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \quad (4)$$

(Petkovšek *et al.* 1996) and

$$xL'_n(x) = nL_n(x) - nL_{n-1}(x). \quad (5)$$

The first few Laguerre polynomials are

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= -x+1 \\ L_2(x) &= \tfrac{1}{2}(x^2-4x+2) \\ L_3(x) &= \tfrac{1}{6}(-x^3+9x^2-18x+6). \end{aligned}$$

Solutions to the associated LAGUERRE DIFFERENTIAL EQUATION with $\nu \neq 0$ are called associated Laguerre polynomials $L_n^k(x)$. In terms of the normal Laguerre polynomials,

$$L_n(x) = L_n^0(x). \quad (6)$$

The Rodrigues formula for the associated Laguerre polynomials is

$$\begin{aligned} L_n^k(x) &= \frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+k}) \\ &= (-1)^n \frac{d^n}{dx^n}[L_{n+k}(x)] \quad (7) \\ &= \sum_{m=0}^{\infty}(-1)^m \frac{(n+k)!}{(n-m)!(k+m)!m!}x^m \quad (8) \end{aligned}$$

and the GENERATING FUNCTION is

$$g(x,z) = \frac{\exp\left(-\frac{xz}{1-z}\right)}{(1-z)^{k+1}}$$
$$1+(k+1-x)z+\tfrac{1}{2}[x^2-2(k+2)x+(k+1)(k+2)]z^2+\ldots. \quad (9)$$

The associated Laguerre polynomials are orthogonal over $[0,\infty)$ with respect to the WEIGHTING FUNCTION $x^n e^{-x}$.

$$\int_0^\infty e^{-x}x^k L_n^k(x)L_m^k(x)\,dx = \frac{(n+k)!}{n!}\delta_{mn}, \quad (10)$$

where δ_{mn} is the KRONECKER DELTA. They also satisfy

$$\int_0^\infty e^{-x}x^{k+1}[L_n^k(x)]^2\,dx = \frac{(n+k)!}{n!}(2n+k+1). \quad (11)$$

RECURRENCE RELATIONS include

$$\sum_{\nu=0}^{n} L_\nu^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) \quad (12)$$

and

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x). \quad (13)$$

The DERIVATIVE is given by

$$\begin{aligned} \frac{d}{dx}L_n^{(\alpha)}(x) &= -L_{n-1}^{(\alpha+1)}(x) \\ &= x^{-1}[nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \quad (14) \end{aligned}$$

In terms of the CONFLUENT HYPERGEOMETRIC FUNCTION,

$$L_n^k(x) = \frac{(k+1)_n}{n!}\,{}_1F_1(-b;k+1;x). \quad (15)$$

An interesting identity is

$$\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)}{\Gamma(n+\alpha+1)}w^n = e^w(xw)^{-\alpha/2}J_\alpha(2\sqrt{xw}), \quad (16)$$

where $\Gamma(z)$ is the GAMMA FUNCTION and $J_\alpha(z)$ is the BESSEL FUNCTION OF THE FIRST KIND (Szegő 1975, p. 102). An integral representation is

$$e^{-x}x^{\alpha/2}L_n^{(\alpha)}(x) = \frac{1}{n!}\int_0^\infty e^{-t}t^{n+\alpha/2}J_\alpha(2\sqrt{tx})\,dt \quad (17)$$

for $n = 0, 1, \ldots$ and $\alpha > -1$. The DISCRIMINANT is

$$D_n^{(\alpha)} = \prod_{\nu=1}^{n}\nu^{\nu-2n+2}(\nu+\alpha)^{\nu-1} \quad (18)$$

(Szegő 1975, p. 143). The KERNEL POLYNOMIAL is

$$K_n^{(\alpha)}(x,y) = \frac{n+1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n}^{-1}$$
$$\frac{L_n^{(\alpha)}(x)L_{n+1}^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(x)L_n(\alpha)(y)}{x-y}, \quad (19)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT (Szegő 1975, p. 101).

The first few associated Laguerre polynomials are

$$\begin{aligned} L_0^k(x) &= 1 \\ L_1^k(x) &= -x+k+1 \\ L_2^k(x) &= \tfrac{1}{2}[x^2-2(k+2)x+(k+1)(k+2)] \\ L_3^k(x) &= \tfrac{1}{6}[-x^3+3(k+3)x^2-3(k+2)(k+3)x \\ &\quad +(k+1)(k+2)(k+3)]. \end{aligned}$$

see also SONINE POLYNOMIAL

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Orthogonal Polynomials." Ch. 22 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 771–802, 1972.

Arfken, G. "Laguerre Functions." §13.2 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 721–731, 1985.
Chebyshev, P. L. "Sur le développement des fonctions à une seule variable." *Bull. Ph.-Math., Acad. Imp. Sc. St. Pétersbourg* **1**, 193–200, 1859.
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Laguerre, E. de. "Sur l'intégrale $\int_x^{+\infty} x^{-1}e^{-x}\,dx$." *Bull. Soc. math. France* **7**, 72–81, 1879. Reprinted in *Oeuvres, Vol. 1.* New York: Chelsea, pp. 428–437, 1971.
Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, pp. 61–62, 1996.
Sansone, G. "Expansions in Laguerre and Hermite Series." Ch. 4 in *Orthogonal Functions, rev. English ed.* New York: Dover, pp. 295–385, 1991.
Spanier, J. and Oldham, K. B. "The Laguerre Polynomials $L_n(x)$." Ch. 23 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 209–216, 1987.
Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., 1975.

Laguerre Quadrature

A GAUSSIAN QUADRATURE-like FORMULA for numerical estimation of integrals. It fits exactly all POLYNOMIALS of degree $2m-1$.

References
Chandrasekhar, S. *Radiative Transfer.* New York: Dover, p. 61, 1960.

Laguerre's Repeated Fraction

The CONTINUED FRACTION

$$\frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n}=\frac{n}{x+}\,\frac{n^2-1}{3x+}\,\frac{n^2-2^2}{5x+\ldots}.$$

References
Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, p. 13, 1959.

Laisant's Recurrence Formula

The RECURRENCE RELATION

$$(n-1)A_{n+1}=(n^2-1)A_n+(n+1)A_{n-1}+4(-1)^n$$

with $A(1)=A(2)=1$ which solves the MARRIED COUPLES PROBLEM.

see also MARRIED COUPLES PROBLEM

Lakshmi Star

see STAR OF LAKSHMI

Lal's Constant

Let $P(N)$ denote the number of PRIMES of the form n^2+1 for $1\le n\le N$, then

$$P(N)\sim 0.68641\,\mathrm{li}(N), \tag{1}$$

where $\mathrm{li}(N)$ is the LOGARITHMIC INTEGRAL (Shanks 1960, pp. 321–332). Let $Q(N)$ denote the number of PRIMES of the form n^4+1 for $1\le n\le N$, then

$$Q(N)\sim \tfrac{1}{4}s_1\,\mathrm{li}(N)=0.66974\,\mathrm{li}(N) \tag{2}$$

(Shanks 1961, 1962). Let $R(N)$ denote the number of pairs of PRIMES $(n-1)^2+1$ and $(n+1)^2+1$ for $n\le N-1$, then

$$R(N)\sim 0.48762\,\mathrm{li}_2(N), \tag{3}$$

where

$$\mathrm{li}_2(N)\equiv\int_2^N \frac{dn}{(\ln n)^2} \tag{4}$$

(Shanks 1960, pp. 201–203). Finally, let $S(N)$ denote the number of pairs of PRIMES $(n-1)^4+1$ and $(n+1)^4+1$ for $n\le N-1$, then

$$S(N)\sim\lambda\,\mathrm{li}_2(N) \tag{5}$$

(Lal 1967), where λ is called Lal's constant. Shanks (1967) showed that $\lambda\approx 0.79220$.

References
Lal, M. "Primes of the Form n^4+1." *Math. Comput.* **21**, 245–247, 1967.
Shanks, D. "On the Conjecture of Hardy and Littlewood Concerning the Number of Primes of the Form n^2+a." *Math. Comput.* **14**, 321–332, 1960.
Shanks, D. "On Numbers of the Form n^4+1." *Math. Comput.* **15**, 186–189, 1961.
Shanks, D. Corrigendum to "On the Conjecture of Hardy and Littlewood Concerning the Number of Primes of the Form n^2+a." *Math. Comput.* **16**, 513, 1962.
Shanks, D. "Lal's Constant and Generalization." *Math. Comput.* **21**, 705–707, 1967.

Lam's Problem

Given an 111×111 MATRIX, fill 11 spaces in each row in such a way that all columns also have 11 spaces filled. Furthermore, each pair of rows must have exactly one filled space in the same column. This problem is equivalent to finding a PROJECTIVE PLANE of order 10. Using a computer program, Lam showed that no such arrangement exists.

see also PROJECTIVE PLANE

Laman's Theorem

Let a GRAPH G have exactly $2n-3$ EDGES, where n is the number of VERTICES in G. Then G is "generically" RIGID in $\mathbb{R}^2$ IFF $e'\le 2n'-3$ for every SUBGRAPH of G having n' VERTICES and r' EDGES.

see also RIGID

References
Laman, G. "On Graphs and Rigidity of Plane Skeletal Structures." *J. Engineering Math.* **4**, 331–340, 1970.

Lambda Calculus

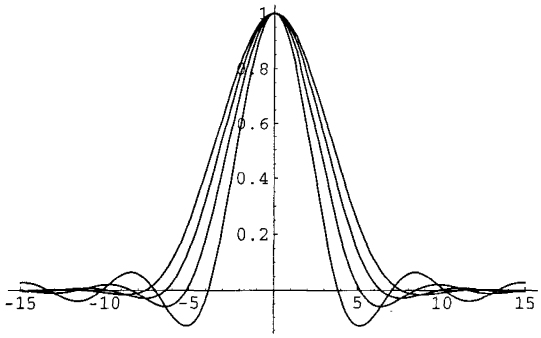
Developed by Alonzo Church and Stephen Kleene to address the COMPUTABLE NUMBER problem. In the lambda calculus, λ is defined as the ABSTRACTION OPERATOR. Three theorems of lambda calculus are λ-conversion, α-conversion, and η-conversion.

see also ABSTRACTION OPERATOR, COMPUTABLE NUMBER

References

Penrose, R. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics.* Oxford, England: Oxford University Press, pp. 66–70, 1989.

Lambda Function



The lambda function defined by Jahnke and Emden (1945) is

$$\Lambda_\nu(z) \equiv \Gamma(\nu+1)\frac{J_\nu(z)}{(\frac{1}{2}z)^\nu} \quad (1)$$

$$\Lambda_1(z) \equiv \frac{J_1(z)}{\frac{z}{2}} = 2\,\mathrm{jinc}(z), \quad (2)$$

where $J_1(z)$ is a BESSEL FUNCTION OF THE FIRST KIND and $\mathrm{jinc}(z)$ is the JINC FUNCTION.

A two-variable lambda function defined by Gradshteyn and Ryzhik (1979) is

$$\lambda(x,y) \equiv \int_0^y \frac{\Gamma(u+1)\,du}{x^u}, \quad (3)$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

see also AIRY FUNCTIONS, DIRICHLET LAMBDA FUNCTION, ELLIPTIC LAMBDA FUNCTION, JINC FUNCTION, LAMBDA HYPERGEOMETRIC FUNCTION, MANGOLDT FUNCTION, MU FUNCTION, NU FUNCTION

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1079, 1979.

Jahnke, E. and Emde, F. *Tables of Functions with Formulae and Curves, 4th ed.* New York: Dover, 1945.

Lambda Group

The set of linear fractional transformations w which satisfy

$$w(t) = \frac{at+b}{ct+d},$$

where a and d are ODD and b and c are EVEN. Also called the THETA SUBGROUP. It is a SUBGROUP of the GAMMA GROUP.

see also GAMMA GROUP

Lambda Hypergeometric Function

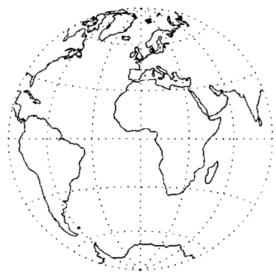
$$\lambda(t) = 16q\prod_{n=1}^\infty \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8, \quad (1)$$

where q is the NOME. The lambda hypergeometric functions satisfy the recurrence relationships

$$\lambda(t+2) = \lambda(t) \quad (2)$$

$$\lambda\left(\frac{t}{2t+1}\right) = \lambda(t). \quad (3)$$

Lambert Azimuthal Equal-Area Projection



$$x = k'\cos\phi\sin(\lambda-\lambda_0) \quad (1)$$

$$y = k'[\cos\phi_1\sin\phi - \sin\phi_1\cos\phi\cos(\lambda-\lambda_0)], \quad (2)$$

where

$$k' = \sqrt{\frac{2}{1+\sin\phi_1\sin\phi+\cos\phi_1\cos\phi\cos(\lambda-\lambda_0)}}. \quad (3)$$

The inverse FORMULAS are

$$\phi = \sin^{-1}\left(\cos c\sin\phi_1 + \frac{y\sin c\cos\phi_1}{\rho}\right) \quad (4)$$

$$\lambda = \lambda_0 + \tan^{-1}\left(\frac{x\sin c}{\rho\cos\phi_1\cos c - y\sin\phi_1\sin c}\right), \quad (5)$$

where

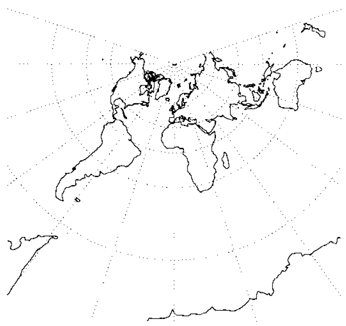
$$\rho = \sqrt{x^2+y^2} \quad (6)$$

$$c = 2\sin^{-1}(\tfrac{1}{2}\rho). \quad (7)$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 182–190, 1987.

Lambert Conformal Conic Projection



$$x = \rho \sin[n(\lambda - \lambda_0)] \quad (1)$$

$$y = \rho_0 - \rho \cos[n(\lambda - \lambda_0)], \quad (2)$$

where

$$\rho = F \cot^n(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) \quad (3)$$

$$\rho_0 = F \cot^n(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi_0) \quad (4)$$

$$F = \frac{\cos\phi_1 \tan^n(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi_1)}{n} \quad (5)$$

$$n = \frac{\ln(\cos\phi_1 \sec\phi_2)}{\ln[\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi_2)\cot(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi_1)]}. \quad (6)$$

The inverse FORMULAS are

$$\phi = 2\tan^{-1}\left[\left(\frac{F}{\rho}\right)^{1/n}\right] - \tfrac{1}{2}\pi \quad (7)$$

$$\lambda = \lambda_0 + \frac{\theta}{n}, \quad (8)$$

where

$$\rho = \operatorname{sgn}(n)\sqrt{x^2 + (\rho_0 - y)^2} \quad (9)$$

$$\theta = \tan^{-1}\left(\frac{x}{\rho_0 - y}\right). \quad (10)$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 104–110, 1987.

Lambert's Method

A ROOT-finding method also called BAILEY'S METHOD and HUTTON'S METHOD. If $g(x) = x^d - r$, then

$$H_g(x) = \frac{(d-1)x^d + (d+1)r}{(d+1)x^d + (d-1)r}x.$$

References

Scavo, T. R. and Thoo, J. B. "On the Geometry of Halley's Method." *Amer. Math. Monthly* **102**, 417–426, 1995.

Lambert Series

A series of the form

$$F(x) \equiv \sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n} \quad (1)$$

for $|x| < 1$. Then

$$F(x) = \sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} x^{mn} = \sum_{N=1}^{\infty} b_N x^N, \quad (2)$$

where

$$b_N \equiv \sum_{n|N} a_n. \quad (3)$$

Some beautiful series of this type include

$$\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1 - x^n} = x \quad (4)$$

$$\sum_{n=1}^{\infty} \frac{\phi(n)x^n}{1 - x^n} = \frac{x}{(1-x)^2} \quad (5)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} d(n)x^n \quad (6)$$

$$\sum_{n=1}^{\infty} \frac{n^k x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_k(n)x^n \quad (7)$$

$$\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}x^n}{1 - x^n} = \sum_{n=1}^{\infty} r(n)x^n, \quad (8)$$

where $\mu(n)$ is the MÖBIUS FUNCTION, $\phi(n)$ is the TOTIENT FUNCTION, $d(n) = \sigma_0(n)$ is the number of divisors of n, $\sigma_k(n)$ is the DIVISOR FUNCTION, and $r(n)$ is the number of representations of n in the form $n = A^2 + B^2$ where A and B are rational integers (Hardy and Wright 1979).

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Number Theoretic Functions." §24.3.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 826–827, 1972.

Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, pp. 257–258, 1979.

Lambert's Transcendental Equation

An equation proposed by Lambert (1758) and studied by Euler in 1779 (Euler 1921).

$$x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}.$$

When $\alpha \to \beta$, the equation becomes

$$\ln x = vx^\beta,$$

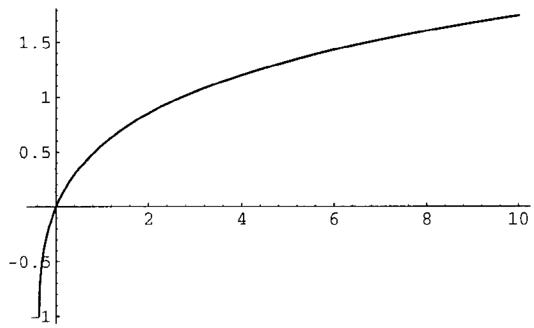
which has the solution

$$x = \exp\left[-\frac{W(-\beta v)}{\beta}\right],$$

where W is LAMBERT'S W-FUNCTION.

References
Corless, R. M.; Gonnet, G. H.; Hare, D. E. G.; and Jeffrey, D. J. "On Lambert's W Function." `ftp://watdragon.uwaterloo.ca/cs-archive/CS-93-03/W.ps.Z`.
de Bruijn, N. G. *Asymptotic Methods in Analysis.* Amsterdam, Netherlands: North-Holland, pp. 27–28, 1961.
Euler, L. "De Serie Lambertina Plurismique Eius Insignibus Proprietatibus." *Leonhardi Euleri Opera Omnia, Ser. 1. Opera Mathematica,* Bd. 6, 1921.
Lambert, J. H. "Observations variae in Mathesin Puram." *Acta Helvitica, physico-mathematico-anatomico-botanico-medica* **3**, 128–168, 1758.

Lambert's W-Function



The inverse of the function

$$f(W) = We^W, \tag{1}$$

also called the OMEGA FUNCTION. The function is implemented as the *Mathematica*® (Wolfram Research, Champaign, IL) function `ProductLog[z]`. $W(1)$ is called the OMEGA CONSTANT and can be considered a sort of "GOLDEN RATIO" of exponentials since

$$\exp[-W(1)] = W(1), \tag{2}$$

giving

$$\ln\left[\frac{1}{W(1)}\right] = W(1). \tag{3}$$

Lambert's W-Function has the series expansion

$$W(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!} x^n = x - x^2 + \tfrac{3}{2}x^3 - \tfrac{8}{3}x^4 + \tfrac{125}{24}x^5 - \tfrac{54}{5}x^6 + \tfrac{16807}{720}x^7 + \ldots. \tag{4}$$

The LAGRANGE INVERSION THEOREM gives the equivalent series expansion

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n, \tag{5}$$

where $n!$ is a FACTORIAL. However, this series oscillates between ever larger POSITIVE and NEGATIVE values for REAL $z \gtrsim 0.4$, and so cannot be used for practical numerical computation. An asymptotic FORMULA which yields reasonably accurate results for $z \gtrsim 3$ is

$$\begin{aligned} W(z) &= \operatorname{Ln} z - \ln \operatorname{Ln} z \\ &\quad + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km} (\ln \operatorname{Ln} z)^{m+1} (\operatorname{Ln} z)^{-k-m-1} \\ &= L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2+L_2)}{2{L_1}^2} \\ &\quad + \frac{L_2(6 - 9L_2 + 2{L_2}^2)}{6{L_1}^2} \\ &\quad + \frac{L_2(-12 + 36L_2 - 22{L_2}^2 + 3{L_2}^3)}{12{L_1}^4} \\ &\quad + \frac{L_2(60 - 300L_2 + 350{L_2}^2 - 125{L_2}^3 + 12{L_2}^4)}{60{L_1}^5} \\ &\quad + \mathcal{O}\left[\left(\frac{L_2}{L_1}\right)^6\right], \end{aligned} \tag{6}$$

where

$$L_1 = \operatorname{Ln} z \tag{7}$$

$$L_2 = \ln \operatorname{Ln} z \tag{8}$$

(Corless *et al.*), correcting a typographical error in de Bruijn (1961). Another expansion due to Gosper is the DOUBLE SUM

$$W(x) = a + \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{S_1(n,k)}{\left[\ln\left(\frac{x}{a}\right) - a\right]^{k-1} (n-k+1)!} \right\} \times \left[1 - \frac{\ln\left(\frac{x}{a}\right)}{a}\right]^n, \tag{9}$$

where S_1 is a *nonnegative* STIRLING NUMBER OF THE FIRST KIND and a is a first approximation which can be used to select between branches. Lambert's W-function is two-valued for $-1/e \le x < 0$. For $W(x) \ge -1$, the function is denoted $W_0(x)$ or simply $W(x)$, and this is called the principal branch. For $W(x) \le -1$, the function is denoted $W_{-1}(x)$. The DERIVATIVE of W is

$$W'(x) = \frac{1}{[1 + W(x)] \exp[W(x)]} = \frac{W(x)}{x[1 + W(x)]} \tag{10}$$

for $x \ne 0$. For the principal branch when $z > 0$,

$$\ln W(z) = \ln z - W(z). \tag{11}$$

see also ITERATED EXPONENTIAL CONSTANTS, OMEGA CONSTANT

References
Corless, R. M.; Gonnet, G. H.; Hare, D. E. G.; and Jeffrey, D. J. "On Lambert's W Function." `ftp://watdragon.uwaterloo.ca/cs-archive/CS-93-03/W.ps.Z`.
de Bruijn, N. G. *Asymptotic Methods in Analysis.* Amsterdam, Netherlands: North-Holland, pp. 27–28, 1961.

Lamé Curve

A curve with Cartesian equation

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = c$$

first discussed in 1818 by Lamé. If n is a rational, then the curve is algebraic. However, for irrational n, the curve is transcendental. For EVEN INTEGERS n, the curve becomes closer to a rectangle as n increases. For ODD INTEGER values of n, the curve looks like the EVEN case in the POSITIVE quadrant but goes to infinity in both the second and fourth quadrants (MacTutor Archive). The EVOLUTE of an ELLIPSE,

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

n	Curve
$\frac{2}{3}$	astroid
$\frac{5}{2}$	superellipse
3	witch of Agnesi

see also ASTROID, SUPERELLIPSE, WITCH OF AGNESI

References

MacTutor History of Mathematics Archive. "Lamé Curves." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Lame.html.

Lamé's Differential Equation

$$(x^2-b^2)(x^2-c^2)\frac{d^2z}{dx^2} + x(x^2-b^2+x^2-c^2)\frac{dz}{dx} - [m(m+1)x^2 - (b^2+c^2)p]z = 0. \quad (1)$$

(Byerly 1959, p. 255). The solution is denoted $E_m^p(x)$ and is known as a LAMÉ FUNCTION or an ELLIPSOIDAL HARMONIC. Whittaker and Watson (1990, pp. 554–555) give the alternative forms

$$4\Delta_\lambda \frac{d}{d\lambda}\left[\Delta_\lambda \frac{d\Lambda}{d\lambda}\right] = [n(n+1)\lambda + C]\Lambda \quad (2)$$

$$\frac{d^2\Lambda}{d\lambda^2} + \left[\frac{\frac{1}{2}}{a^2+\lambda} + \frac{\frac{1}{2}}{b^2+\lambda} + \frac{\frac{1}{2}}{c^2}\right]\frac{d\Lambda}{d\lambda} = \frac{[n(n+1)\lambda + C]\Lambda}{4\Delta_\lambda} \quad (3)$$

$$\frac{d^2\Lambda}{du^2} = [n(n+1)\wp(u) + C - \tfrac{1}{3}n(n+1)(a^2+b^2+c^2)]\Lambda \quad (4)$$

$$\frac{d^2\Lambda}{d{z_1}^2} = [n(n+1)k^2 \operatorname{sn}^2\alpha + A]\Lambda, \quad (5)$$

where $\wp$ is a WEIERSTRAß ELLIPTIC FUNCTION and

$$\Lambda(\theta) \equiv \prod_{q=1}^m (\theta - \theta_q) \quad (6)$$

$$\Delta_\lambda \equiv \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} \quad (7)$$

$$A \equiv \frac{C - \frac{1}{3}n(n+1)(a^2+b^2+c^2) + e_3 n(n+1)}{e_1 - e_3}. \quad (8)$$

References

Byerly, W. E. *An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics.* New York: Dover, 1959.

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Lamé's Differential Equation (Types)

Whittaker and Watson (1990, pp. 539–540) write Lamé's differential equation for ELLIPSOIDAL HARMONICS of the four types as

$$4\Delta(\theta)\frac{d}{d\theta}\left[F(\theta)\frac{d\Lambda(\theta)}{d\theta}\right] = [2m(2m+1)\theta + C]\Lambda(\theta) \quad (1)$$

$$4\Delta(\theta)\frac{d}{d\theta}\left[F(\theta)\frac{d\Lambda(\theta)}{d\theta}\right] = [(2m+1)(2m+2)\theta + C]\Lambda(\theta) \quad (2)$$

$$4\Delta(\theta)\frac{d}{d\theta}\left[F(\theta)\frac{d\Lambda(\theta)}{d\theta}\right] = [(2m+2)(2m+3)\theta + C]\Lambda(\theta) \quad (3)$$

$$4\Delta(\theta)\frac{d}{d\theta}\left[F(\theta)\frac{d\Lambda(\theta)}{d\theta}\right] = [(2m+3)(2m+4)\theta + C]\Lambda(\theta), \quad (4)$$

where

$$\Delta(\theta) \equiv \sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)} \quad (5)$$

$$\Lambda(\theta) \equiv \prod_{q=1}^m (\theta - \theta_q). \quad (6)$$

References

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Lamé Function

see ELLIPSOIDAL HARMONIC

Lamé's Theorem

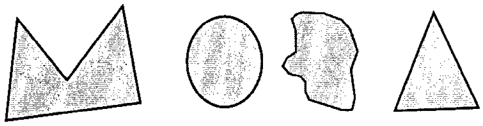
If a is the smallest INTEGER for which there is a smaller INTEGER b such that a and b generate a EUCLIDEAN ALGORITHM remainder sequence with n steps, then a is the FIBONACCI NUMBER F_{n+2}. Furthermore, the number of steps in the EUCLIDEAN ALGORITHM never exceeds 5 times the number of digits in the smaller number.

see also EUCLIDEAN ALGORITHM

References

Honsberger, R. "A Theorem of Gabriel Lamé." Ch. 7 in *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 54–57, 1976.

Lamina



A 2-D planar closed surface L which has a mass M and a surface density $\sigma(x,y)$ (in units of mass per areas squared) such that

$$M = \int_L \sigma(x,y)\,dx\,dy.$$

The CENTER OF MASS of a lamina is called its CENTROID.

see also CENTROID (GEOMETRIC), CROSS-SECTION, SOLID

Laminated Lattice

A LATTICE which is built up of layers of n-D lattices in $(n+1)$-D space. The VECTORS specifying how layers are stacked are called GLUE VECTORS.

see also GLUE VECTOR, LATTICE

References

Conway, J. H. and Sloane, N. J. A. "Laminated Lattices." Ch. 6 in *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, pp. 157–180, 1993.

Lancret Equation

$$ds_N{}^2 = ds_T{}^2 + ds_B{}^2,$$

where N is the NORMAL VECTOR, T is the TANGENT, and B is the BINORMAL VECTOR.

Lancret's Theorem

A NECESSARY and SUFFICIENT condition for a curve to be a HELIX is that the ratio of CURVATURE to TORSION be constant.

Lanczos Approximation

see GAMMA FUNCTION

Lanczos σ Factor

Writing a FOURIER SERIES as

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{m} \operatorname{sinc}\left(\frac{n\pi}{2m}\right)[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

where m is the last term and the $\operatorname{sinc} x$ terms are the Lanczos σ factor, removes the GIBBS PHENOMENON (Acton 1990).

see also FOURIER SERIES, GIBBS PHENOMENON, SINC FUNCTION

References

Acton, F. S. *Numerical Methods That Work, 2nd printing.* Washington, DC: Math. Assoc. Amer., p. 228, 1990.

Landau Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let F be the set of COMPLEX analytic functions f defined on an open region containing the closure of the unit disk $D = \{z : |z| < 1\}$ satisfying $f(0) = 0$ and $df/dz(0) = 1$. For each f in F, let $l(f)$ be the SUPREMUM of all numbers r such that $f(D)$ contains a disk of radius r. Then

$$L \equiv \inf\{l(f) : f \in F\}.$$

This constant is called the Landau constant, or the BLOCH-LANDAU CONSTANT. Robinson (1938, unpublished) and Rademacher (1943) derived the bounds

$$\tfrac{1}{2} < L \leq \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} = 0.5432588\ldots,$$

where $\Gamma(z)$ is the GAMMA FUNCTION, and conjectured that the second inequality is actually an equality,

$$L = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} = 0.5432588\ldots.$$

see also BLOCH CONSTANT

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/bloch/bloch.html`.

Rademacher, H. "On the Bloch-Landau Constant." *Amer. J. Math.* **65**, 387–390, 1943.

Landau-Kolmogorov Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $||f||$ be the SUPREMUM of $|f(x)|$, a real-valued function f defined on $(0,\infty)$. If f is twice differentiable and both f and f'' are bounded, Landau (1913) showed that

$$||f'|| \leq 2||f||^{1/2}||f''||^{1/2}, \tag{1}$$

where the constant 2 is the best possible. Schoenberg (1973) extended the result to the nth derivative of f defined on $(0,\infty)$ if both f and $f^{(n)}$ are bounded,

$$||f^{(k)}|| \leq C(n,k)||f||^{1-k/n}||f^{(n)}||^{k/n}. \tag{2}$$

An explicit FORMULA for $C(n,k)$ is not known, but particular cases are

$$C(3,1) = \left(\frac{243}{8}\right)^{1/3} \tag{3}$$

$$C(3,2) = 24^{1/3} \tag{4}$$

$$C(4,1) = 4.288\ldots \tag{5}$$

$$C(4,2) = 5.750\ldots \tag{6}$$

$$C(4,3) = 3.708\ldots. \tag{7}$$

Let $||f||$ be the SUPREMUM of $|f(x)|$, a real-valued function f defined on $(-\infty, \infty)$. If f is twice differentiable and both f and f'' are bounded, Hadamard (1914) showed that

$$||f'|| \leq \sqrt{2}||f||^{1/2}||f''||^{1/2}, \tag{8}$$

where the constant $\sqrt{2}$ is the best possible. Kolmogorov (1962) determined the best constants $C(n, k)$ for

$$||f^{(k)}|| \leq C(n,k)||f||^{1-k/n}||f^{(n)}||^{k/n} \tag{9}$$

in terms of the FAVARD CONSTANTS

$$a_n = \frac{4}{\pi}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{2j+1}\right]^{n+1} \tag{10}$$

by

$$C(n,k) = a_{n-k}{a_n}^{-1+k/n}. \tag{11}$$

Special cases derived by Shilov (1937) are

$$C(3,1) = \left(\frac{9}{8}\right)^{1/3} \tag{12}$$

$$C(3,2) = 3^{1/3} \tag{13}$$

$$C(4,1) = \left(\frac{512}{375}\right)^{1/4} \tag{14}$$

$$C(4,2) = \sqrt{\frac{6}{5}} \tag{15}$$

$$C(4,3) = \left(\frac{24}{5}\right)^{1/4} \tag{16}$$

$$C(5,1) = \left(\frac{1953125}{1572864}\right)^{1/5} \tag{17}$$

$$C(5,2) = \left(\frac{125}{72}\right)^{1/5}. \tag{18}$$

For a real-valued function f defined on $(-\infty, \infty)$, define

$$||f|| = \sqrt{\int_{-\infty}^{\infty} [f(x)]^2\, dx}. \tag{19}$$

If f is n differentiable and both f and $f^{(n)}$ are bounded, Hardy *et al.* (1934) showed that

$$||f^{(k)}|| \leq ||f||^{1-k/n}\,||f^{(n)}||^{k/n}, \tag{20}$$

where the constant 1 is the best possible for all n and $0 < k < n$.

For a real-valued function f defined on $(0, \infty)$, define

$$||f|| = \sqrt{\int_{0}^{\infty} [f(x)]^2\, dx}. \tag{21}$$

If f is twice differentiable and both f and f'' are bounded, Hardy *et al.* (1934) showed that

$$||f'|| \leq \sqrt{2}\,||f||^{1/2}||f^{(n)}||^{1/2}, \tag{22}$$

where the constant $\sqrt{2}$ is the best possible. This inequality was extended by Ljubic (1964) and Kupcov (1975) to

$$||f^{(k)}|| \leq C(n,k)\,||f||^{1-k/n}||f^{(n)}||^{k/n} \tag{23}$$

where $C(n, k)$ are given in terms of zeros of POLYNOMIALS. Special cases are

$$C(3,1) = C(3,2) = 3^{1/2}[2(2^{1/2}-1)]^{-1/3} = 1.84420\ldots \tag{24}$$

$$C(4,1) = C(4,3) = \sqrt{\frac{3^{1/4}+3^{-3/4}}{a}} = 2.27432\ldots \tag{25}$$

$$C(4,2) = \sqrt{\frac{2}{b}} = 2.97963\ldots \tag{26}$$

$$C(4,3) = \left(\frac{24}{5}\right)^{1/4} \tag{27}$$

$$C(5,1) = C(5,4) = 2.70247\ldots \tag{28}$$

$$C(5,2) = C(5,3) = 4.37800\ldots, \tag{29}$$

where a is the least POSITIVE ROOT of

$$x^8 - 6x^4 - 8x^2 + 1 = 0 \tag{30}$$

and b is the least POSITIVE ROOT of

$$x^4 - 2x^2 - 4x + 1 = 0 \tag{31}$$

(Franco *et al.* 1985, Neta 1980). The constants $C(n, 1)$ are given by

$$C(n,1) = \sqrt{\frac{(n-1)^{1/n}+(n+1)^{-1+1/n}}{c}}, \tag{32}$$

where c is the least POSITIVE ROOT of

$$\int_0^c \int_0^\infty \frac{dx\,dy}{(x^{2n}-yx^2+1)\sqrt{y}} = \frac{\pi^2}{2n}. \tag{33}$$

An explicit FORMULA of this type is not known for $k > 1$.

The cases $p = 1, 2, \infty$ are the only ones for which the best constants have exact expressions (Kwong and Zettl 1992, Franco *et al.* 1983).

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Landau-Ramanujan Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $S(x)$ denote the number of POSITIVE INTEGERS not exceeding x which can be expressed as a sum of two squares, then

$$\lim_{x\to\infty} \frac{\sqrt{\ln x}}{x} S(x) = K, \tag{1}$$

as proved by Landau (1908) and stated by Ramanujan. The value of K (also sometimes called λ) is

$$K = \sqrt{\tfrac{1}{2} \prod_{\substack{p \text{ a prime} \\ \equiv 3 \pmod 4}} \frac{1}{1-p^{-2}}} = 0.764223653\ldots \tag{2}$$

(Hardy 1940, Berndt 1994). Ramanujan found the approximate value $K = 0.764$. Flajolet and Vardi (1996) give a beautiful FORMULA with fast convergence

$$K = \frac{1}{\sqrt{2}} \prod_{n=1}^{\infty} \left[\left(1 - \frac{1}{2^{2^n}}\right) \frac{\zeta(2^n)}{\beta(2^n)}\right]^{1/(2^n+1)}, \tag{3}$$

where

$$\beta(s) \equiv \frac{1}{4^s}[\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4})] \tag{4}$$

is the DIRICHLET BETA FUNCTION, and $\zeta(z,a)$ is the HURWITZ ZETA FUNCTION. Landau proved the even stronger fact

$$\lim_{x\to\infty} \frac{(\ln x)^{3/2}}{Kx} \left[S(x) - \frac{Kx}{\sqrt{\ln x}}\right] = C, \tag{5}$$

where

$$C \equiv \frac{1}{2}\left[1 - \ln\left(\frac{\pi e^\gamma}{L}\right)\right] - \frac{1}{4}\frac{d}{ds}\left[\ln\left(\prod_{\substack{p \text{ prime} \\ p=4k+3}} \frac{1}{p^{-2s}}\right)\right]_{s=1}$$
$$= 0.581948659\ldots. \tag{6}$$

Here,

$$L = 5.2441151086\ldots \tag{7}$$

is the ARC LENGTH of a LEMNISCATE with $a = 1$ (the LEMNISCATE CONSTANT to within a factor of 2 or 4), and γ is the EULER-MASCHERONI CONSTANT.

References

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Landau Symbol

Let $f(z)$ be a function $\neq 0$ in an interval containing $z = 0$. Let $g(z)$ be another function also defined in this interval such that $g(z)/f(z) \to 0$ as $z \to 0$. Then $g(z)$ is said to be $\mathcal{O}(f(z))$.

Landen's Formula

$$\frac{\vartheta_3(z,t)\vartheta_4(z,t)}{\vartheta_4(2z,2t)} = \frac{\vartheta_3(0,t)\vartheta_4(0,t)}{\vartheta_4(0,2t)} = \frac{\vartheta_2(z,t)\vartheta_1(z,t)}{\vartheta_1(2z,2t)},$$

where ϑ_i are THETA FUNCTIONS. This transformation was used by Gauss to show that ELLIPTIC INTEGRALS could be computed using the ARITHMETIC-GEOMETRIC MEAN.

Landen's Transformation

If $x \sin \alpha = \sin(2\beta - \alpha)$, then

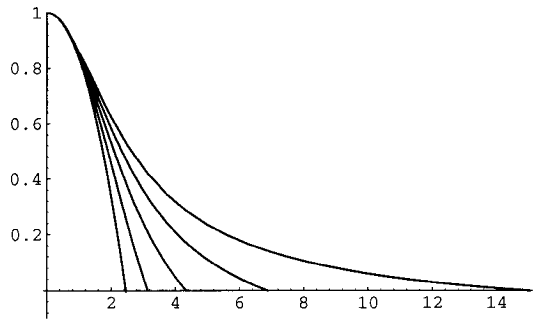
$$(1+x) \int_0^\alpha \frac{d\phi}{\sqrt{1 - x^2 \sin^2 \phi}} = 2 \int_0^\beta \frac{d\phi}{\sqrt{1 - \frac{4x}{(1+x)^2} \sin^2 \phi}}.$$

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, GAUSS'S TRANSFORMATION

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Ascending Landen Transformation" and "Landen's Transformation." §16.14 and 17.5 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 573–574 and 597–598, 1972.

Lane-Emden Differential Equation



A second-order ORDINARY DIFFERENTIAL EQUATION arising in the study of stellar interiors. It is given by

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0 \tag{1}$$

$$\frac{1}{\xi^2} \left(2\xi \frac{d\theta}{d\xi} + \xi^2 \frac{d^2\theta}{d\xi^2} \right) + \theta^n = \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi} \frac{d\theta}{d\xi} + \theta^n = 0. \tag{2}$$

It has the BOUNDARY CONDITIONS

$$\theta(0) = 1 \tag{3}$$

$$\left[\frac{d\theta}{d\xi} \right]_{\xi=0} = 0. \tag{4}$$

Solutions $\theta(\xi)$ for $n = 0$, 1, 2, 3, and 4 are shown above. The cases $n = 0$, 1, and 5 can be solved analytically (Chandrasekhar 1967, p. 91); the others must be obtained numerically.

For $n = 0$ ($\gamma = \infty$), the LANE-EMDEN DIFFERENTIAL EQUATION is

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + 1 = 0 \tag{5}$$

(Chandrasekhar 1967, pp. 91–92). Directly solving gives

$$\frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\xi^2 \tag{6}$$

$$\int d \left(\xi^2 \frac{d\theta}{d\xi^2} \right) = - \int \xi^2 \, d\xi \tag{7}$$

$$\xi^2 \frac{d\theta}{d\xi} = c_1 - \tfrac{1}{3}\xi^3 \tag{8}$$

$$\frac{d\theta}{d\xi} = \frac{c_1 - \frac{1}{3}\xi^3}{\xi^2} \tag{9}$$

$$\theta(\xi) = \int d\theta = \int \frac{c_1 - \frac{1}{3}\xi^3}{\xi^2} \, d\xi \tag{10}$$

$$\theta(\xi) = \theta_0 - c_1 \xi^{-1} - \tfrac{1}{6}\xi^2. \tag{11}$$

The BOUNDARY CONDITION $\theta(0) = 1$ then gives $\theta_0 = 1$ and $c_1 = 0$, so

$$\theta_1(\xi) = 1 - \tfrac{1}{6}\xi^2, \tag{12}$$

and $\theta_1(\xi)$ is PARABOLIC.

For $n = 1$ ($\gamma = 2$), the differential equation becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta = 0 \tag{13}$$

$$\frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta \xi^2 = 0, \tag{14}$$

which is the SPHERICAL BESSEL DIFFERENTIAL EQUATION

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + [k^2 r^2 - n(n+1)]R = 0 \tag{15}$$

with $k = 1$ and $n = 0$, so the solution is

$$\theta(\xi) = A j_0(\xi) + B n_0(\xi). \tag{16}$$

Applying the BOUNDARY CONDITION $\theta(0) = 1$ gives

$$\theta_2(\xi) = j_0(\xi) = \frac{\sin \xi}{\xi}, \tag{17}$$

where $j_0(x)$ is a SPHERICAL BESSEL FUNCTION OF THE FIRST KIND (Chandrasekhar 1967, pp. 92).

For $n = 5$, make Emden's transformation

$$\theta = A x^\omega z \tag{18}$$

$$\omega = \frac{2}{n-1}, \tag{19}$$

which reduces the Lane-Emden equation to

$$\frac{d^2 z}{dt^2} + (2\omega - 1)\frac{dz}{dt} + \omega(\omega - 1)z + A^{n-1} z^n = 0 \tag{20}$$

(Chandrasekhar 1967, p. 90). After further manipulation (not reproduced here), the equation becomes

$$\frac{d^2 z}{dt^2} = \tfrac{1}{4} z(1 - z^4) \tag{21}$$

and then, finally,

$$\theta_3(\xi)(1 + \tfrac{1}{3}\xi^2)^{-1/2}. \tag{22}$$

References

Chandrasekhar, S. *An Introduction to the Study of Stellar Structure.* New York: Dover, pp. 84–182, 1967.

Langford's Problem

Arrange copies of the n digits 1, ..., n such that there is one digit between the 1s, two digits between the 2s, etc. For example, the $n = 3$ solution is 312132 and the $n = 4$ solution is 41312432. Solutions exist only if $n \equiv 0, 3 \pmod 4$. The number of solutions for $n = 3$, 4, 5, ... are 1, 1, 0, 0, 26, 150, 0, 0, 17792, 108144, ... (Sloane's A014552).

References

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Langlands Program

A grand unified theory of mathematics which includes the search for a generalization of ARTIN RECIPROCITY (known as LANGLANDS RECIPROCITY) to non-Abelian Galois extensions of NUMBER FIELDS. Langlands proposed in 1970 that the mathematics of algebra and analysis are intimately related. He was a co-recipient of the 1996 Wolf Prize for this formulation.

see also ARTIN RECIPROCITY, LANGLANDS RECIPROCITY

References

American Mathematical Society. "Langlands and Wiles Share Wolf Prize." *Not. Amer. Math. Soc.* **43**, 221–222, 1996.

Knapp, A. W. "Group Representations and Harmonic Analysis from Euler to Langlands." *Not. Amer. Math. Soc.* **43**, 410–415, 1996.

Langlands Reciprocity

The conjecture that the ARTIN L-FUNCTION of any n-D GALOIS GROUP representation is an L-FUNCTION obtained from the GENERAL LINEAR GROUP $GL_1(\mathbb{A})$.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Langton's Ant

A CELLULAR AUTOMATON. The COHEN-KUNG THEOREM guarantees that the ant's trajectory is unbounded.

see also CELLULAR AUTOMATON, COHEN-KUNG THEOREM

References

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Laplace-Beltrami Operator

A self-adjoint elliptic differential operator defined somewhat technically as

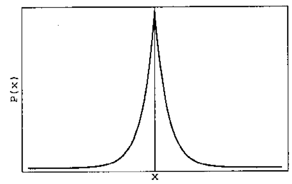
$$\Delta = d\delta + \delta d,$$

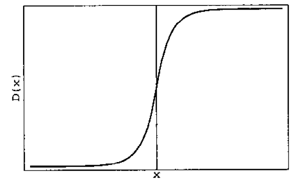
where d is the EXTERIOR DERIVATIVE and d and δ are adjoint to each other with respect to the INNER PRODUCT.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 628, 1980.

Laplace Distribution





Also called the DOUBLE EXPONENTIAL DISTRIBUTION. It is the distribution of differences between two independent variates with identical EXPONENTIAL DISTRIBUTIONS (Abramowitz and Stegun 1972, p. 930).

$$P(x) = \frac{1}{2b} e^{-|x-\mu|/b} \tag{1}$$

$$D(x) = \tfrac{1}{2}[1 + \operatorname{sgn}(x-\mu)(1 - e^{-|x-\mu|/b})]. \tag{2}$$

The MOMENTS about the MEAN μ_n are related to the MOMENTS about 0 by

$$\mu_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mu'_j \mu^{n-j}, \tag{3}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, so

$$\begin{aligned} \mu_n &= \sum_{j=0}^{n} \sum_{k=0}^{\lfloor j/2 \rfloor} (-1)^{n-j} \binom{n}{j} \binom{j}{2k} b^{2k} \mu^{n-2k} \Gamma(2k+1) \\ &= \begin{cases} n! b^n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd,} \end{cases} \end{aligned} \tag{4}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and $\Gamma(2k+1)$ is the GAMMA FUNCTION.

The MOMENTS can also be computed using the CHARACTERISTIC FUNCTION,

$$\phi(t) \equiv \int_{-\infty}^{\infty} e^{itx} P(x)\, dx = \frac{1}{2b} \int_{-\infty}^{\infty} e^{itx} e^{-|x-\mu|/b}\, dx. \tag{5}$$

Using the FOURIER TRANSFORM OF THE EXPONENTIAL FUNCTION

$$\mathcal{F}[e^{-2\pi k_0 |x|}] = \frac{1}{\pi} \frac{k_0}{k^2 + {k_0}^2} \tag{6}$$

gives

$$\phi(t) = \frac{e^{i\mu t}}{2b} \frac{\frac{2}{b}}{t^2 + \left(\frac{1}{b}\right)^2} = \frac{e^{i\mu t}}{1 + b^2 t^2}. \tag{7}$$

The MOMENTS are therefore

$$\mu_n = (-i)^n \phi(0) = (-i)^n \left[\frac{d^n \phi}{dt^n}\right]_{t=0}. \quad (8)$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \mu \quad (9)$$
$$\sigma^2 = 2b^2 \quad (10)$$
$$\gamma_1 = 0 \quad (11)$$
$$\gamma_2 = 3. \quad (12)$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, 1972.

Laplace's Equation

The scalar form of Laplace's equation is the PARTIAL DIFFERENTIAL EQUATION

$$\nabla^2 \psi = 0. \quad (1)$$

It is a special case of the HELMHOLTZ DIFFERENTIAL EQUATION

$$\nabla^2 \psi + k^2 \psi = 0 \quad (2)$$

with $k = 0$, or POISSON'S EQUATION

$$\nabla^2 \psi = -4\pi\rho \quad (3)$$

with $\rho = 0$. The vector Laplace's equation is given by

$$\nabla^2 \mathbf{F} = \mathbf{0}. \quad (4)$$

A FUNCTION ψ which satisfies Laplace's equation is said to be HARMONIC. A solution to Laplace's equation has the property that the average value over a spherical surface is equal to the value at the center of the SPHERE (GAUSS'S HARMONIC FUNCTION THEOREM). Solutions have no local maxima or minima. Because Laplace's equation is linear, the superposition of any two solutions is also a solution.

A solution to Laplace's equation is uniquely determined if (1) the value of the function is specified on all boundaries (DIRICHLET BOUNDARY CONDITIONS) or (2) the normal derivative of the function is specified on all boundaries (NEUMANN BOUNDARY CONDITIONS).

Laplace's equation can be solved by SEPARATION OF VARIABLES in all 11 coordinate systems that the HELMHOLTZ DIFFERENTIAL EQUATION can. In addition, separation can be achieved by introducing a multiplicative factor in two additional coordinate systems. The separated form is

$$\psi = \frac{X_1(u_1)X_2(u_2)X_3(u_3)}{R(u_1,u_2,u_3)}, \quad (5)$$

and setting

$$\frac{h_1h_2h_3}{{h_i}^2} = g_i(u_{i+1},u_{i+2})f_i(u_i)R^2, \quad (6)$$

where h_i are SCALE FACTORS, gives the Laplace's equation

$$\sum_{i=1}^3 \frac{1}{{h_i}^2 X_i}\left[\frac{1}{f_i}\frac{d}{du_i}\left(f_i\frac{dX_i}{du_i}\right)\right] = \sum_{i=1}^3 \frac{1}{{h_i}^2 R}\left[\frac{1}{f_i}\frac{\partial}{\partial u_i}\left(f_i\frac{\partial R}{\partial u_i}\right)\right]. \quad (7)$$

If the right side is equal to $-{k_1}^2/F(u_1,u_2,u_3)$, where k_1 is a constant and F is any function, and if

$$h_1h_2h_3 = Sf_1f_2f_3R^2F, \quad (8)$$

where S is the STÄCKEL DETERMINANT, then the equation can be solved using the methods of the HELMHOLTZ DIFFERENTIAL EQUATION. The two systems where this is the case are BISPHERICAL and TOROIDAL, bringing the total number of separable systems for Laplace's equation to 13 (Morse and Feshbach 1953, pp. 665–666).

In 2-D BIPOLAR COORDINATES, Laplace's equation is separable, although the HELMHOLTZ DIFFERENTIAL EQUATION is not.

see also BOUNDARY CONDITIONS, HARMONIC EQUATION, HELMHOLTZ DIFFERENTIAL EQUATION, PARTIAL DIFFERENTIAL EQUATION, POISSON'S EQUATION, SEPARATION OF VARIABLES, STÄCKEL DETERMINANT

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Laplace's Equation—Bipolar Coordinates

In 2-D BIPOLAR COORDINATES, LAPLACE'S EQUATION is

$$\frac{(\cosh v - \cos u)^2}{a^2}\left(\frac{\partial F^2}{\partial u^2} + \frac{\partial F^2}{\partial v^2}\right) = 0, \quad (1)$$

which simplifies to

$$\frac{\partial F^2}{\partial u^2} + \frac{\partial F^2}{\partial v^2} = 0, \quad (2)$$

so LAPLACE'S EQUATION is separable, although the HELMHOLTZ DIFFERENTIAL EQUATION is not.

Laplace's Equation—Bispherical Coordinates

$$\left[\frac{-\cos u\cot^2 u+3\cosh v\cot^2 u}{\cosh v-\cos u}\right.$$
$$\left.-\frac{3\cosh^2 v\cot u\csc u+\cosh^3 v\csc^2 u}{\cosh v-\cos u}\right]\frac{\partial}{\partial\phi^2}$$
$$+(\cos u-\cosh v)\sinh v\frac{\partial}{\partial v}+(\cosh^2 v-\cos u)^2\frac{\partial^2}{\partial v^2}$$
$$+(\cosh v-\cos u)(\cosh v\cot u-\sin u-\cos u\cot u)\frac{\partial}{\partial u}$$
$$+(\cosh^2 v-\cos u)^2\frac{\partial^2}{\partial u^2}=0. \quad (1)$$

Let

$$F(u,v,\phi)=\sqrt{\cosh u-\cos v}\,U(u)V(v)\Phi(\phi), \quad (2)$$

then LAPLACE'S EQUATION is partially separable, although the HELMHOLTZ DIFFERENTIAL EQUATION is not.

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 665–666, 1953.

Laplace's Equation—Toroidal Coordinates

$$\nabla^2 f=\frac{(\cosh v-\cos u)^3}{a^2}\frac{\partial}{\partial u}\left(\frac{1}{\cosh v-\cos u}\frac{\partial f}{\partial u}\right)$$
$$+\frac{(\cosh v-\cos u)^3}{a^2\sinh v}\frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v-\cos u}\frac{\partial f}{\partial v}\right)$$
$$+\frac{(\cosh v-\cos u)^2}{a^2\sinh v}\frac{\partial^2 f}{\partial\phi^2} \quad (1)$$
$$=\left[\frac{-3\cos\coth^2 v+\cosh v\coth^2 v}{\cosh v-\cos u}\right.$$
$$\left.+\frac{3\cos^2 u\coth v\operatorname{csch} v-\cos^3 u\operatorname{csch}^2 v}{\cosh v-\cos u}\right]\frac{\partial^2}{\partial\phi^2}$$
$$+(\cos u-\cosh v)\sin u\frac{\partial}{\partial u}+(\cosh v-\cos u)^2\frac{\partial^2}{\partial u^2}$$
$$+(\cosh v-\cos u)(\cosh v\coth v-\sinh v-\cos u\coth v)\frac{\partial}{\partial v}$$
$$+(\cosh^2 v-\cos u)^2\frac{\partial^2}{\partial v^2}. \quad (2)$$

Let

$$f(\xi,\eta,\phi)=\sqrt{\cosh\eta-\cos\xi}\,X(\xi)H(\eta)\Psi(\psi), \quad (3)$$

then

$$X(\xi)=\frac{\sin}{\cos}(n\xi) \quad (4)$$

$$\Psi(\psi)=\frac{\sin}{\cos}(m\phi), \quad (5)$$

and the equation in η becomes

$$\frac{1}{\sinh\eta}\frac{d}{d\eta}\left(\sinh\eta\frac{dH}{d\eta}\right)-\frac{m^2}{\sinh^2\eta}H-(n^2-\tfrac{1}{4})H=0. \quad (6)$$

LAPLACE'S EQUATION is partially separable, although the HELMHOLTZ DIFFERENTIAL EQUATION is not.

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Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 666, 1953.

Laplace's Integral

$$P_n(x)=\frac{1}{\pi}\int_0^p i\frac{du}{\left(x+\sqrt{x^2-1}\cos u\right)^{n+1}}$$
$$=\frac{1}{\pi}\int_0^\pi(x+\sqrt{x^2-1}\cos u)^n\,du.$$

Laplace Limit

The value $e=0.6627434193\ldots$ (Sloane's A033259) for which Laplace's formula for solving KEPLER'S EQUATION begins diverging. The constant is defined as the value e at which the function

$$f(x)=\frac{x\exp(\sqrt{1+x^2})}{1+\sqrt{1+x^2}}$$

equals $f(\lambda)=1$. The CONTINUED FRACTION of e is given by [0, 1, 1, 1, 27, 1, 1, 1, 8, 2, 154, ...] (Sloane's A033260). The positions of the first occurrences of n in the CONTINUED FRACTION of e are 2, 10, 35, 13, 15, 32, 101, 9, ... (Sloane's A033261). The incrementally largest terms in the CONTINUED FRACTION are 1, 27, 154, 1601, 2135, ... (Sloane's A033262), which occur at positions 2, 5, 11, 19, 1801, ... (Sloane's A033263).

see also ECCENTRIC ANOMALY, KEPLER'S EQUATION

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Sloane, N. J. A. Sequences A033259, A033260, A033261, A033262, and A033263 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Laplace-Mehler Integral

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{2\pi}(\cos\theta + i\sin\theta\cos\phi)^n\,d\phi$$
$$= \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos[(n+\frac{1}{2})\phi]}{\sqrt{\cos\phi-\cos\theta}}\,d\phi$$
$$= \frac{\sqrt{2}}{\pi}\int_\theta^\pi \frac{\sin[(n+\frac{1}{2})\phi]}{\sqrt{\cos\theta-\cos\phi}}\,d\phi.$$

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Laplace Series

A function $f(\theta,\phi)$ expressed as a double sum of SPHERICAL HARMONICS is called a Laplace series. Taking f as a COMPLEX FUNCTION,

$$f(\theta,\phi) = \sum_{l=0}^\infty \sum_{m=-1}^{l} a_{lm} Y_l^m(\theta,\phi). \tag{1}$$

Now multiply both sides by $Y_{l'}^{m'*}\sin\theta$ and integrate over $d\theta$ and $d\phi$.

$$\int_0^{2\pi}\int_0^\pi f(\theta,\phi)Y_{l'}^{m'*}\sin\theta\,d\theta\,d\phi$$
$$= \sum_{l=0}^\infty\sum_{m=-l}^{l} a_{lm}\int_0^{2\pi}\int_0^\pi Y_{l'}^{m'*}(\theta,\phi)Y_l^m(\theta,\phi)\sin\theta\,d\theta\,d\phi. \tag{2}$$

Now use the ORTHOGONALITY of the SPHERICAL HARMONICS

$$\int_0^{2\pi}\int_0^\pi Y_l^m(\theta,\phi)Y_{l'}^{m'*}\sin\theta\,d\theta\,d\phi = \delta_{mm'}\delta_{ll'}, \tag{3}$$

so (2) becomes

$$\int_0^{2\pi}\int_0^\pi f(\theta,\phi)Y_{l'}^{m'*}\sin\theta\,d\theta\,d\phi$$
$$= \sum_{l=0}^\infty\sum_{m=-1}^{l} a_{lm}\delta_{mm'}\delta_{ll'} = a_{lm}, \tag{4}$$

where δ_{mn} is the KRONECKER DELTA.

For a REAL series, consider

$$f(\theta,\phi)$$
$$= \sum_{l=0}^\infty\sum_{m=-1}^{l}[C_l^m\cos(m\phi)+S_l^m\sin(m\phi)]P_l^m(\cos\theta). \tag{5}$$

Proceed as before, using the orthogonality relationships

$$\int_0^{2\pi}\int_0^\pi P_l^m(\cos\theta)\cos(m\phi)P_{l'}^{m'}(\cos\theta)$$
$$\times\cos(m'\phi)\sin(\theta)\,d\theta\,d\phi = -\frac{2\pi(l+m)!}{(2l+1)(l-m)!}\delta_{mm'}\delta_{ll'} \tag{6}$$

$$\int_0^{2\pi}\int_0^\pi P_l^m(\cos\theta)\sin(m\phi)P_{l'}^{m'}(\cos\theta)$$
$$\times\sin(m'\phi)\sin\theta\,d\theta\,d\phi = -\frac{2\pi(l+m)!}{(2l+1)(l-m)!}\delta_{mm'}\delta_{ll'}. \tag{7}$$

So C_l^m and S_l^m are given by

$$C_l^m = -\frac{(2l+1)(l-m)!}{2\pi(l+m)!}$$
$$\times\int_0^{2\pi}\int_0^\pi f(\theta,\phi)P_l^m\cos\theta\cos(m\phi)\sin\theta\,d\theta\,d\phi \tag{8}$$
$$S_l^m = -\frac{(2l+1)(l-m)!}{2\pi(l+m)!}$$
$$\times\int_0^{2\pi}\int_0^\pi f(\theta,\phi)P_l^m\cos\theta\sin(m\phi)\sin\theta\,d\theta\,d\phi. \tag{9}$$

Laplace-Stieltjes Transform

An integral transform which is often written as an ordinary LAPLACE TRANSFORM involving the DELTA FUNCTION.

see also LAPLACE TRANSFORM

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Laplace Transform

The Laplace transform is an INTEGRAL TRANSFORM perhaps second only to the FOURIER TRANSFORM in its utility in solving physical problems. Due to its useful properties, the Laplace transform is particularly useful in solving linear ORDINARY DIFFERENTIAL EQUATIONS such as those arising in the analysis of electronic circuits.

The (one-sided) Laplace transform $\mathcal{L}$ (not to be confused with the LIE DERIVATIVE) is defined by

$$\mathcal{L}(s) = \mathcal{L}(f(t)) \equiv \int_0^\infty f(t)e^{-st}\,dt, \tag{1}$$

where $f(t)$ is defined for $t \geq 0$. A two-sided Laplace transform is sometimes also defined by

$$\mathcal{L}(s) = \mathcal{L}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt. \quad (2)$$

The Laplace transform existence theorem states that, if $f(t)$ is piecewise CONTINUOUS on every finite interval in $[0,\infty)$ satisfying

$$|f(t)| \leq Me^{at} \quad (3)$$

for all $t \in [0,\infty)$, then $\mathcal{L}(f(t))$ exists for all $s > a$. The Laplace transform is also UNIQUE, in the sense that, given two functions $F_1(t)$ and $F_2(t)$ with the same transform so that

$$\mathcal{L}[F_1(t)] = \mathcal{L}[F_2(t)] \equiv f(s), \quad (4)$$

then LERCH'S THEOREM guarantees that the integral

$$\int_0^a N(t)\,dt = 0 \quad (5)$$

vanishes for all $a > 0$ for a NULL FUNCTION defined by

$$N(t) \equiv F_1(t) - F_2(t). \quad (6)$$

The Laplace transform is LINEAR since

$$\begin{aligned}\mathcal{L}[af(t)+bg(t)] &= \int_0^{\infty}[af(t)+bg(t)]e^{-st}\,dt\\ &= a\int_0^{\infty} f(t)e^{-st}\,dt + b\int_0^{\infty} g(t)e^{-st}\,dt\\ &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]. \quad (7)\end{aligned}$$

The inverse Laplace transform is given by the BROMWICH INTEGRAL (see also DUHAMEL'S CONVOLUTION PRINCIPLE). A table of several important Laplace transforms follows.

$f(t)$	$\mathcal{L}[f(t)]$	Range
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$n \in \mathbb{Z} > 0$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$a > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$s > \|a\|$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\delta(t-c)$	e^{-cs}	
$H_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$J_0(t)$	$\frac{1}{\sqrt{s^2+1}}$	

In the above table, $J_0(t)$ is the zeroth order BESSEL FUNCTION OF THE FIRST KIND, $\delta(t)$ is the DELTA FUNCTION, and $H_c(t)$ is the HEAVISIDE STEP FUNCTION. The Laplace transform has many important properties.

The Laplace transform of a CONVOLUTION is given by

$$\mathcal{L}(f(t)*g(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t)) \quad (8)$$

$$\mathcal{L}^{-1}(F(s)G(s)) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G(s)). \quad (9)$$

Now consider DIFFERENTIATION. Let $f(t)$ be continuously differentiable $n-1$ times in $[0,\infty)$. If $|f(t)| \leq Me^{at}$, then

$$\begin{aligned}\mathcal{L}[f^{(n)}(t)] &= s^n\mathcal{L}(f(t)) - s^{n-1}f(0)\\ &\quad - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \quad (10)\end{aligned}$$

This can be proved by INTEGRATION BY PARTS,

$$\begin{aligned}\mathcal{L}[f'(t)] &= \lim_{a\to 0}\int_0^a e^{-st}f'(t)\,dt\\ &= \lim_{a\to 0}[e^{-st}f(t)]_0^a + s\int_0^a e^{-st}f(t)\,dt]\\ &= \lim_{a\to 0}[e^{-sa}f(a) - f(0) + s\int_0^a e^{-st}f(t)\,dt]\\ &= s\mathcal{L}[f(t)] - f(0). \quad (11)\end{aligned}$$

Continuing for higher order derivatives then gives

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \quad (12)$$

This property can be used to transform differential equations into algebraic equations, a procedure known as the HEAVISIDE CALCULUS, which can then be inverse transformed to obtain the solution. For example, applying the Laplace transform to the equation

$$f''(t) + a_1 f'(t) + a_0 f(t) = 0 \quad (13)$$

gives

$$\begin{aligned}\{s^2\mathcal{L}[f(t)] - sf(0) - f'(0)\} + a_1\{s\mathcal{L}[f(t)] - f(0)\}\\ + a_0\mathcal{L}[f(t)] = 0 \quad (14)\end{aligned}$$

$$\mathcal{L}[f(t)](s^2+a_1 s+a_0) - sf(0) - f'(0) - a_1 f(0) = 0, \quad (15)$$

which can be rearranged to

$$\mathcal{L}[f(t)] = \frac{sf(0) + [f'(0) + a_1 f(0)]}{s^2 + a_1 s + a_0}. \quad (16)$$

If this equation can be inverse Laplace transformed, then the original differential equation is solved.

Consider EXPONENTIATION. If $\mathcal{L}(f(t)) = F(s)$ for $s > \alpha$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$ for $s > a + \alpha$.

$$F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}\,dt = \int_0^\infty [f(t)e^{at}]e^{-st}\,dt = \mathcal{L}(e^{at}f(t)). \tag{17}$$

Consider INTEGRATION. If $f(t)$ is piecewise continuous and $|f(t)| \leq Me^{at}$, then

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\mathcal{L}[f(t)]. \tag{18}$$

The inverse transform is known as the BROMWICH INTEGRAL, or sometimes the FOURIER-MELLIN INTEGRAL.

see also BROMWICH INTEGRAL, FOURIER-MELLIN INTEGRAL, FOURIER TRANSFORM, INTEGRAL TRANSFORM, LAPLACE-STIELTJES TRANSFORM, OPERATIONAL MATHEMATICS

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Laplacian

The Laplacian operator for a SCALAR function ϕ is defined by

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]\phi \tag{1}$$

in VECTOR notation, where the h_i are the SCALE FACTORS of the coordinate system. In TENSOR notation, the Laplacian is written

$$\nabla^2\phi = (g^{\lambda\kappa}\phi_{;\lambda})_{;\kappa} = g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^\lambda \partial x^\kappa} - \Gamma^\lambda \frac{\partial\phi}{\partial x^\lambda} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\frac{\partial\phi}{\partial x^i}\right), \tag{2}$$

where $g_{;\kappa}$ is a COVARIANT DERIVATIVE and

$$\Gamma^\lambda \equiv \tfrac{1}{2}g^{\mu\nu}g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\mu}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right). \tag{3}$$

The finite difference form is

$$\nabla^2\psi(x,y,z) = \frac{1}{h^2}[\psi(x+h,y,z) + \psi(x-h,y,z) + \psi(x,y+h,z) + \psi(x,y-h,z) + \psi(x,y,z+h) + \psi(x,y,z-h) - 6\psi(x,y,z)]. \tag{4}$$

For a pure radial function $g(r)$,

$$\nabla^2 g(r) \equiv \nabla\cdot[\nabla g(r)] = \nabla\cdot\left[\frac{\partial g(r)}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial g(r)}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial g(r)}{\partial\phi}\hat{\boldsymbol{\phi}}\right] = \nabla\cdot\left(\hat{\mathbf{r}}\frac{dg}{dr}\right). \tag{5}$$

Using the VECTOR DERIVATIVE identity

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + (\nabla f)\cdot(\mathbf{A}), \tag{6}$$

so

$$\nabla^2 g(r) \equiv \nabla\cdot[\nabla g(r)] = \frac{dg}{dr}\nabla\cdot\hat{\mathbf{r}} + \nabla\left(\frac{dg}{dr}\right)\cdot\hat{\mathbf{r}} = \frac{2}{r}\frac{dg}{dr} + \frac{d^2g}{dr^2}. \tag{7}$$

Therefore, for a radial POWER law,

$$\nabla^2 r^n = \frac{2}{r}nr^{n-1} + n(n-1)r^{n-2} = [2n + n(n-1)]r^{n-2} = n(n+1)r^{n-2}. \tag{8}$$

A VECTOR Laplacian can also be defined for a VECTOR $\mathbf{A}$ by

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) \tag{9}$$

in vector notation. In tensor notation, $\mathbf{A}$ is written A_μ, and the identity becomes

$$\nabla^2 A_\mu = A_{\mu;\lambda}{}^{;\lambda} = (g^{\lambda\kappa}A_{\mu;\lambda})_{;\kappa} = g^\lambda{}\kappa_{;\kappa}A_{\mu;\lambda} + g^{\lambda\kappa}A_{\mu;\lambda\kappa}. \tag{10}$$

Similarly, a TENSOR Laplacian can be given by

$$\nabla^2 A_{\alpha\beta} = A_{\alpha\beta;\lambda}{}^{;\lambda}. \tag{11}$$

An identity satisfied by the Laplacian is

$$\nabla^2|\mathbf{x}\mathsf{A}| = \frac{|\mathsf{A}|_2{}^2 - |(\mathbf{x}\mathsf{A})\mathsf{A}^{\mathrm{T}}|^2}{|\mathbf{x}\mathsf{A}|^3}, \tag{12}$$

where $|\mathsf{A}|_2$ is the HILBERT-SCHMIDT NORM, $\mathbf{x}$ is a row VECTOR, and A^{T} is the MATRIX TRANSPOSE of A.

To compute the LAPLACIAN of the inverse distance function $1/r$, where $r \equiv |\mathbf{r}-\mathbf{r}'|$, and integrate the LAPLACIAN over a volume,

$$\int_V \nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3\mathbf{r}. \tag{13}$$

This is equal to

$$\begin{aligned}\int_V \nabla^2 \frac{1}{r} d^3\mathbf{r} &= \int_V \nabla \cdot \left(\nabla \frac{1}{r}\right) d^3\mathbf{r} = \int_S \left(\nabla \frac{1}{r}\right) \cdot d\mathbf{a} \\ &= \int_S \frac{\partial}{\partial r}\left(\frac{1}{r}\right) \hat{\mathbf{r}} \cdot d\mathbf{a} = \int_S -\frac{1}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{a} \\ &= -4\pi \frac{R^2}{r^2},\end{aligned} \tag{14}$$

where the integration is over a small SPHERE of RADIUS R. Now, for $r > 0$ and $R \to 0$, the integral becomes 0. Similarly, for $r = R$ and $R \to 0$, the integral becomes -4π. Therefore,

$$\nabla^2 \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}'), \tag{15}$$

where δ is the DELTA FUNCTION.

see also ANTILAPLACIAN

Laplacian Determinant Expansion by Minors

see DETERMINANT EXPANSION BY MINORS

Large Number

There are a wide variety of large numbers which crop up in mathematics. Some are contrived, but some actually arise in proofs. Often, it is possible to prove existence theorems by deriving some potentially huge upper limit which is frequently greatly reduced in subsequent versions (e.g., GRAHAM'S NUMBER, KOLMOGOROV-ARNOLD-MOSER THEOREM, MERTENS CONJECTURE, SKEWES NUMBER, WANG'S CONJECTURE).

Large decimal numbers beginning with 10^9 are named according to two mutually conflicting nomenclatures: the American system (in which the prefix stands for n in 10^{3+3n}) and the British system (in which the prefix stands for n in 10^{6n}). The following table gives the names assigned to various POWERS of 10 (Woolf 1982).

American	British	Power of 10
million	million	10^6
billion	milliard	10^9
trillion	billion	10^{12}
quadrillion		10^{15}
quintillion	trillion	10^{18}
sextillion		10^{21}
septillion	quadrillion	10^{24}
octillion		10^{27}
nonillion	quintillion	10^{30}
decillion		10^{33}
undecillion	sexillion	10^{36}
duodecillion		10^{39}
tredecillion	septillion	10^{42}
quattuordecillion		10^{45}
quindecillion	octillion	10^{48}
sexdecillion		10^{51}
septendecillion	nonillion	10^{54}
octodecillion		10^{57}
novemdecillion	decillion	10^{60}
vigintillion		10^{63}
	undecillion	10^{66}
	duodecillion	10^{72}
	tredecillion	10^{78}
	quattuordecillion	10^{84}
	quindecillion	10^{90}
	sexdecillion	10^{96}
	septendecillion	10^{102}
	octodecillion	10^{108}
	novemdecillion	10^{114}
	vigintillion	10^{120}
centillion		10^{303}
	centillion	10^{600}

see also 10, ACKERMANN NUMBER, ARROW NOTATION, BILLION, CIRCLE NOTATION, EDDINGTON NUMBER, G-FUNCTION, GÖBEL'S SEQUENCE, GOOGOL, GOOGOLPLEX, GRAHAM'S NUMBER, HUNDRED, HYPERFACTORIAL, JUMPING CHAMPION, LAW OF TRULY LARGE NUMBERS, MEGA, MEGISTRON, MILLION, MONSTER GROUP, MOSER, n-PLEX, POWER TOWER, SKEWES NUMBER, SMALL NUMBER, STEINHAUS-MOSER NOTATION, STRONG LAW OF LARGE NUMBERS, SUPERFACTORIAL, THOUSAND, WEAK LAW OF LARGE NUMBERS, ZILLION

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Large Prime

see GIGANTIC PRIME, LARGE NUMBER, TITANIC PRIME

Laspeyres' Index

The statistical INDEX

$$P_L \equiv \frac{\sum p_n q_0}{\sum p_0 q_0},$$

where p_n is the price per unit in period n and q_n is the quantity produced in period n.

see also INDEX

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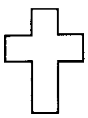
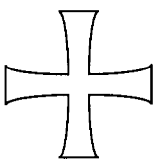
Latin Cross



An irregular DODECAHEDRON CROSS in the shape of a dagger †. The six faces of a CUBE can be cut along seven EDGES and unfolded into a Latin cross (i.e., the Latin cross is the NET of the CUBE). Similarly, eight hypersurfaces of a HYPERCUBE can be cut along 17 SQUARES and unfolded to form a 3-D Latin cross.



Another cross also called the Latin cross is illustrated above. It is a GREEK CROSS with flared ends, and is also known as the crux immissa or cross patée.

see also CROSS, DISSECTION, DODECAHEDRON, GREEK CROSS

Latin Rectangle

A $k \times n$ Latin rectangle is a $k \times n$ MATRIX with elements $a_{ij} \in \{1, 2, \ldots, n\}$ such that entries in each row and column are distinct. If $k = n$, the special case of a LATIN SQUARE results. A normalized Latin rectangle has first row $\{1, 2, \ldots, n\}$ and first column $\{1, 2, \ldots, k\}$. Let $L(k,n)$ be the number of normalized $k \times n$ Latin rectangles, then the total number of $k \times n$ Latin rectangles is

$$N(k,n) = \frac{n!(n-1)!L(k,n)}{(n-k)!}$$

(McKay and Rogoyski 1995), where $n!$ is a FACTORIAL. Kerewala (1941) found a RECURRENCE RELATION for $L(3,n)$, and Athreya, Pranesachar, and Singhi (1980) found a summation FORMULA for $L(4,n)$.

The asymptotic value of $L(o(n^{6/7}), n)$ was found by Godsil and McKay (1990). The numbers of $k \times n$ Latin rectangles are given in the following table from McKay and Rogoyski (1995). The entries $L(1,n)$ and $L(n,n)$ are omitted, since

$$L(1,n) = 1$$
$$L(n,n) = L(n-1,n),$$

but $L(1,1)$ and $L(2,1)$ are included for clarity. The values of $L(k,n)$ are given as a "wrap-around" series by Sloane's A001009.

n	k	$L(k,n)$
1	1	1
2	1	1
3	2	1
4	2	3
4	3	4
5	2	11
5	3	46
5	4	56
6	2	53
6	3	1064
6	4	6552
6	5	9408
7	2	309
7	3	36792
7	4	1293216
7	5	11270400
7	6	16942080
8	2	2119
8	3	1673792
8	4	420906504
8	5	27206658048
8	6	335390189568
8	7	535281401856
9	2	16687
9	3	103443808
9	4	207624560256
9	5	112681643983776
9	6	12962605404381184
9	7	224382967916691456
9	8	377597570964258816
10	2	148329
10	3	8154999232
10	4	147174521059584
10	5	746988383076286464
10	6	870735405591003709440
10	7	177144296983054185922560
10	8	4292039421591854273003520
10	9	7580721482160132811489280

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McKay, B. D. and Rogoyski, E. "Latin Squares of Order 10." *Electronic J. Combinatorics* **2**, N3, 1–4, 1995. `http://www.combinatorics.org/Volume_2/volume2.html#N3`.
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Latin Square

An $n \times n$ Latin square is a LATIN RECTANGLE with $k = n$. Specifically, a Latin square consists of n sets of the numbers 1 to n arranged in such a way that no orthogonal (row or column) contains the same two numbers. The numbers of Latin squares of order $n = 1$, 2, ... are 1, 2, 12, 576, ... (Sloane's A002860). A pair of Latin squares is said to be orthogonal if the n^2 pairs formed by juxtaposing the two arrays are all distinct.

Two of the Latin squares of order 3 are

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix},$$

which are orthogonal. Two of the 576 Latin squares of order 4 are

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}.$$

A normalized, or reduced, Latin square is a Latin square with the first row and column given by $\{1, 2, \ldots, n\}$. General FORMULAS for the number of *normalized* Latin squares $L(n,n)$ are given by Nechvatal (1981), Gessel (1987), and Shao and Wei (1992). The total number of Latin squares of order n can then be computed from

$$N(n,n) = n!(n-1)!L(n,n) = n!(n-1)!L(n-1,n).$$

The numbers of normalized Latin square of order $n = 1$, 2, ..., are 1, 1, 1, 4, 56, 9408, ... (Sloane's A000315). McKay and Rogoyski (1995) give the number of normalized LATIN RECTANGLES $L(k,n)$ for $n = 1, \ldots, 10$, as well as estimates for $L(n,n)$ with $n = 11, 12, \ldots, 15$.

n	$L(n,n)$
11	5.36×10^{33}
12	1.62×10^{44}
13	2.51×10^{56}
14	2.33×10^{70}
15	1.5×10^{86}

see also EULER SQUARE, KIRKMAN TRIPLE SYSTEM, PARTIAL LATIN SQUARE, QUASIGROUP

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Latin-Graeco Square

see EULER SQUARE

Latitude

The latitude of a point on a SPHERE is the elevation of the point from the PLANE of the equator. The latitude δ is related to the COLATITUDE (the polar angle in SPHERICAL COORDINATES) by $\delta = \phi - 90°$. More generally, the latitude of a point on an ELLIPSOID is the ANGLE between a LINE PERPENDICULAR to the surface of the ELLIPSOID at the given point and the PLANE of the equator (Snyder 1987).

The equator therefore has latitude 0°, and the north and south poles have latitude ±90°, respectively. Latitude is also called GEOGRAPHIC LATITUDE or GEODETIC LATITUDE in order to distinguish it from several subtly different varieties of AUXILIARY LATITUDES.

The shortest distance between any two points on a SPHERE is the so-called GREAT CIRCLE distance, which can be directly computed from the latitudes and LONGITUDES of the two points.

see also AUXILIARY LATITUDE, COLATITUDE, CONFORMAL LATITUDE, GREAT CIRCLE, ISOMETRIC LATITUDE, LATITUDE, LONGITUDE, SPHERICAL COORDINATES

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Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, p. 13, 1987.

Lattice

A lattice is a system K such that $\forall A \in K$, $A \subset A$, and if $A \subset B$ and $B \subset A$, then $A = B$, where $=$ here means "is included in." Lattices offer a natural way to formalize and study the ordering of objects using a general concept known as the POSET (partially ordered set). The study of lattices is called LATTICE THEORY. Note that this type of lattice is an abstraction of the regular array of points known as LATTICE POINTS.

The following inequalities hold for any lattice:

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$$

(Grätzer 1971, p. 35). The first three are the distributive inequalities, and the last is the modular identity.

see also DISTRIBUTIVE LATTICE, INTEGRATION LATTICE, LATTICE THEORY, MODULAR LATTICE, TORIC VARIETY

References
Grätzer, G. *Lattice Theory: First Concepts and Distributive Lattices.* San Francisco, CA: W. H. Freeman, 1971.

Lattice Algebraic System

A generalization of the concept of set unions and intersections.

Lattice Animal

A distinct (including reflections and rotations) arrangement of adjacent squares on a grid, also called fixed POLYOMINOES.

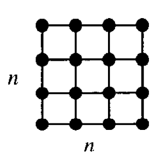
see also PERCOLATION THEORY, POLYOMINO

Lattice Distribution

A DISCRETE DISTRIBUTION of a random variable such that every possible value can be represented in the form $a + bn$, where $a, b \neq 0$ and n is an INTEGER.

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 927, 1972.

Lattice Graph



The lattice graph with n nodes on a side is denoted $L(n)$.

see also TRIANGULAR GRAPH

Lattice Groups

In the plane, there are 17 lattice groups, eight of which are pure translation. In $\mathbb{R}^3$, there are 32 POINT GROUPS and 230 SPACE GROUPS. In $\mathbb{R}^4$, there are 4783 space lattice groups.

see also POINT GROUPS, SPACE GROUPS, WALLPAPER GROUPS

Lattice Path

A path composed of connected horizontal and vertical line segments, each passing between adjacent LATTICE POINTS. A lattice path is therefore a SEQUENCE of points P_0, P_1, ..., P_n with $n \geq 0$ such that each P_i is a LATTICE POINT and P_{i+1} is obtained by offsetting one unit east (or west) or one unit north (or south).

The number of paths of length $a + b$ from the ORIGIN (0,0) to a point (a, b) which are restricted to east and north steps is given by the BINOMIAL COEFFICIENT $\binom{a+b}{a}$.

see also BALLOT PROBLEM, GOLYGON, KINGS PROBLEM, LATTICE POINT, p-GOOD PATH, RANDOM WALK

References
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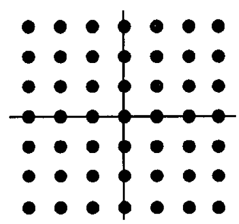
Lattice Point



A POINT at the intersection of two or more grid lines in a ruled array. (The array of grid lines could be oriented to form unit cells in the shape of a square, rectangle, hexagon, etc.) However, unless otherwise specified, lattice points are generally taken to refer to points in a square array, i.e., points with coordinates $(m, n, \ldots)$, where m, n, ... are INTEGERS.

An n-D $\mathbb{Z}[\omega]$-lattice L_n lattice can be formally defined as a free $\mathbb{Z}[\omega]$-MODULE in complex n-D space $\mathbb{C}^n$.

The FRACTION of lattice points VISIBLE from the ORIGIN, as derived in Castellanos (1988, pp. 155–156), is

$$\frac{N'(r)}{N(r)} = \frac{\frac{24}{\pi^2}r^2 + \mathcal{O}(r \ln r)}{4r^2 + \mathcal{O}(r)} = \frac{\frac{6}{\pi^2} + \mathcal{O}\left(\frac{\ln r}{r}\right)}{1 + \mathcal{O}\left(\frac{1}{r}\right)} = \frac{6}{\pi^2}.$$

Therefore, this is also the probability that two randomly picked integers will be RELATIVELY PRIME to one another.

For $2 \leq n \leq 32$, it is possible to select $2n$ lattice points with $x, y \in [1, n]$ such that no three are in a straight

LINE. The number of distinct solutions (not counting reflections and rotations) for $n = 1, 2, \ldots,$ are 1, 1, 4, 5, 11, 22, 57, 51, 156 ... (Sloane's A000769). For large n, it is conjectured that it is only possible to select at most $(c+\epsilon)n$ lattice points with no three COLLINEAR, where

$$c = (2\pi^2/3)^{1/3} \approx 1.85$$

(Guy and Kelly 1968; Guy 1994, p. 242). The number of the n^2 lattice points $x, y \in [1, n]$ which can be picked with no four CONCYCLIC is $\mathcal{O}(n^{2/3} - \epsilon)$ (Guy 1994, p. 241).

A special set of POLYGONS defined on the regular lattice are the GOLYGONS. A NECESSARY and SUFFICIENT condition that a linear transformation transforms a lattice to itself is that it be UNIMODULAR. M. Ajtai has shown that there is no efficient ALGORITHM for finding any fraction of a set of spanning vectors in a lattice having the shortest lengths unless there is an efficient algorithm for all of them (of which none is known). This result has potential applications to cryptography and authentication (Cipra 1996).

see also BARNES-WALL LATTICE, BLICHFELDT'S THEOREM, BROWKIN'S THEOREM, CIRCLE LATTICE POINTS, COXETER-TODD LATTICE, EHRHART POLYNOMIAL, GAUSS'S CIRCLE PROBLEM, GOLYGON, INTEGRATION LATTICE, JARNICK'S INEQUALITY, LATTICE PATH, LATTICE SUM, LEECH LATTICE, MINKOWSKI CONVEX BODY THEOREM, MODULAR LATTICE, N-CLUSTER, NOSARZEWSKA'S INEQUALITY, PICK'S THEOREM, POSET, RANDOM WALK, SCHINZEL'S THEOREM, SCHRÖDER NUMBER, VISIBLE POINT, VORONOI POLYGON

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Lattice Reduction

The process finding a reduced set of basis vectors for a given LATTICE having certain special properties. Lattice reduction is implemented in Mathematica® (Wolfram Research, Champaign, IL) using the function `LatticeReduce`. Lattice reduction algorithms are used in a number of modern number theoretical applications, including in the discovery of a SPIGOT ALGORITHM for PI.

see also INTEGER RELATION, PSLQ ALGORITHM

Lattice Sum

Cubic lattice sums include the following:

$$b_2(2s) \equiv \sum_{i,j=-\infty}^{\infty}{}' \frac{(-1)^{i+j}}{(i^2+j^2)^s} \tag{1}$$

$$b_3(2s) \equiv \sum_{i,j,k=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k}}{(i^2+j^2+k^2)^s} \tag{2}$$

$$b_n(2s) \equiv \sum_{k_1,\ldots,k_n=-\infty}^{\infty}{}' \frac{(-1)^{k_1+\ldots+k_n}}{({k_1}^2+\ldots+{k_n}^2)^s}, \tag{3}$$

where the prime indicates that summation over $(0,0,0)$ is excluded. As shown in Borwein and Borwein (1987, pp. 288–301), these have closed forms for even n

$$b_2(2s) = -4\beta(s)\eta(s) \tag{4}$$

$$b_4(2s) = -8\eta(s)\eta(s-1) \tag{5}$$

$$b_8(2s) = -16\zeta(s)\eta(s-3), \quad \text{for } \Re[s] > 1 \tag{6}$$

where $\beta(z)$ is the DIRICHLET BETA FUNCTION, $\eta(z)$ is the DIRICHLET ETA FUNCTION, and $\zeta(z)$ is the RIEMANN ZETA FUNCTION. The lattice sums evaluated at $s = 1$ are called the MADELUNG CONSTANTS. Borwein and Borwein (1986) prove that $b_8(2)$ converges (the closed form for $b_8(2s)$ above does not apply for $s = 1$), but its value has not been computed.

For hexagonal sums, Borwein and Borwein (1987, p. 292) give

$$h_2(2s) \equiv \frac{4}{3} \sum_{m,n=-\infty}^{\infty} \frac{\sin[(n+1)\theta]\sin[(m+1)\theta] - \sin(n\theta)\sin[(m-1)\theta]}{\left[(n+\frac{1}{2}m)^2 + 3(\frac{1}{2}m)^2\right]^s}, \tag{7}$$

where $\theta \equiv 2\pi/3$. This MADELUNG CONSTANT is expressible in closed form for $s = 1$ as

$$h_2(2) = \pi \ln 3\sqrt{3}. \tag{8}$$

see also BENSON'S FORMULA, MADELUNG CONSTANTS

References

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Lattice Theory

Lattice theory is the study of sets of objects known as LATTICES. It is an outgrowth of the study of BOOLEAN ALGEBRAS, and provides a framework for unifying the study of classes or ordered sets in mathematics. Its study was given a great boost by a series of papers and subsequent textbook written by Birkhoff (1967).

see also LATTICE

References
Birkhoff, G. *Lattice Theory, 3rd ed.* Providence, RI: Amer. Math. Soc., 1967.
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Latus Rectum

Twice the SEMILATUS RECTUM.

see also PARABOLA

Laurent Polynomial

A Laurent polynomial with COEFFICIENTS in the FIELD $\mathbb{F}$ is an algebraic object that is typically expressed in the form

$$\ldots + a_{-n}t^{-n} + a_{-(n-1)}t^{-(n-1)} + \ldots + a_{-1}t^{-1} + a_0 + a_1 t + \ldots + a_n t^n + \ldots,$$

where the a_i are elements of $\mathbb{F}$, and only finitely many of the a_i are NONZERO. A Laurent polynomial is an algebraic object in the sense that it is treated as a POLYNOMIAL except that the indeterminant "t" can also have NEGATIVE POWERS.

Expressed more precisely, the collection of Laurent polynomials with COEFFICIENTS in a FIELD $\mathbb{F}$ form a RING, denoted $\mathbb{F}[t, t^{-1}]$, with RING operations given by componentwise addition and multiplication according to the relation

$$at^n \cdot bt^m = abt^{n+m}$$

for all n and m in the INTEGERS. Formally, this is equivalent to saying that $\mathbb{F}[t, t^{-1}]$ is the GROUP RING of the INTEGERS and the FIELD $\mathbb{F}$. This corresponds to $\mathbb{F}[t]$ (the POLYNOMIAL ring in one variable for $\mathbb{F}$) being the GROUP RING or MONOID ring for the MONOID of natural numbers and the FIELD $\mathbb{F}$.

see also POLYNOMIAL

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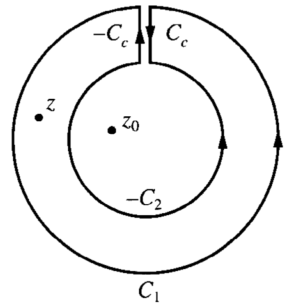
Laurent Series



Let there be two circular contours C_2 and C_1, with the radius of C_1 larger than that of C_2. Let z_0 be interior to C_1 and C_2, and z be between C_1 and C_2. Now create a cut line C_c between C_1 and C_2, and integrate around the path $C \equiv C_1 + C_c - C_2 - C_c$, so that the plus and minus contributions of C_c cancel one another, as illustrated above. From the CAUCHY INTEGRAL FORMULA,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{z'-z}\,dz' + \frac{1}{2\pi i}\int_{C_c} \frac{f(z')}{z'-z}\,dz' \\ &\quad - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_c} \frac{f(z')}{z'-z}\,dz' \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{z'-z}\,dz'. \end{aligned} \quad (1)$$

Now, since contributions from the cut line in opposite directions cancel out,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' \\ &\quad - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}\,dz' \\ &\quad - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z-z_0)\left(\frac{z'-z_0}{z-z_0}-1\right)}\,dz' \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}\,dz' \\ &\quad + \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z-z_0)\left(1-\frac{z'-z_0}{z-z_0}\right)}\,dz'. \end{aligned} \quad (2)$$

For the first integral, $|z'-z_0| > |z-z_0|$. For the second, $|z'-z_0| < |z-z_0|$. Now use the TAYLOR EXPANSION (valid for $|t| < 1$)

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad (3)$$

to obtain

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\left[\int_{C_1}\frac{f(z')}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz'\right.\\&\quad\left.+\int_{C_2}\frac{f(z')}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n dz'\right]\\&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'\\&\quad+\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^{-n-1}\int_{C_2}(z'-z_0)^n f(z')\,dz'\\&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'\\&\quad+\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\int_{C_2}(z'-z_0)^{n+1} f(z')\,dz',\end{aligned} \tag{4}$$

where the second term has been re-indexed. Re-indexing again,

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'\\&\quad+\frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n\int_{C_2}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'.\end{aligned} \tag{5}$$

Now, use the CAUCHY INTEGRAL THEOREM, which requires that any CONTOUR INTEGRAL of a function which encloses no POLES has value 0. But $1/(z'-z_0)^{n+1}$ is never singular inside C_2 for $n \geq 0$, and $1/(z'-z_0)^{n+1}$ is never singular inside C_1 for $n \leq -1$. Similarly, there are no POLES in the closed cut $C_c - C_c$. We can therefore replace C_1 and C_2 in the above integrals by C without altering their values, so

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'\\&\quad+\frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n\int_{C}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'\\&=\frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}(z-z_0)^n\int_{C}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'\\&\equiv\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n.\end{aligned} \tag{6}$$

The only requirement on C is that it encloses z, so we are free to choose any contour γ that does so. The RESIDUES a_n are therefore defined by

$$a_n \equiv \frac{1}{2\pi i}\int_{\gamma}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'. \tag{7}$$

see also MACLAURIN SERIES, RESIDUE (COMPLEX ANALYSIS), TAYLOR SERIES

References

Arfken, G. "Laurent Expansion." §6.5 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 376–384, 1985.

Morse, P. M. and Feshbach, H. "Derivatives of Analytic Functions, Taylor and Laurent Series." §4.3 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 374–398, 1953.

Law

A law is a mathematical statement which always holds true. Whereas "laws" in physics are generally experimental observations backed up by theoretical underpinning, laws in mathematics are generally THEOREMS which can formally be proven true under the stated conditions. However, the term is also sometimes used in the sense of an empirical observation, e.g., BENFORD'S LAW.

see also ABSORPTION LAW, BENFORD'S LAW, CONTRADICTION LAW, DE MORGAN'S DUALITY LAW, DE MORGAN'S LAWS, ELLIPTIC CURVE GROUP LAW, EXCLUDED MIDDLE LAW, EXPONENT LAWS, GIRKO'S CIRCULAR LAW, LAW OF COSINES, LAW OF SINES, LAW OF TANGENTS, LAW OF TRULY LARGE NUMBERS, MORRIE'S LAW, PARALLELOGRAM LAW, PLATEAU'S LAWS, QUADRATIC RECIPROCITY LAW, STRONG LAW OF LARGE NUMBERS, STRONG LAW OF SMALL NUMBERS, SYLVESTER'S INERTIA LAW, TRICHOTOMY LAW, VECTOR TRANSFORMATION LAW, WEAK LAW OF LARGE NUMBERS, ZIPF'S LAW

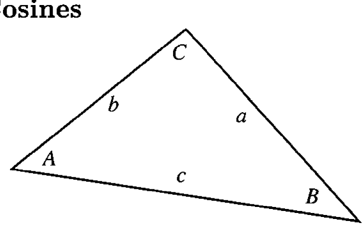
Law of Anomalous Numbers

see BENFORD'S LAW

Law of Cancellation

see CANCELLATION LAW

Law of Cosines



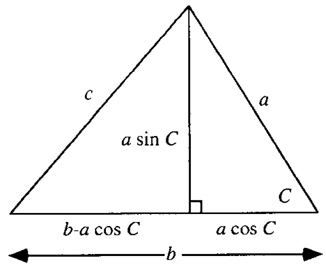
Let a, b, and c be the lengths of the legs of a TRIANGLE opposite ANGLES A, B, and C. Then the law of cosines states

$$c^2 = a^2 + b^2 - 2ab\cos C. \tag{1}$$

This law can be derived in a number of ways. The definition of the DOT PRODUCT incorporates the law of cosines, so that the length of the VECTOR from $\mathbf{X}$ to $\mathbf{Y}$ is given by

$$\begin{aligned}|\mathbf{X}-\mathbf{Y}|^2 &= (\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}-\mathbf{Y}) &(2)\\&=\mathbf{X}\cdot\mathbf{X}-2\mathbf{X}\cdot\mathbf{Y}+\mathbf{Y}\cdot\mathbf{Y} &(3)\\&=|\mathbf{X}|^2+|\mathbf{Y}|^2-2|\mathbf{X}|\,|\mathbf{Y}|\cos\theta, &(4)\end{aligned}$$

where θ is the ANGLE between $\mathbf{X}$ and $\mathbf{Y}$.



The formula can also be derived using a little geometry and simple algebra. From the above diagram,

$$\begin{aligned} c^2 &= (a\sin C)^2 + (b - a\cos C)^2 \\ &= a^2\sin^2 c + b^2 - 2ab\cos C + a^2\cos^2 C \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned} \tag{5}$$

The law of cosines for the sides of a SPHERICAL TRIANGLE states that

$$\cos a = \cos b\cos c + \sin b\sin c\cos A \tag{6}$$

$$\cos b = \cos c\cos a + \sin c\sin a\cos B \tag{7}$$

$$\cos c = \cos a\cos b + \sin a\sin b\cos C \tag{8}$$

(Beyer 1987). The law of cosines for the angles of a SPHERICAL TRIANGLE states that

$$\cos A = -\cos B\cos C + \sin B\sin C\cos a \tag{9}$$

$$\cos B = -\cos C\cos A + \sin C\sin A\cos b \tag{10}$$

$$\cos C = -\cos A\cos B + \sin A\sin B\cos c \tag{11}$$

(Beyer 1987).

see also LAW OF SINES, LAW OF TANGENTS

References

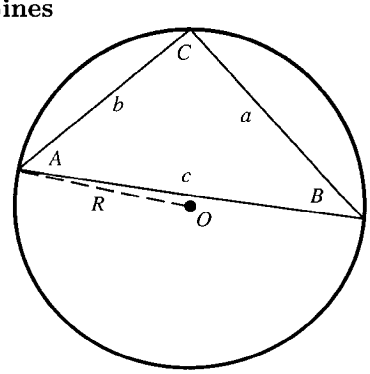
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 79, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 148–149, 1987.

Law of Large Numbers

see LAW OF TRULY LARGE NUMBERS, STRONG LAW OF LARGE NUMBERS, WEAK LAW OF LARGE NUMBERS

Law of Sines



Let a, b, and c be the lengths of the LEGS of a TRIANGLE opposite ANGLES A, B, and C. Then the law of sines states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \tag{1}$$

where R is the radius of the CIRCUMCIRCLE. Other related results include the identities

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0 \tag{2}$$

$$a = b\cos C + c\cos B, \tag{3}$$

the LAW OF COSINES

$$\cos A = \frac{c^2 + b^2 - a^2}{2bc}, \tag{4}$$

and the LAW OF TANGENTS

$$\frac{a+b}{a-b} = \frac{\tan[\frac{1}{2}(A+B)]}{\tan[\frac{1}{2}(A-B)]}. \tag{5}$$

The law of sines for oblique SPHERICAL TRIANGLES states that

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{6}$$

see also LAW OF COSINES, LAW OF TANGENTS

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 79, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 148, 1987.

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 1–3, 1967.

Law of Small Numbers

see STRONG LAW OF SMALL NUMBERS

Law of Tangents

Let a TRIANGLE have sides of lengths a, b, and c and let the ANGLES opposite these sides by A, B, and C. The law of tangents states

$$\frac{a-b}{a+b} = \frac{\tan[\frac{1}{2}(A-B)]}{\tan[\frac{1}{2}(A+B)]}.$$

An analogous result for oblique SPHERICAL TRIANGLES states that

$$\frac{\tan[\frac{1}{2}(a-b)]}{\tan[\frac{1}{2}(a+b)]} = \frac{\tan[\frac{1}{2}(A-B)]}{\tan[\frac{1}{2}(A+B)]}.$$

see also LAW OF COSINES, LAW OF SINES

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 79, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 145 and 149, 1987.

Law of Truly Large Numbers

With a large enough sample, any outrageous thing is likely to happen (Diaconis and Mosteller 1989). Littlewood (1953) considered an event which occurs one in a million times to be "surprising." Taking this definition, close to 100,000 surprising events are "expected" each year in the United States alone and, in the world at large, "we can be absolutely sure that we will see incredibly remarkable events" (Diaconis and Mosteller 1989).

see also COINCIDENCE, STRONG LAW OF LARGE NUMBERS, STRONG LAW OF SMALL NUMBERS, WEAK LAW OF LARGE NUMBERS

References

Diaconis, P. and Mosteller, F. "Methods of Studying Coincidences." *J. Amer. Statist. Assoc.* **84**, 853–861, 1989.

Littlewood, J. E. *Littlewood's Miscellany*. Cambridge, England: Cambridge University Press, 1986.

Lax-Milgram Theorem

Let ϕ be a bounded COERCIVE bilinear FUNCTIONAL on a HILBERT SPACE H. Then for every bounded linear FUNCTIONAL f on H, there exists a unique $x_f \in H$ such that

$$f(x) = \phi(x, x_f)$$

for all $x \in H$.

References

Debnath, L. and Mikusiński, P. *Introduction to Hilbert Spaces with Applications*. San Diego, CA: Academic Press, 1990.

Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics*. New York: Springer-Verlag, 1995.

Lax Pair

A pair of linear OPERATORS L and A associated with a given PARTIAL DIFFERENTIAL EQUATION which can be used to solve the equation. However, it turns out to be very difficult to find the L and A corresponding to a given equation, so it is actually simpler to postulate a given L and A and determine to which PARTIAL DIFFERENTIAL EQUATION they correspond (Infeld and Rowlands 1990).

see also PARTIAL DIFFERENTIAL EQUATION

References

Infeld, E. and Rowlands, G. "Integrable Equations in Two Space Dimensions as Treated by the Zakharov Shabat Methods." §7.10 in *Nonlinear Waves, Solitons, and Chaos*. Cambridge, England: Cambridge University Press, pp. 216–223, 1990.

Layer

see p-LAYER

Le Cam's Identity

Let S_n be the sum of n random variates X_i with a BERNOULLI DISTRIBUTION with $P(X_i = 1) = p_i$. Then

$$\sum_{k=0}^{\infty} \left| P(S_n = k) - \frac{e^{-\lambda}\lambda^k}{k!} \right| < 2 \sum_{i=1}^{n} {p_i}^2,$$

where

$$\lambda \equiv \sum_{i=1}^{n} p_i.$$

see also BERNOULLI DISTRIBUTION

References

Cox, D. A. "Introduction to Fermat's Last Theorem." *Amer. Math. Monthly* **101**, 3–14, 1994.

Leading Digit Phenomenon

see BENFORD'S LAW

Leading Order Analysis

A procedure for determining the behavior of an nth order ORDINARY DIFFERENTIAL EQUATION at a REMOVABLE SINGULARITY without actually solving the equation. Consider

$$\frac{d^n y}{dz^n} = F\left(\frac{d^{n-1}y}{dz^{n-1}}, \ldots, \frac{dy}{dx}, y, z\right), \tag{1}$$

where F is ANALYTIC in z and rational in its other arguments. Proceed by making the substitution

$$y(z) \equiv a(z - z_0)^\alpha \tag{2}$$

with $\alpha < 1$. For example, in the equation

$$\frac{d^2 y}{dz^2} = 6y^2 + Ay, \tag{3}$$

making the substitution gives

$$a\alpha(\alpha-1)(z-z_0)^{\alpha-2} = 6a^2(z-z_0)^{2\alpha} + Aa(az-z_0)^\alpha. \tag{4}$$

The most singular terms (those with the most NEGATIVE exponents) are called the "dominant balance terms," and must balance exponents and COEFFICIENTS at the SINGULARITY. Here, the first two terms are dominant, so

$$\alpha - 2 = 2\alpha \Rightarrow \alpha = -2 \tag{5}$$

$$6a = 6a^2 \Rightarrow a = 1, \tag{6}$$

and the solution behaves as $y(z) = (z - z_0)^{-2}$. The behavior in the NEIGHBORHOOD of the SINGULARITY is given by expansion in a LAURENT SERIES, in this case,

$$y(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{j-2}. \tag{7}$$

Plugging this series in yields

$$\sum_{j=0}^{\infty} a_j(j-2)(j-3)(z-z_0)^{j-4}$$
$$= 6\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j a_k (z-z_0)^{j+k-4} + A\sum_{j=0}^{\infty} a_j(z-z_0)^{j-2}. \quad (8)$$

This gives RECURRENCE RELATIONS, in this case with a_6 arbitrary, so the $(z-z_0)^6$ term is called the resonance or KOVALEVSKAYA EXPONENT. At the resonances, the COEFFICIENT will always be arbitrary. If no resonance term is present, the POLE present is not ordinary, and the solution must be investigated using a PSI FUNCTION.

see also PSI FUNCTION

References
Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, p. 330, 1989.

Leaf (Foliation)

Let M^n be an n-MANIFOLD and let $\mathsf{F} = \{F_\alpha\}$ denote a PARTITION of M into DISJOINT path-connected SUBSETS. Then if F is a FOLIATION of M, each F_α is called a leaf and is not necessarily closed or compact.

see also FOLIATION

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 284, 1976.

Leaf (Tree)

An unconnected end of a TREE.

see also BRANCH, CHILD, FORK, ROOT (TREE), TREE

Leakage

see ALIASING

Least Bound

see SUPREMUM

Least Common Multiple

The least common multiple of two numbers n_1 and n_2 is denoted $\mathrm{LCM}(n_1, n_2)$ or $[n_1, n_2]$ and can be obtained by finding the PRIME factorization of each

$$n_1 = {p_1}^{a_1} \cdots {p_n}^{a_n} \quad (1)$$

$$n_2 = {p_1}^{b_1} \cdots {p_n}^{b_n}, \quad (2)$$

where the p_is are all PRIME FACTORS of n_1 and n_2, and if p_i does not occur in one factorization, then the corresponding exponent is 0. The least common multiple is then

$$\mathrm{LCM}(n_1, n_2) = [n_1, n_2] = \prod_{i=1}^{n} {p_i}^{\max(a_i, b_i)}. \quad (3)$$

Let m be a common multiple of a and b so that

$$m = ha = kb. \quad (4)$$

Write $a = a_1(a,b)$ and $b = b_1(a,b)$, where a_1 and b_1 are RELATIVELY PRIME by definition of the GREATEST COMMON DIVISOR $(a_1, b_1) = 1$. Then $ha_1 = kb_1$, and from the DIVISION LEMMA (given that ha_1 is DIVISIBLE by b and $(b_1, a_1) = 0$), we have h is DIVISIBLE by b_1, so

$$h = nb_1 \quad (5)$$

$$m = ha = nb_1 a = n\frac{ab}{(a,b)}. \quad (6)$$

The smallest m is given by $n = 1$,

$$\mathrm{LCM}(a,b) = \frac{ab}{\mathrm{GCD}(a,b)}, \quad (7)$$

so

$$\mathrm{GCD}(a,b)\ \mathrm{LCM}(a,b) = ab \quad (8)$$

$$(a,b)[a,b] = ab. \quad (9)$$

The LCM is IDEMPOTENT

$$[a,a] = a, \quad (10)$$

COMMUTATIVE

$$[a,b] = [b,a], \quad (11)$$

ASSOCIATIVE

$$[a,b,c] = [[a,b],c] = [a,[b,c]], \quad (12)$$

DISTRIBUTIVE

$$[ma, mb, mc] = m[a,b,c], \quad (13)$$

and satisfies the ABSORPTION LAW

$$(a,[a,b]) = a. \quad (14)$$

It is also true that

$$[ma, mb] = \frac{(ma)(mb)}{(ma, mb)} = m\frac{ab}{(a,b)} = m[a,b]. \quad (15)$$

see also GREATEST COMMON DIVISOR, MANGOLDT FUNCTION, RELATIVELY PRIME

References
Guy, R. K. "Density of a Sequence with L.C.M. of Each Pair Less than x." §E2 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 200–201, 1994.

Least Deficient Number

A number for which

$$\sigma(n) = 2n - 1.$$

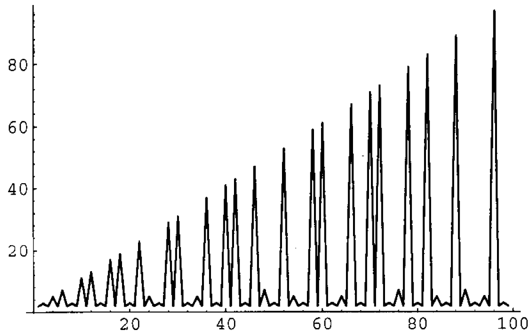
All POWERS of 2 are least deficient numbers.

see also DEFICIENT NUMBER, QUASIPERFECT NUMBER

Least Period

The smallest n for which a point x_0 is a PERIODIC POINT of a function f so that $f^n(x_0) = x_0$. For example, for the FUNCTION $f(x) = -x$, all points x have period 2 (including $x = 0$). However, $x = 0$ has a *least* period of 1. The analogous concept exists for a PERIODIC SEQUENCE, but not for a PERIODIC FUNCTION. The least period is also called the EXACT PERIOD.

Least Prime Factor



For an INTEGER $n \geq 2$, let $\operatorname{lpf}(x)$ denote the LEAST PRIME FACTOR of n, i.e., the number p_1 in the factorization

$$n = p_1{}^{a_1} \cdots p_k{}^{a_k},$$

with $p_i < p_j$ for $i < j$. For $n = 2, 3, \ldots$, the first few are 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, 2, 3, ... (Sloane's A020639). The above plot of the least prime factor function can be seen to resemble a jagged terrain of mountains, which leads to the appellation of "TWIN PEAKS" to a PAIR of INTEGERS (x, y) such that

1. $x < y$,
2. $\operatorname{lpf}(x) = \operatorname{lpf}(y)$,
3. For all z, $x < z < y$ IMPLIES $\operatorname{lpf}(z) < \operatorname{lpf}(x)$.

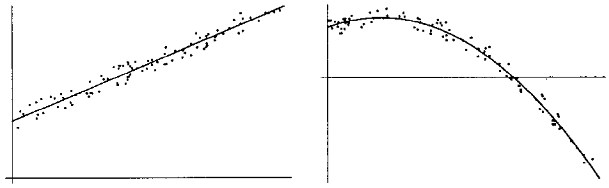
The least *multiple* prime factors for SQUAREFUL integers are 2, 2, 3, 2, 2, 3, 2, 2, 5, 3, 2, 2, 2, ... (Sloane's A046027).

see also ALLADI-GRINSTEAD CONSTANT, DISTINCT PRIME FACTORS, ERDŐS-SELFRIDGE FUNCTION, FACTOR, GREATEST PRIME FACTOR, LEAST COMMON MULTIPLE, MANGOLDT FUNCTION, PRIME FACTORS, TWIN PEAKS

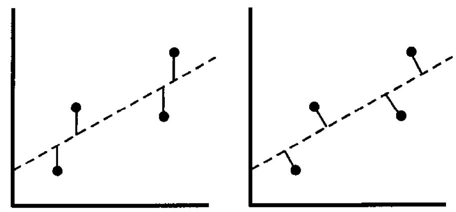
References

Sloane, N. J. A. Sequence A020639 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Least Squares Fitting



A mathematical procedure for finding the best fitting curve to a given set of points by minimizing the sum of the squares of the offsets ("the residuals") of the points from the curve. The sum of the *squares* of the offsets is used instead of the offset absolute values because this allows the residuals to be treated as a continuous differentiable quantity. However, because squares of the offsets are used, outlying points can have a disproportionate effect on the fit, a property which may or may not be desirable depending on the problem at hand.



vertical offsets *perpendicular offsets*

In practice, the *vertical* offsets from a line are almost always minimized instead of the *perpendicular* offsets. This allows uncertainties of the data points along the x- and y-axes to be incorporated simply, and also provides a much simpler analytic form for the fitting parameters than would be obtained using a fit based on perpendicular distances. In addition, the fitting technique can be easily generalized from a best-fit *line* to a best-fit *polynomial* when sums of vertical distances are used (which is not the case using perpendicular distances). For a reasonable number of noisy data points, the difference between vertical and perpendicular fits is quite small.

The *linear* least squares fitting technique is the simplest and most commonly applied form of LINEAR REGRESSION and provides a solution to the problem of finding the best fitting *straight* line through a set of points. In fact, if the functional relationship between the two quantities being graphed is known to within additive or multiplicative constants, it is common practice to transform the data in such a way that the resulting line *is* a straight line, say by plotting T vs. $\sqrt{\ell}$ instead of t vs. ℓ. For this reason, standard forms for EXPONENTIAL, LOGARITHMIC, and POWER laws are often explicitly computed. The formulas for linear least squares fitting were independently derived by Gauss and Legendre.

For NONLINEAR LEAST SQUARES FITTING to a number of unknown parameters, linear least squares fitting may be applied iteratively to a linearized form of the function until convergence is achieved. Depending on the type of fit and initial parameters chosen, the nonlinear fit may have good or poor convergence properties. If uncertainties (in the most general case, error ellipses) are given for the points, points can be weighted differently in order to give the high-quality points more weight.

The residuals of the best-fit line for a set of n points using *unsquared perpendicular* distances d_i of points (x_i, y_i) are given by

$$R_\perp \equiv \sum_{i=1}^{n} d_i. \tag{1}$$

Since the perpendicular distance from a line $y = a + bx$ to point i is given by

$$d_i = \frac{|y_i - (a + bx_i)|}{\sqrt{1+b^2}}, \tag{2}$$

the function to be minimized is

$$R_\perp \equiv \sum_{i=1}^{n} \frac{|y_i - (a + bx_i)|}{\sqrt{1+b^2}}. \tag{3}$$

Unfortunately, because the absolute value function does not have continuous derivatives, minimizing $R_\perp$ is not amenable to analytic solution. However, if the *square* of the perpendicular distances

$$R_\perp^2 \equiv \sum_{i=1}^{n} \frac{[y_i - (a + bx_i)]^2}{1+b^2} \tag{4}$$

is minimized instead, the problem can be solved in closed form. $R_\perp^2$ is a minimum when (suppressing the indices)

$$\frac{\partial R_\perp^2}{\partial a} = \frac{2}{1+b^2} \sum [y - (a+bx)](-1) = 0 \tag{5}$$

and

$$\frac{\partial R_\perp^2}{\partial b} = \frac{2}{1+b^2} \sum [y - (a+bx)](-x) + \sum \frac{[y-(a+bx)]^2(-1)(2b)}{(1+b^2)^2} = 0. \tag{6}$$

The former gives

$$a = \frac{\sum y - b \sum x}{n} = \bar{y} - b\bar{x}, \tag{7}$$

and the latter

$$(1+b^2)\sum [y-(a+bx)]x + b\sum [y-(a+bx)]^2 = 0. \tag{8}$$

But

$$\begin{aligned}[y-(a+bx)]^2 &= y^2 - 2(a+bx)y + (a+bx)^2 \\ &= y^2 - 2ay - 2bxy + a^2 + 2abx + b^2x^2,\end{aligned} \tag{9}$$

so (8) becomes

$$\begin{aligned}&(1+b^2)\left(\sum xy - a\sum x - b\sum x^2\right) \\ &+b\left(\sum y^2 - 2a\sum y - 2b\sum xy + a^2\sum 1\right. \\ &\left.+2ab\sum x + b^2\sum x^2\right) = 0\end{aligned} \tag{10}$$

$$\begin{aligned}&[(1+b^2)(-b) + b(b^2)]\sum x^2 + [(1+b^2) - 2b^2]\sum xy \\ &+b\sum y^2 + [-a(1+b^2) + 2ab^2]\sum x - 2ab\sum y \\ &+ba^2\sum 1 = 0\end{aligned} \tag{11}$$

$$-b\sum x^2 + (1-b^2)\sum xy + b\sum y^2 + a(b^2-1)\sum x - 2ab\sum y + ba^2 n = 0. \tag{12}$$

Plugging (7) into (12) then gives

$$\begin{aligned}&-b\sum x^2 + (1-b^2)\sum xy + b\sum y^2 \\ &+\frac{1}{n}(b^2-1)\left(\sum y - b\sum x\right)\sum x \\ &-\frac{2}{n}\left(\sum y - b\sum x\right)b\sum y + \frac{1}{n}b\left(\sum y - b\sum x\right)^2 \\ &= 0\end{aligned} \tag{13}$$

After a fair bit of algebra, the result is

$$b^2 + \frac{\sum y^2 - \sum x^2 + \frac{1}{n}\left[\left(\sum x\right)^2 - \left(\sum y\right)^2\right]}{\frac{1}{n}\sum x \sum y - \sum xy} b - 1 = 0. \tag{14}$$

So define

$$\begin{aligned}B &\equiv \frac{1}{2}\frac{\left[\sum y^2 - \frac{1}{n}\left(\sum y\right)^2\right] - \left[\sum x^2 - \frac{1}{n}\left(\sum x\right)^2\right]}{\frac{1}{n}\sum x\sum y - \sum xy} \\ &= \frac{1}{2}\frac{\left(\sum y^2 - n\bar{y}^2\right) - \left(\sum x^2 - n\bar{x}^2\right)}{n\sum x \sum y - \sum xy},\end{aligned} \tag{15}$$

and the QUADRATIC FORMULA gives

$$b = -B \pm \sqrt{B^2+1}, \tag{16}$$

with a found using (7). Note the rather unwieldy form of the best-fit parameters in the formulation. In addition, minimizing $R_\perp^2$ for a second- or higher-order POLYNOMIAL leads to polynomial equations having *higher* order, so this formulation cannot be extended.

Vertical least squares fitting proceeds by finding the sum of the *squares* of the *vertical* deviations R^2 of a set of n data points

$$R^2 \equiv \sum [y_i - f(x_i, a_1, a_2, \ldots, a_n)]^2 \tag{17}$$

from a function f. Note that this procedure does *not* minimize the actual deviations from the line (which would be measured perpendicular to the given function). In addition, although the *unsquared* sum of distances might seem a more appropriate quantity to minimize, use of the absolute value results in discontinuous derivatives which cannot be treated analytically. The square deviations from each point are therefore summed, and the resulting residual is then minimized to find the best fit line. This procedure results in outlying points being given disproportionately large weighting.

The condition for R^2 to be a minimum is that

$$\frac{\partial(R^2)}{\partial a_i} = 0 \tag{18}$$

for $i = 1, \ldots, n$. For a linear fit,

$$f(a,b) = a + bx, \tag{19}$$

so

$$R^2(a,b) \equiv \sum_{i=1}^{n}[y_i - (a + bx_i)]^2 \tag{20}$$

$$\frac{\partial(R^2)}{\partial a} = -2\sum_{i=1}^{n}[y_i - (a + bx_i)] = 0 \tag{21}$$

$$\frac{\partial(R^2)}{\partial b} = -2\sum_{i=1}^{n}[y_i - (a + bx_i)]x_i = 0. \tag{22}$$

These lead to the equations

$$na + b\sum x = \sum y \tag{23}$$

$$a\sum x + b\sum x^2 = \sum xy, \tag{24}$$

where the subscripts have been dropped for conciseness. In MATRIX form,

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}, \tag{25}$$

so

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}. \tag{26}$$

The 2×2 MATRIX INVERSE is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{n\sum x^2 - \left(\sum x\right)^2} \begin{bmatrix} \sum y \sum x^2 - \sum x \sum xy \\ n\sum xy - \sum x \sum y \end{bmatrix}, \tag{27}$$

so

$$a = \frac{\sum y \sum x^2 - \sum x \sum xy}{n\sum x^2 - \left(\sum x\right)^2} \tag{28}$$

$$= \frac{\bar{y}\sum x^2 - \bar{x}\sum xy}{\sum x^2 - n\bar{x}^2} \tag{29}$$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} \tag{30}$$

$$= \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2} \tag{31}$$

(Kenney and Keeping 1962). These can be rewritten in a simpler form by defining the sums of squares

$$\mathrm{ss}_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \left(\sum x^2 - n\bar{x}^2\right) \tag{32}$$

$$\mathrm{ss}_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \left(\sum y^2 - n\bar{y}^2\right) \tag{33}$$

$$\mathrm{ss}_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \left(\sum xy - n\bar{x}\bar{y}\right), \tag{34}$$

which are also written as

$$\sigma_x^2 = \mathrm{ss}_{xx} \tag{35}$$

$$\sigma_y^2 = \mathrm{ss}_{yy} \tag{36}$$

$$\mathrm{cov}(x,y) = \mathrm{ss}_{xy}. \tag{37}$$

Here, $\mathrm{cov}(x,y)$ is the COVARIANCE and σ_x^2 and σ_y^2 are variances. Note that the quantities $\sum xy$ and $\sum x^2$ can also be interpreted as the DOT PRODUCTS

$$\sum x^2 = \mathbf{x} \cdot \mathbf{x} \tag{38}$$

$$\sum xy = \mathbf{x} \cdot \mathbf{y}. \tag{39}$$

In terms of the sums of squares, the REGRESSION COEFFICIENT b is given by

$$b = \frac{\mathrm{cov}(x,y)}{{\sigma_x}^2} = \frac{\mathrm{ss}_{xy}}{\mathrm{ss}_{xx}}, \tag{40}$$

and a is given in terms of b using (24) as

$$a = \bar{y} - b\bar{x}. \tag{41}$$

The overall quality of the fit is then parameterized in terms of a quantity known as the CORRELATION COEFFICIENT, defined by

$$r^2 = \frac{{\mathrm{ss}_{xy}}^2}{\mathrm{ss}_{xx}\mathrm{ss}_{yy}}, \tag{42}$$

which gives the proportion of ss_{yy} which is accounted for by the regression.

The STANDARD ERRORS for a and b are

$$\mathrm{SE}(a) = s\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\mathrm{ss}_{xx}}} \tag{43}$$

$$\mathrm{SE}(b) = \frac{s}{\sqrt{\mathrm{ss}_{xx}}}. \tag{44}$$

Let $\hat{y}_i$ be the vertical coordinate of the best-fit line with x-coordinate x_i, so

$$\hat{y}_i \equiv a + bx_i, \tag{45}$$

then the error between the actual vertical point y_i and the fitted point is given by

$$e_i \equiv y_i - \hat{y}_i. \tag{46}$$

Now define s^2 as an estimator for the variance in e_i,

$$s^2 = \sum_{i=1}^{n} \frac{{e_i}^2}{n-2}. \tag{47}$$

Then s can be given by

$$s = \sqrt{\frac{\mathrm{ss}_{yy} - b\,\mathrm{ss}_{xy}}{n-2}} = \sqrt{\frac{\mathrm{ss}_{yy} - \frac{s_{xy}^2}{s_{xx}}}{n-2}} \tag{48}$$

(Acton 1966, pp. 32–35; Gonick and Smith 1993, pp. 202–204).

Generalizing from a straight line (i.e., first degree polynomial) to a kth degree POLYNOMIAL

$$y = a_0 + a_1 x + \ldots + a_k x^k, \tag{49}$$

the residual is given by

$$R^2 \equiv \sum_{i=1}^{n} [y_i - (a_0 + a_1 x_i + \ldots + a_k {x_i}^k)]^2. \tag{50}$$

The PARTIAL DERIVATIVES (again dropping superscripts) are

$$\frac{\partial(R^2)}{\partial a_0} = -2\sum [y - (a_0 + a_1 x + \ldots + a_k x^k)] = 0 \tag{51}$$

$$\frac{\partial(R^2)}{\partial a_1} = -2\sum [y - (a_0 + a_1 x + \ldots + a_k x^k)]x = 0 \tag{52}$$

$$\frac{\partial(R^2)}{\partial a_k} = -2\sum [y - (a_0 + a_1 x + \ldots + a_k x^k)]x^k = 0. \tag{53}$$

These lead to the equations

$$a_0 n + a_1 \sum x + \ldots + a_k \sum x^k = \sum y \tag{54}$$

$$a_0 \sum x + a_1 \sum x^2 + \ldots + a_k \sum x^{k+1} = \sum xy \tag{55}$$

$$a_0 \sum x^k + a_1 \sum x^{k+1} + \ldots + a_k \sum x^{2k} = \sum x^k y \tag{56}$$

or, in MATRIX form

$$\begin{bmatrix} n & \sum x & \cdots & \sum x^k \\ \sum x & \sum x^2 & \cdots & \sum x^{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum x^k & \sum x^{k+1} & \cdots & \sum x^{2k} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \vdots \\ \sum x^k y \end{bmatrix}. \tag{57}$$

This is a VANDERMONDE MATRIX. We can also obtain the MATRIX for a least squares fit by writing

$$\begin{bmatrix} 1 & x_1 & \cdots & {x_1}^k \\ 1 & x_2 & \cdots & {x_2}^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & {x_n}^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{58}$$

Premultiplying both sides by the TRANSPOSE of the first MATRIX then gives

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ {x_1}^k & {x_2}^k & \cdots & {x_n}^k \end{bmatrix} \begin{bmatrix} 1 & x_1 & \cdots & {x_1}^k \\ 1 & x_2 & \cdots & {x_2}^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & {x_n}^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ {x_1}^k & {x_2}^k & \cdots & {x_n}^k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \tag{59}$$

so

$$\begin{bmatrix} n & \sum x & \cdots & \sum x^n \\ \sum x & \sum x^2 & \cdots & \sum x^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum x^n & \sum x^{n+1} & \cdots & \sum x^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \vdots \\ \sum x^k y \end{bmatrix}. \tag{60}$$

As before, given m points (x_i, y_i) and fitting with POLYNOMIAL COEFFICIENTS $a_0, \ldots, a_n$ gives

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^n \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & {x_m}^2 & \cdots & {x_m}^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_0 \\ \vdots \\ a_n \end{bmatrix}, \tag{61}$$

In MATRIX notation, the equation for a polynomial fit is given by

$$\mathbf{y} = \mathsf{X}\mathbf{a}. \tag{62}$$

This can be solved by premultiplying by the MATRIX TRANSPOSE X^{T},

$$\mathsf{X}^{\mathrm{T}}\mathbf{y} = \mathsf{X}^{\mathrm{T}}\mathsf{X}\mathbf{a}. \tag{63}$$

This MATRIX EQUATION can be solved numerically, or can be inverted directly if it is well formed, to yield the solution vector

$$\mathbf{a} = (\mathsf{X}^{\mathrm{T}}\mathsf{X})^{-1}\mathsf{X}^{\mathrm{T}}\mathbf{y}. \tag{64}$$

Setting $m = 1$ in the above equations reproduces the linear solution.

see also CORRELATION COEFFICIENT, INTERPOLATION, LEAST SQUARES FITTING—EXPONENTIAL, LEAST SQUARES FITTING—LOGARITHMIC, LEAST SQUARES FITTING—POWER LAW, MOORE-PENROSE GENERALIZED MATRIX INVERSE, NONLINEAR LEAST SQUARES FITTING, REGRESSION COEFFICIENT, SPLINE

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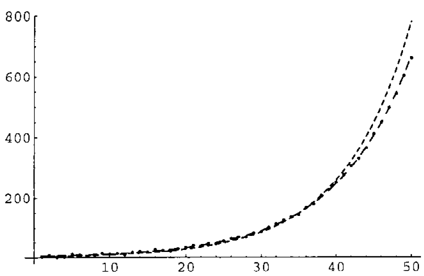
Least Squares Fitting—Exponential



To fit a functional form

$$y = Ae^{Bx}, \tag{1}$$

take the LOGARITHM of both sides

$$\ln y = \ln A + B \ln x. \tag{2}$$

The best-fit values are then

$$a = \frac{\sum \ln y \sum x^2 - \sum x \sum x \ln y}{n \sum x^2 - \left(\sum x\right)^2} \tag{3}$$

$$b = \frac{n \sum x \ln y - \sum x \sum \ln y}{n \sum x^2 - \left(\sum x\right)^2}, \tag{4}$$

where $B \equiv b$ and $A \equiv \exp(a)$.

This fit gives greater weights to small y values so, in order to weight the points equally, it is often better to minimize the function

$$\sum y(\ln y - a - bx)^2. \tag{5}$$

Applying LEAST SQUARES FITTING gives

$$a \sum y + b \sum xy = \sum y \ln y \tag{6}$$

$$a \sum xy + b \sum x^2 y = \sum xy \ln y \tag{7}$$

$$\begin{bmatrix} \sum y & \sum xy \\ \sum xy & \sum x^2 y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y \ln y \\ \sum xy \ln y \end{bmatrix}. \tag{8}$$

Solving for a and b,

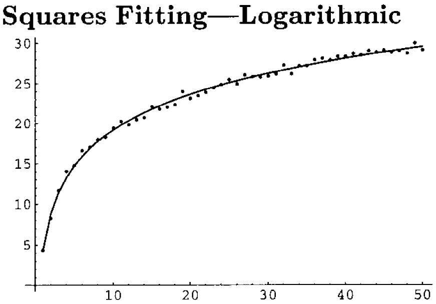
$$a = \frac{\sum(x^2 y) \sum(y \ln y) - \sum(xy) \sum(xy \ln y)}{\sum y \sum(x^2 y) - \left(\sum xy\right)^2} \tag{9}$$

$$b = \frac{\sum y \sum(xy \ln y) - \sum(xy) \sum(y \ln y)}{\sum y \sum(x^2 y) - \left(\sum xy\right)^2}. \tag{10}$$

In the plot above, the short-dashed curve is the fit computed from (3) and (4) and the long-dashed curve is the fit computed from (9) and (10).

see also LEAST SQUARES FITTING, LEAST SQUARES FITTING—LOGARITHMIC, LEAST SQUARES FITTING—POWER LAW

Least Squares Fitting—Logarithmic



Given a function of the form

$$y = a + b \ln x, \tag{1}$$

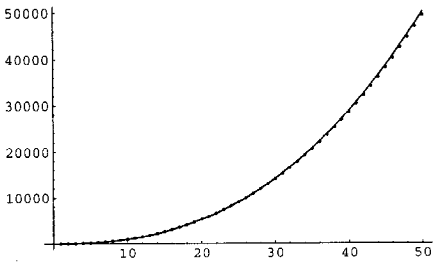
the COEFFICIENTS can be found from LEAST SQUARES FITTING as

$$b = \frac{n \sum(y \ln x) - \sum y \sum(\ln x)}{n \sum[(\ln x)^2] - \left[\sum(\ln x)\right)^2} \tag{2}$$

$$a = \frac{\sum y - b \sum(\ln x)}{n}. \tag{3}$$

see also LEAST SQUARES FITTING, LEAST SQUARES FITTING—EXPONENTIAL, LEAST SQUARES FITTING—POWER LAW

Least Squares Fitting—Power Law



Given a function of the form

$$y = Ax^B, \tag{1}$$

LEAST SQUARES FITTING gives the COEFFICIENTS as

$$b = \frac{n\sum(\ln x \ln y) - \sum(\ln x)\sum(\ln y)}{n\sum[(\ln x)^2] - \left(\sum \ln x\right)^2} \tag{2}$$

$$a = \frac{\sum(\ln y) - b\sum(\ln x)}{n}, \tag{3}$$

where $B \equiv b$ and $A \equiv \exp(a)$.

see also LEAST SQUARES FITTING, LEAST SQUARES FITTING—EXPONENTIAL, LEAST SQUARES FITTING—LOGARITHMIC

Least Upper Bound

see SUPREMUM

Lebesgue Constants (Fourier Series)

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Assume a function f is integrable over the interval $[-\pi, \pi]$ and $S_n(f, x)$ is the nth partial sum of the FOURIER SERIES of f, so that

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt \tag{1}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt \tag{2}$$

and

$$S_n(f,x) = \tfrac{1}{2}a_0 + \left\{\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right\}. \tag{3}$$

If

$$|f(x)| \leq 1 \tag{4}$$

for all x, then

$$S_n(f,x) \leq \frac{1}{\pi}\int_0^{\pi} \frac{|\sin[\frac{1}{2}(2n+1)\theta]|}{\sin(\frac{1}{2}\theta)}\,d\theta = L_n, \tag{5}$$

and L_n is the smallest possible constant for which this holds for all continuous f. The first few values of L_n are

$$L_0 = 1 \tag{6}$$

$$L_1 = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} = 1.435991124\ldots \tag{7}$$

$$L_2 = 1.642188435\ldots \tag{8}$$

$$L_3 = 1.778322862. \tag{9}$$

Some FORMULAS for L_n include

$$L_n = \frac{1}{2n+1} + \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k}\tan\left(\frac{\pi k}{2n+1}\right) = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{(2n+1)k}\frac{1}{4k^2-1}\frac{1}{2j-1} \tag{10}$$

(Zygmund 1959) and integral FORMULAS include

$$L_n = 4\int_0^{\infty}\frac{\tanh[(2n+1)x]}{\tanh x}\frac{dx}{\pi^2+4x^2} = \frac{4}{\pi^2}\int_0^{\infty}\frac{\sinh[(2n+1)x]}{\sinh x}\ln\{\coth[\tfrac{1}{2}(2n+1)x]\}\,dx \tag{11}$$

(Hardy 1942). For large n,

$$\frac{4}{\pi^2}\ln n < L_n < 3 + \frac{4}{\pi^2}\ln n. \tag{12}$$

This result can be generalized for an r-differentiable function satisfying

$$\left|\frac{d^r f}{dx^r}\right| \leq 1 \tag{13}$$

for all x. In this case,

$$|f(x) - S_n(f,x)| \leq L_{n,r} = \frac{4}{\pi^2}\frac{\ln n}{n^r} + \mathcal{O}\left(\frac{1}{n^r}\right), \tag{14}$$

where

$$L_{n,r} = \begin{cases} \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{\infty}\frac{\sin(kx)}{k^r}\right|dx & \text{for } r \geq 1 \text{ odd} \\ \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\sum_{k=n+1}^{\infty}\frac{\cos(kx)}{k^r}\right|dx & \text{for } r \geq 1 \text{ even} \end{cases} \tag{15}$$

(Kolmogorov 1935, Zygmund 1959).

Watson (1930) showed that

$$\lim_{n\to\infty}\left[L_n - \frac{4}{\pi^2}\ln(2n+1)\right] = c, \tag{16}$$

where

$$c = \frac{8}{\pi^2}\left(\sum_{k=1}^{\infty}\frac{\ln k}{4k^2-1}\right) - \frac{4}{\pi^2}\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad (17)$$

$$= \frac{8}{\pi^2}\left[\sum_{j=0}^{\infty}\frac{\lambda(2j+2)-1}{2j+1}\right] + \frac{4}{\pi^2}(2\ln 2+\gamma) \quad (18)$$

$$= 0.9894312738\ldots, \quad (19)$$

where $\Gamma(z)$ is the GAMMA FUNCTION, $\lambda(z)$ is the DIRICHLET LAMBDA FUNCTION, and γ is the EULER-MASCHERONI CONSTANT.

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Lebesgue Constants (Lagrange Interpolation)

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Define the nth Lebesgue constant for the LAGRANGE INTERPOLATING POLYNOMIAL by

$$\Lambda_n(X) \equiv \max_{-1\leq x\leq 1}\sum_{k=1}^{n}\left|\prod_{j\neq k}\frac{x-x_j}{x_k-x_j}\right|. \quad (1)$$

It is true that

$$\Lambda_n > \frac{4}{\pi^2}\ln n - 1. \quad (2)$$

The efficiency of a Lagrange interpolation is related to the rate at which Λ_n increases. Erdős (1961) proved that there exists a POSITIVE constant such that

$$\Lambda_n > \frac{2}{\pi}\ln n - C \quad (3)$$

for all n. Erdős (1961) further showed that

$$\Lambda_n < \frac{2}{\pi}\ln n + 4, \quad (4)$$

so (3) cannot be improved upon.

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Lebesgue Covering Dimension

An important DIMENSION and one of the first dimensions investigated. It is defined in terms of covering sets, and is therefore also called the COVERING DIMENSION. Another name for the Lebesgue covering dimension is the TOPOLOGICAL DIMENSION.

A SPACE has Lebesgue covering dimension m if for every open COVER of that space, there is an open COVER that refines it such that the refinement has order at most $m+1$. Consider how many elements of the cover contain a given point in a base space. If this has a maximum over all the points in the base space, then this maximum is called the order of the cover. If a SPACE does not have Lebesgue covering dimension m for any m, it is said to be infinite dimensional.

Results of this definition are:

1. Two homeomorphic spaces have the same dimension,
2. $\mathbb{R}^n$ has dimension n,
3. A TOPOLOGICAL SPACE can be embedded as a closed subspace of a EUCLIDEAN SPACE IFF it is locally compact, Hausdorff, second countable, and is finite dimensional (in the sense of the LEBESGUE DIMENSION), and
4. Every compact metrizable m-dimensional TOPOLOGICAL SPACE can be embedded in $\mathbb{R}^{2m+1}$.

see also LEBESGUE MINIMAL PROBLEM

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Lebesgue Dimension

see LEBESGUE COVERING DIMENSION

Lebesgue Integrable

A real-valued function f defined on the reals $\mathbb{R}$ is called Lebesgue integrable if there exists a SEQUENCE of STEP FUNCTIONS $\{f_n\}$ such that the following two conditions are satisfied:

1. $\sum_{n=1}^{\infty}\int|f_n| < \infty$,
2. $f(x) = \sum_{n=1}^{\infty}$ for every $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty}\int|f_n| < \infty$.

Here, the above integral denotes the ordinary RIEMANN INTEGRAL. Note that this definition avoids explicit use of the LEBESGUE MEASURE.

see also INTEGRAL, LEBESGUE INTEGRAL, RIEMANN INTEGRAL, STEP FUNCTION

Lebesgue Integral

The LEBESGUE INTEGRAL is defined in terms of upper and lower bounds using the LEBESGUE MEASURE of a SET. It uses a LEBESGUE SUM $S_n = \eta_i \mu(E_i)$ where η_i is the value of the function in subinterval i, and $\mu(E_i)$ is the LEBESGUE MEASURE of the SET E_i of points for which values are approximately η_i. This type of integral covers a wider class of functions than does the RIEMANN INTEGRAL.

see also A-INTEGRABLE, COMPLETE FUNCTIONS, INTEGRAL

References

Kestelman, H. "Lebesgue Integral of a Non-Negative Function" and "Lebesgue Integrals of Functions Which Are Sometimes Negative." Chs. 5–6 in *Modern Theories of Integration, 2nd rev. ed.* New York: Dover, pp. 113–160, 1960.

Lebesgue Measurability Problem

A problem related to the CONTINUUM HYPOTHESIS which was solved by Solovay (1970) using the INACCESSIBLE CARDINALS AXIOM. It has been proven by Shelah and Woodin (1990) that use of this AXIOM is essential to the proof.

see also CONTINUUM HYPOTHESIS, INACCESSIBLE CARDINALS AXIOM, LEBESGUE MEASURE

References

Shelah, S. and Woodin, H. "Large Cardinals Imply that Every Reasonable Definable Set of Reals is Lebesgue Measurable." *Israel J. Math.* **70**, 381–394, 1990.

Solovay, R. M. "A Model of Set-Theory in which Every Set of Reals is Lebesgue Measurable." *Ann. Math.* **92**, 1–56, 1970.

Lebesgue Measure

An extension of the classical notions of length and AREA to more complicated sets. Given an open set $S \equiv \sum_k (a_k, b_k)$ containing DISJOINT intervals,

$$\mu_L(S) \equiv \sum_k (b_k - a_k).$$

Given a CLOSED SET $S' \equiv [a,b] - \sum_k (a_k, b_k)$,

$$\mu_L(S') \equiv (b-a) - \sum_k (b_k - a_k).$$

A LINE SEGMENT has Lebesgue measure 1; the CANTOR SET has Lebesgue measure 0. The MINKOWSKI MEASURE of a bounded, CLOSED SET is the same as its Lebesgue measure (Ko 1995).

see also CANTOR SET, MEASURE, RIESZ-FISCHER THEOREM

References

Kestelman, H. "Lebesgue Measure." Ch. 3 in *Modern Theories of Integration, 2nd rev. ed.* New York: Dover, pp. 67–91, 1960.

Ko, K.-I. "A Polynomial-Time Computable Curve whose Interior has a Nonrecursive Measure." *Theoret. Comput. Sci.* **145**, 241–270, 1995.

Lebesgue Minimal Problem

Find the plane LAMINA of least AREA A which is capable of covering any plane figure of unit GENERAL DIAMETER. A UNIT CIRCLE is too small, but a HEXAGON circumscribed on the UNIT CIRCLE is too large. More specifically, the AREA is bounded by

$$0.8257\ldots = \tfrac{1}{8}\pi + \tfrac{1}{4}\sqrt{3} < A < \tfrac{2}{3}(3 - \sqrt{3}) = 0.8454\ldots$$

(Pal 1920).

see also AREA, BORSUK'S CONJECTURE, DIAMETER (GENERAL), KAKEYA NEEDLE PROBLEM

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 99, 1987.

Coxeter, H. S. M. "Lebesgue's Minimal Problem." *Eureka* **21**, 13, 1958.

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Yaglom, I. M. and Boltyanskii, V. G. *Convex Figures.* New York: Holt, Rinehart, & Winston, pp. 18 and 100, 1961.

Lebesgue-Radon Integral

see LEBESGUE-STIELTJES INTEGRAL

Lebesgue Singular Integrals

$$\mathcal{U}_n(f) = \int_a^b f(x) K_n(x)\,dx,$$

where $\{K_n(x)\}$ is a SEQUENCE of CONTINUOUS FUNCTIONS.

Lebesgue-Stieltjes Integral

Let $\alpha(x)$ be a monotone increasing function and define an INTERVAL $I = (x_1, x_2)$. Then define the NONNEGATIVE function

$$U(I) = \alpha(x_2 + 0) - \alpha(x_1 + 0).$$

The LEBESGUE INTEGRAL with respect to a MEASURE constructed using $U(I)$ is called the Lebesgue-Stieltjes integral, or sometimes the LEBESGUE-RADON INTEGRAL.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 326, 1980.

Lebesgue Sum

$$S_n = \eta_i \mu(E_i),$$

where $\mu(E_i)$ is the MEASURE of the SET E_i of points on the x-axis for which $f(x) \approx \eta_i$.

Leech Lattice

A 24-D Euclidean lattice. An AUTOMORPHISM of the Leech lattice modulo a center of two leads to the CONWAY GROUP Co_1. Stabilization of the 1- and 2-D sublattices leads to the CONWAY GROUPS Co_2 and Co_3, the HIGMAN-SIMS GROUP HS and the MCLAUGHLIN GROUP McL.

The Leech lattice appears to be the densest HYPERSPHERE PACKING in 24-D, and results in each HYPERSPHERE touching 195,560 others.

see also BARNES-WALL LATTICE, CONWAY GROUPS, COXETER-TODD LATTICE, HIGMAN-SIMS GROUP, HYPERSPHERE, HYPERSPHERE PACKING, KISSING NUMBER, MCLAUGHLIN GROUP

References

Conway, J. H. and Sloane, N. J. A. "The 24-Dimensional Leech Lattice Λ_{24}," "A Characterization of the Leech Lattice," "The Covering Radius of the Leech Lattice," "Twenty-Three Constructions for the Leech Lattice," "The Cellular of the Leech Lattice," "Lorentzian Forms for the Leech Lattice." §4.11, Ch. 12, and Chs. 23–26 in *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, pp. 131–135, 331–336, and 478–526, 1993.

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Wilson, R. A. "Vector Stabilizers and Subgroups of Leech Lattice Groups." *J. Algebra* **127**, 387–408, 1989.

Lefshetz Fixed Point Formula

see LEFSHETZ TRACE FORMULA

Lefshetz's Theorem

Each DOUBLE POINT assigned to an irreducible curve whose GENUS is NONNEGATIVE imposes exactly one condition.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 104, 1959.

Lefshetz Trace Formula

A formula which counts the number of FIXED POINTS for a topological transformation.

Leg

The leg of a TRIANGLE is one of its sides.

see also HYPOTENUSE, TRIANGLE

Legendre Addition Theorem

see SPHERICAL HARMONIC ADDITION THEOREM

Legendre's Chi-Function

The function defined by

$$\chi_\nu(z) = \sum_{k=0}^\infty \frac{z^{2k+1}}{(2k+1)^\nu}$$

for integral $\nu = 2, 3, \ldots$. It is related to the POLYLOGARITHM by

$$\begin{aligned}\chi_\nu(z) &= \tfrac{1}{2}[\mathrm{Li}_\nu(z) - \mathrm{Li}_\nu(-z)] \\ &= \mathrm{Li}_\nu(z) - 2^{-\nu}\,\mathrm{Li}_\nu(z^2).\end{aligned}$$

see also POLYLOGARITHM

References

Cvijović, D. and Klinowski, J. "Closed-Form Summation of Some Trigonometric Series." *Math. Comput.* **64**, 205–210, 1995.

Lewin, L. *Polylogarithms and Associated Functions.* Amsterdam, Netherlands: North-Holland, pp. 282–283, 1981.

Legendre's Constant

The number 1.08366 in Legendre's guess at the PRIME NUMBER THEOREM

$$\pi(n) \sim \frac{n}{\ln n - 1.08366}.$$

This expression is correct to leading term only.

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 147, 1983.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 28–29, 1991.

Legendre Differential Equation

The second-order ORDINARY DIFFERENTIAL EQUATION

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0, \qquad (1)$$

which can be rewritten

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + l(l+1)y = 0. \qquad (2)$$

The above form is a special case of the associated Legendre differential equation with $m = 0$. The Legendre differential equation has REGULAR SINGULAR POINTS at -1, 1, and ∞. It can be solved using a series expansion,

$$y = \sum_{n=0}^\infty a_n x^n \qquad (3)$$

$$y' = \sum_{n=0}^\infty n a_n x^{n-1} \qquad (4)$$

$$y'' = \sum_{n=0}^\infty n(n-1)a_n x^{n-2}. \qquad (5)$$

Plugging in,

$$(1-x^2)\sum_{n=0}^\infty n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^\infty n a_n x^{n-1} + l(l+1)\sum_{n=0}^\infty a_n x^n = 0 \qquad (6)$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$
$$-2x\sum_{n=0}^{\infty} na_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0 \quad (7)$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$
$$-2\sum_{n=0}^{\infty} na_n x^n + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0 \quad (8)$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n$$
$$-2\sum_{n=0}^{\infty} na_n x^n + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0 \quad (9)$$

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2}$$
$$+[-n(n-1) - 2n + l(l+1)]a_n\} = 0, \quad (10)$$

so each term must vanish and

$$(n+1)(n+2)a_{n+2} - n(n+1) + l(l+1)]a_n = 0 \quad (11)$$

$$\begin{aligned} a_{n+2} &= \frac{n(n+1) - l(l+1)}{(n+1)(n+2)} a_n \\ &= -\frac{[l+(n+1)](l-n)}{(n+1)(n+2)} a_n. \end{aligned} \quad (12)$$

Therefore,

$$a_2 = -\frac{l(l+1)}{1\cdot 2} a_0 \quad (13)$$

$$\begin{aligned} a_4 &= -\frac{(l-2)(l+3)}{3\cdot 4} a_2 \\ &= (-1)^2 \frac{[(l-2)l][(l+1)(l+3)]}{1\cdot 2\cdot 3\cdot 4} a_0 \end{aligned} \quad (14)$$

$$\begin{aligned} a_6 &= -\frac{(l-4)(l+5)}{5\cdot 6} a_4 \\ &= (-1)^3 \frac{[(l-4)(l-2)l][(l+1)(l+3)(l+5)]}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} a_0, \end{aligned} \quad (15)$$

so the EVEN solution is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{[(l-2n+2)\cdots(l-2)l][(l+1)(l+3)\cdots(l+2n-1)]}{(2n)!} x^{2n}. \quad (16)$$

Similarly, the ODD solution is

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \times \frac{[(l-2n+1)\cdots(l-3)(l-1)][(l+2)(l+4)\cdots(l+2n)]}{(2n+1)!} x^{2m+1}. \quad (17)$$

If l is an EVEN INTEGER, the series y_1 reduces to a POLYNOMIAL of degree l with only EVEN POWERS of x and the series y_2 diverges. If l is an ODD INTEGER, the series y_2 reduces to a POLYNOMIAL of degree l with only ODD POWERS of x and the series y_1 diverges. The general solution for an INTEGER l is given by the LEGENDRE POLYNOMIALS

$$P_n(x) = c_n \begin{cases} y_1(x) & \text{for } l \text{ even} \\ y_2(x) & \text{for } l \text{ odd,} \end{cases} \quad (18)$$

where c_n is chosen so that $P_n(1) = 1$. If the variable x is replaced by $\cos\theta$, then the Legendre differential equation becomes

$$\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{dx} + l(l+1)y = 0, \quad (19)$$

as is derived for the associated Legendre differential equation with $m = 0$.

The *associated* Legendre differential equation is

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0 \quad (20)$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0. \quad (21)$$

The solutions to this equation are called the associated Legendre polynomials. Writing $x \equiv \cos\theta$, first establish the identities

$$\frac{dy}{dx} = \frac{dy}{d(\cos\theta)} = -\frac{1}{\sin\theta}\frac{dy}{d\theta} \quad (22)$$

$$x\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}, \quad (23)$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dy}{d\theta}\right) \\ &= \frac{1}{\sin\theta}\left(\frac{-\cos\theta}{\sin^2\theta}\right)\frac{dy}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2}, \end{aligned} \quad (24)$$

and

$$1 - x^2 = 1 - \cos^2\theta = \sin^2\theta. \quad (25)$$

Therefore,

$$(1-x^2)\frac{d^2y}{dx^2} = \sin^2\theta\frac{1}{\sin\theta}\left(\frac{-\cos\theta}{\sin^2\theta}\right)\frac{dy}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2}$$
$$= \frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}. \qquad (26)$$

Plugging (22) into (26) and the result back into (21) gives

$$\left(\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}\right)$$
$$+2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0 \quad (27)$$

$$\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0. \qquad (28)$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 332, 1972.

Legendre Duplication Formula

GAMMA FUNCTIONS of argument $2z$ can be expressed in terms of GAMMA FUNCTIONS of smaller arguments. From the definition of the BETA FUNCTION,

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 u^{m-1}(1-u)^{n-1}\,du. \quad (1)$$

Now, let $m = n \equiv z$, then

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = \int_0^1 u^{z-1}(1-u)^{z-1}\,du \qquad (2)$$

and $u \equiv (1+x)/2$, so $du = dx/2$ and

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = \int_0^1 \left(\frac{1+x}{2}\right)^{z-1}\left(1-\frac{1+x}{2}\right)^{z-1}\left(\tfrac{1}{2}\,dx\right)$$
$$= \frac{1}{2}\int_0^1 \left(\frac{1+x}{2}\right)^{z-1}\left(\frac{1-x}{2}\right)^{z-1}dx$$
$$= \frac{1}{2^{1+2(z-1)}}\int_0^1 (1-x^2)^{z-1}\,dx$$
$$= 2^{1-2z}\int_0^1 (1-x^2)^{z-1}\,dx. \qquad (3)$$

Now, use the BETA FUNCTION identity

$$B(m,n) = 2\int_0^1 x^{2z-1}(1-x^2)^{z-1}\,dx \qquad (4)$$

to write the above as

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z}B(\tfrac{1}{2},z) = 2^{1-2z}\frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(z+\frac{1}{2})}. \qquad (5)$$

Solving for $\Gamma(2z)$,

$$\Gamma(2z) = \frac{\Gamma(z)\Gamma(z+\frac{1}{2})2^{2z-1}}{\Gamma(\frac{1}{2})} = \frac{\Gamma(z)\Gamma(z+\frac{1}{2})2^{2z-1}}{\sqrt{\pi}}$$
$$= (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}), \qquad (6)$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

see also GAMMA FUNCTION, GAUSS MULTIPLICATION FORMULA

References

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Legendre's Factorization Method

A PRIME FACTORIZATION ALGORITHM in which a sequence of TRIAL DIVISORS is chosen using a QUADRATIC SIEVE. By using QUADRATIC RESIDUES of N, the QUADRATIC RESIDUES of the factors can also be found.

see also PRIME FACTORIZATION ALGORITHMS, QUADRATIC RESIDUE, QUADRATIC SIEVE FACTORIZATION METHOD, TRIAL DIVISOR

Legendre's Formula

Counts the number of POSITIVE INTEGERS less than or equal to a number x which are not divisible by any of the first a PRIMES,

$$\phi(x,a) = \lfloor x\rfloor - \sum\left\lfloor\frac{x}{p_i}\right\rfloor + \sum\left\lfloor\frac{x}{p_ip_j}\right\rfloor$$
$$-\sum\left\lfloor\frac{x}{p_ip_jp_k}\right\rfloor + \ldots, \qquad (1)$$

where $\lfloor x\rfloor$ is the FLOOR FUNCTION. Taking $a = x$ gives

$$\phi(x,x) = \pi(x) - \pi(\sqrt{x}) + 1 = \lfloor x\rfloor - \sum_{p_i\le\sqrt{x}}\left\lfloor\frac{x}{p_i}\right\rfloor$$
$$+\sum_{p_i<p_j\le\sqrt{x}}\left\lfloor\frac{x}{p_ip_j}\right\rfloor - \sum_{p_i<p_j<p_k\le\sqrt{x}}\left\lfloor\frac{x}{p_ip_jp_k}\right\rfloor + \ldots, \qquad (2)$$

where $\pi(n)$ is the PRIME COUNTING FUNCTION. Legendre's formula holds since one more than the number of PRIMES in a range equals the number of INTEGERS minus the number of composites in the interval.

Legendre's formula satisfies the RECURRENCE RELATION

$$\phi(x,a) = \phi(x,a-1) - \phi\left(\frac{x}{p_a},a-1\right). \qquad (3)$$

Let $m_k \equiv p_1 p_2 \cdots p_k$, then

$$\begin{aligned}\phi(m_k, k) &= \lfloor m_k \rfloor - \sum \left\lfloor \frac{m_k}{p_i} \right\rfloor + \sum \left\lfloor \frac{m_k}{p_i p_j} \right\rfloor - \ldots \\ &= m_k - \sum \frac{m_k}{p_i} + \sum \frac{m_k}{p_i p_j} - \ldots \\ &= m_k \left(1 - \frac{1}{p-1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\ &= \prod_{i=1}^{k} (p_i - 1) = \phi(m_k), \end{aligned} \tag{4}$$

where $\phi(n)$ is the TOTIENT FUNCTION, and

$$\phi(s m_k + t, k) = s\phi(m_k) + \phi(t, k), \tag{5}$$

where $0 \leq t \leq m_k$. If $t > m_k/2$, then

$$\phi(t, k) = \phi(m_k) - \phi(m_k - t - 1, k). \tag{6}$$

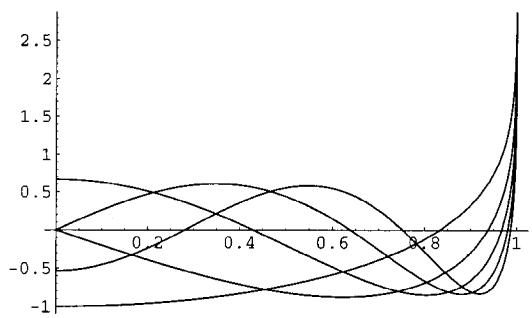
Note that $\phi(n, n)$ is not practical for computing $\pi(n)$ for large arguments. A more efficient modification is MEISSEL'S FORMULA.

see also LEHMER'S FORMULA, MAPES' METHOD, MEISSEL'S FORMULA, PRIME COUNTING FUNCTION

Legendre Function of the First Kind

see LEGENDRE POLYNOMIAL

Legendre Function of the Second Kind



A solution to the LEGENDRE DIFFERENTIAL EQUATION which is singular at the origin. The Legendre functions of the second kind satisfy the same RECURRENCE RELATION as the LEGENDRE FUNCTIONS OF THE FIRST KIND. The first few are

$$\begin{aligned} Q_0 &= \tfrac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \\ Q_1 &= \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1 \\ Q_2 &= \frac{3x^2 - 1}{4} \ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2} \\ Q_3 &= \frac{5x^3 - 3x}{4} \ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}. \end{aligned}$$

The associated Legendre functions of the second kind have DERIVATIVE about 0 of

$$\left[\frac{dQ_\nu^\mu(x)}{dx}\right]_{x=0} = \frac{2^\mu \sqrt{\pi}\, \cos[\frac{1}{2}\pi(\nu+\mu)]\Gamma(\frac{1}{2}\nu + \frac{1}{2}\mu + 1)}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2})}$$

(Abramowitz and Stegun 1972, p. 334). The logarithmic derivative is

$$\left[\frac{d \ln Q_\lambda^\mu(z)}{dz}\right]_{z=0} = 2\exp\{\tfrac{1}{2}\pi i \operatorname{sgn}(\Im[z])\} \frac{[\frac{1}{2}(\lambda+\mu)]![\frac{1}{2}(\lambda-\mu)]!}{[\frac{1}{2}(\lambda+\mu-1)]![\frac{1}{2}(\lambda-\mu-1)]!}$$

(Binney and Tremaine 1987, p. 654).

References

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Spanier, J. and Oldham, K. B. "The Legendre Functions $P_\nu(x)$ and $Q_\nu(x)$." Ch. 59 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 581–597, 1987.

Legendre-Gauss Quadrature

Also called "the" GAUSSIAN QUADRATURE or LEGENDRE QUADRATURE. A GAUSSIAN QUADRATURE over the interval $[-1, 1]$ with WEIGHTING FUNCTION $W(x) = 1$. The ABSCISSAS for quadrature order n are given by the roots of the LEGENDRE POLYNOMIALS $P_n(x)$, which occur symmetrically about 0. The weights are

$$w_i = -\frac{A_{n+1}\gamma_n}{A_n P_n'(x_i) P_{n+1}(x_i)} = \frac{A_n}{A_{n-1}} \frac{\gamma_{n-1}}{P_{n-1}(x_i) P_n'(x_i)}, \tag{1}$$

where A_n is the COEFFICIENT of x^n in $P_n(x)$. For LEGENDRE POLYNOMIALS,

$$A_n = \frac{(2n)!}{2^n (n!)^2}, \tag{2}$$

so

$$\begin{aligned} \frac{A_{n+1}}{A_n} &= \frac{[2(n+1)]!}{2^{n+1}[(n+1)!]^2} \frac{2^n (n!)^2}{(2n)!} \\ &= \frac{(2n+1)(2n+2)}{2(n+1)^2} = \frac{2n+1}{n+1}. \end{aligned} \tag{3}$$

Additionally,

$$\gamma_n = \frac{2}{2n+1}, \tag{4}$$

so

$$w_i = -\frac{2}{(n+1)P_{n+1}(x_i)P_n'(x_i)} = \frac{2}{nP_{n-1}(x_i)P_n'(x_i)}. \tag{5}$$

Using the RECURRENCE RELATION

$$\begin{aligned}(1-x^2)P_n'(x) &= nxP_n(x) + nP_{n-1}(x) &(6)\\ &= (n+1)xP_n(x) - (n+1)P_{n+1}(x) &(7)\end{aligned}$$

gives

$$w_i = \frac{2}{(1-x_i{}^2)[P_n'(x_i)]^2} = \frac{2(1-x_i{}^2)}{(n+1)^2[P_{n+1}(x_i)]^2}. \tag{8}$$

The error term is

$$E = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi). \tag{9}$$

Beyer (1987) gives a table of ABSCISSAS and weights up to $n = 16$, and Chandrasekhar (1960) up to $n = 8$ for n EVEN.

n	x_i	w_i
2	± 0.57735	1.000000
3	0	0.888889
	± 0.774597	0.555556
4	± 0.339981	0.652145
	± 0.861136	0.347855
5	0	0.568889
	± 0.538469	0.478629
	± 0.90618	0.236927

The ABSCISSAS and weights can be computed analytically for small n.

n	x_i	w_i
2	$\pm\frac{1}{3}\sqrt{3}$	1
3	0	$\frac{8}{9}$
	$\pm\frac{1}{5}\sqrt{15}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}$	
	$\pm\sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$	

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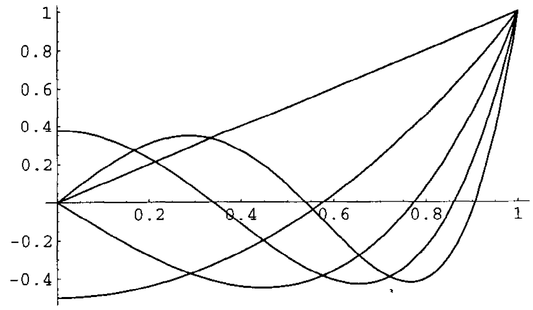
Legendre-Jacobi Elliptic Integral

Any of the three standard forms in which an ELLIPTIC INTEGRAL can be expressed.

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC INTEGRAL OF THE SECOND KIND, ELLIPTIC INTEGRAL OF THE THIRD KIND

Legendre Polynomial



The LEGENDRE FUNCTIONS OF THE FIRST KIND are solutions to the LEGENDRE DIFFERENTIAL EQUATION. If l is an INTEGER, they are POLYNOMIALS. They are a special case of the ULTRASPHERICAL FUNCTIONS with $\alpha = 1/2$. The Legendre polynomials $P_n(x)$ are illustrated above for $x \in [0,1]$ and $n = 1, 2, \ldots, 5$.

The Rodrigues FORMULA provides the GENERATING FUNCTION

$$P_l(x) = \frac{1}{2^l l!}\frac{dl}{dx^l}(x^2-1)^l, \tag{1}$$

which yields upon expansion

$$P_l(x) = \frac{1}{2^l}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k(2l-2k)!}{k!(l-k)!(l-2k)!}x^{l-2k}, \tag{2}$$

where $\lfloor r\rfloor$ is the FLOOR FUNCTION. The GENERATING FUNCTION is

$$g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{3}$$

Take $\partial g/\partial t$,

$$-\tfrac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}. \tag{4}$$

Multiply (4) by $2t$,

$$-t(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} 2nP_n(x)t^n \tag{5}$$

and add (3) and (5),

$$(1-2xt+t^2)^{-3/2}[(2xt-2t^2)+(1-2xt+t^2)] = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n \tag{6}$$

$$(1-2xt+t^2)^{-3/2}(1-t^2) = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n. \tag{7}$$

This expansion is useful in some physical problems, including expanding the Heyney-Greenstein phase function and computing the charge distribution on a SPHERE. They satisfy the RECURRENCE RELATION

$$(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0. \qquad (8)$$

The Legendre polynomials are orthogonal over $(-1,1)$ with WEIGHTING FUNCTION 1 and satisfy

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}, \qquad (9)$$

where δ_{mn} is the KRONECKER DELTA.

A COMPLEX GENERATING FUNCTION is

$$P_l(x) = \frac{1}{2\pi i}\int (1-2zx+z^2)^{-1/2} z^{-l-1}\,dz, \qquad (10)$$

and the Schläfli integral is

$$P_l(x) = \frac{(-1)^l}{2^l}\frac{1}{2\pi i}\int \frac{(1-z^2)^l}{(z-x)^{l+1}}\,dz. \qquad (11)$$

Additional integrals (Byerly 1959, p. 172) include

$$\int_0^1 P_m(x)\,dx = \begin{cases} 0 & m \text{ even} \neq 0 \\ (-1)^{(m-1)/2}\frac{m!!}{m(m+1)(m-1)!!} & m \text{ odd} \end{cases} \qquad (12)$$

$$\int_0^1 P_m(x)P_n(x)\,dx = \begin{cases} 0 & m, n \text{ both even or odd } m \neq n \\ (-1)^{(m+n+1)/2}\frac{m!n!}{2^{m+n+1}(m-n)(m+n+1)(\frac{1}{2}m)!\{[\frac{1}{2}(n-1)]!\}^2} & m \text{ even, } n \text{ odd} \\ \frac{1}{2n+1}, & m = n. \end{cases} \qquad (13)$$

An additional identity is

$$1 - [P_n(x)]^2 = \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu{}^2}\left[\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right]^2 \qquad (14)$$

(Szegő 1975, p. 348).

The first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2-1) \\ P_3(x) &= \tfrac{1}{2}(5x^3-3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3) \\ P_5(x) &= \tfrac{1}{8}(63x^5-70x^3+15x) \\ P_6(x) &= \tfrac{1}{16}(231x^6-315x^4+105x^2-5). \end{aligned}$$

The first few POWERS in terms of Legendre polynomials are

$$\begin{aligned} x &= P_1 \\ x^2 &= \tfrac{1}{3}(P_0+2P_2) \\ x^3 &= \tfrac{1}{5}(3P_1+2P_3) \\ x^4 &= \tfrac{1}{35}(7P_0+20P_2+8P_4) \\ x^5 &= \tfrac{1}{63}(27P_1+28P_3+8P_5) \\ x^6 &= \tfrac{1}{231}(33P_0+110P_2+72P_4+16P_5). \end{aligned}$$

For Legendre polynomials and POWERS up to exponent 12, see Abramowitz and Stegun (1972, p. 798).

The Legendre POLYNOMIALS can also be generated using GRAM-SCHMIDT ORTHONORMALIZATION in the OPEN INTERVAL $(-1,1)$ with the WEIGHTING FUNCTION 1.

$$P_0(x) = 1 \qquad (15)$$

$$\begin{aligned} P_1(x) &= \left[x - \frac{\int_{-1}^1 x\,dx}{\int_{-1}^1 dx}\right]\cdot 1 \\ &= x - \frac{\frac{1}{2}[x^2]_{-1}^1}{[x]_{-1}^1} = x - \frac{\frac{1}{2}(1-1)}{1-(-1)} = x \end{aligned} \qquad (16)$$

$$\begin{aligned} P_2(x) &= \left[x - \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 x^2\,dx}\right] - \left[\frac{\int_{-1}^1 x^2\,dx}{\int_{-1}^1 dx}\right]\cdot 1 \\ &= \left[x - \frac{\frac{1}{4}[x^4]_{-1}^1}{\frac{1}{3}[x^3]_{-1}^1}\right] x - \frac{\frac{1}{3}[x^3]_{-1}^1}{[x]_{-1}^1} = x^2 - \tfrac{1}{3} \end{aligned} \qquad (17)$$

$$\begin{aligned} P_3(x) &= \left[x - \frac{\int_{-1}^1 x(x^2-\frac{1}{3})^2\,dx}{\int_{-1}^1 (x^2-\frac{1}{3})^2\,dx}\right](x^2-\tfrac{1}{3}) \\ &\quad - \left[\frac{\int_{-1}^1 (x^2-\frac{1}{3})^2\,dx}{\int_{-1}^1 x^2\,dx}\right] x \\ &= x\left[x^2 - \tfrac{1}{3} - \frac{(\frac{1}{5}-\frac{2}{9}+\frac{1}{9})x}{\frac{1}{3}}\right] \\ &= x^3 - \tfrac{1}{3}x - 3(\tfrac{1}{5}-\tfrac{1}{9}) \\ &= x^3 - x\left(\tfrac{1}{3}+\tfrac{3}{5}-\tfrac{1}{3}\right) = x^3 - \tfrac{3}{5}x. \end{aligned} \qquad (18)$$

Normalizing so that $P_n(1) = 1$ gives the expected Legendre polynomials.

The "shifted" Legendre polynomials are a set of functions analogous to the Legendre polynomials, but defined on the interval (0, 1). They obey the ORTHOGONALITY relationship

$$\int_0^1 \bar{P}_m(x)\bar{P}_n(x)\,dx = \frac{1}{2n+1}\delta_{mn}. \qquad (19)$$

The first few are

$$\begin{aligned} \bar{P}_0(x) &= 1 \\ \bar{P}_1(x) &= 2x-1 \\ \bar{P}_2(x) &= 6x^2-6x+1 \\ \bar{P}_3(x) &= 20x^3-30x^2+12x-1. \end{aligned}$$

The associated Legendre polynomials $P_l^m(x)$ are solutions to the associated LEGENDRE DIFFERENTIAL EQUATION, where l is a POSITIVE INTEGER and $m = 0, \ldots, l$. They can be given in terms of the unassociated polynomials by

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l, \quad (20)$$

where $P_l(x)$ are the unassociated LEGENDRE POLYNOMIALS. Note that some authors (e.g., Arfken 1985, p. 668) omit the CONDON-SHORTLEY PHASE $(-1)^m$, while others include it (e.g., Abramowitz and Stegun 1972, Press *et al.* 1992, and the `LegendreP[l,m,z]` command of *Mathematica*®). Abramowitz and Stegun (1972, p. 332) use the notation

$$P_{lm}(x) \equiv (-1)^m P_m^l(x) \quad (21)$$

to distinguish these two cases.

Associated polynomials are sometimes called FERRERS' FUNCTIONS (Sansone 1991, p. 246). If $m = 0$, they reduce to the unassociated POLYNOMIALS. The associated Legendre functions are part of the SPHERICAL HARMONICS, which are the solution of LAPLACE'S EQUATION in SPHERICAL COORDINATES. They are ORTHOGONAL over $[-1, 1]$ with the WEIGHTING FUNCTION 1

$$\int_{-1}^{1} P_l^m(x)P_{l'}^m(x)\,dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}, \quad (22)$$

ORTHOGONAL over $[-1, 1]$ with respect to m with the WEIGHTING FUNCTION $(1-x^2)^{-2}$

$$\int_{-1}^{1} P_l^m(x)P_l^{m'}(x)\frac{dx}{1-x^2} = \frac{(l+m)!}{m(l-m)!}\delta_{mm'}. \quad (23)$$

They obey the RECURRENCE RELATIONS

$$(l-m)P_l^m(x) = x(2l-1)P_{l-1}^m(x) - (l+m-1)P_{l-2}^m(x) \quad (24)$$

$$\frac{dP_l^m}{d\theta} = -\sqrt{1-\mu^2}\,\frac{dP_l^m}{d\mu} = \tfrac{1}{2}(l-m+1)(l+m+P_l^{m-1} - P_l^{m+1}) \quad (25)$$

$$(2l+1)\mu P_l^m = (l+m)P_{l-1}^m + (l-m+1)P_{l+1}^m \quad (26)$$

$$(2l+1)\sqrt{1-\mu^2}\,P_l^m = P_{l+1}^{m+1} - P_{l-1}^{m+1}. \quad (27)$$

An identity relating associated POLYNOMIALS with NEGATIVE m to the corresponding functions with POSITIVE m is

$$P_l^{-m} = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m. \quad (28)$$

Additional identities are

$$P_l^l(x) = (-1)^l(2l-1)!!(1-x^2)^{l/2} \quad (29)$$

$$P_{l+1}^l(x) = x(2l+1)P_l^l(x). \quad (30)$$

Written in terms of x and using the convention without a leading factor of $(-1)^m$ (Arfken 1985, p. 669), the first few associated Legendre polynomials are

$$\begin{aligned}
P_0^0(x) &= 1\\
P_1^0(x) &= x\\
P_1^1(x) &= -(1-x^2)^{1/2}\\
P_2^0(x) &= \tfrac{1}{2}(3x^2-1)\\
P_2^1(x) &= -3x(1-x^2)^{1/2}\\
P_2^2(x) &= 3(1-x^2)\\
P_3^0(x) &= \tfrac{1}{2}x(5x^2-3)\\
P_3^1(x) &= \tfrac{3}{2}(1-5x^2)(1-x^2)^{1/2}\\
P_3^2(x) &= 15x(1-x^2)\\
P_3^3(x) &= -15(1-x^2)^{3/2}\\
P_4^0(x) &= \tfrac{1}{8}(35x^4-30x^2+3)\\
P_4^1(x) &= \tfrac{5}{2}x(3-7x^2)(1-x^2)^{1/2}\\
P_4^2(x) &= \tfrac{15}{2}(7x^2-1)(1-x^2)\\
P_4^3(x) &= -105x(1-x^2)^{3/2}\\
P_4^4(x) &= 105(1-x^2)^2\\
P_5^0(x) &= \tfrac{1}{8}x(63x^4-70x^2+15).
\end{aligned}$$

Written in terms $x \equiv \cos\theta$, the first few become

$$\begin{aligned}
P_0^0(\cos\theta) &= 1\\
P_1^{-1}(\cos\theta) &= \tfrac{1}{2}\sin\theta\\
P_1^0(\cos\theta) &= \cos\theta = \mu\\
P_1^1(\cos\theta) &= \sin\theta\\
P_2^{-2}(\cos\theta) &= \tfrac{1}{8}\sin^2\theta\\
P_2^{-1}(\cos\theta) &= \tfrac{1}{2}\sin\theta\cos\theta\\
P_2^0(\cos\theta) &= \tfrac{1}{2}(3\cos^2\theta - 1)\\
P_2^1(\cos\theta) &= 3\sin\theta\cos\theta\\
&= \tfrac{3}{2}\sin^2\theta\\
P_2^2(\cos\theta) &= 3\sin^2\theta\\
&= \tfrac{3}{2}(1-\cos^2\theta)\\
P_3^0(\cos\theta) &= \tfrac{1}{2}\cos\theta(5\cos^2\theta - 3)\\
&= \tfrac{1}{2}\cos\theta(2-5\sin^2\theta)\\
P_3^1(\cos\theta) &= \tfrac{3}{2}(5\cos^2\theta - 1)\sin\theta\\
&= \tfrac{3}{8}(\sin\theta + 5\sin^3\theta).
\end{aligned}$$

The derivative about the origin is

$$\left[\frac{dP_\nu^\mu(x)}{dx}\right]_{x=0} = \frac{2^{\mu+1}\sin[\frac{1}{2}\pi(\nu+\mu)]\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu+1)}{\pi^{1/2}\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2})} \quad (31)$$

(Abramowitz and Stegun 1972, p. 334), and the logarithmic derivative is

$$\left[\frac{d\ln P_\lambda^\mu(z)}{dz}\right]_{z=0} = 2\tan[\tfrac{1}{2}\pi(\lambda+\mu)]\frac{[\frac{1}{2}(\lambda+\mu)]![\frac{1}{2}(\lambda-\mu)]!}{[\frac{1}{2}(\lambda+\mu-1)]![\frac{1}{2}(\lambda-\mu-1)]!}. \tag{32}$$

(Binney and Tremaine 1987, p. 654).

see also CONDON-SHORTLEY PHASE, CONICAL FUNCTION, GEGENBAUER POLYNOMIAL, KINGS PROBLEM, LAPLACE'S INTEGRAL, LAPLACE-MEHLER INTEGRAL, SUPER CATALAN NUMBER, TOROIDAL FUNCTION, TURÁN'S INEQUALITIES

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Legendre Polynomial of the Second Kind

see LEGENDRE FUNCTION OF THE SECOND KIND

Legendre's Quadratic Reciprocity Law

see QUADRATIC RECIPROCITY LAW

Legendre Quadrature

see LEGENDRE-GAUSS QUADRATURE

Legendre Relation

Let $E(k)$ and $K(k)$ be complete ELLIPTIC INTEGRALS OF THE FIRST and SECOND KINDS, with $E'(k)$ and $K'(k)$ the complementary integrals. Then

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \tfrac{1}{2}\pi.$$

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 591, 1972.

Legendre Series

Because the LEGENDRE FUNCTIONS OF THE FIRST KIND form a COMPLETE ORTHOGONAL BASIS, any FUNCTION may be expanded in terms of them

$$f(x) = \sum_{n=0}^\infty a_n P_n(x). \tag{1}$$

Now, multiply both sides by $P_m(x)$ and integrate

$$\int_{-1}^1 P_m(x)f(x)\,dx = \sum_{n=0}^\infty a_n \int_{-1}^1 P_n(x)P_m(x)\,dx. \tag{2}$$

But

$$\int_{-1}^1 P_n(x)P_m(x)\,dx = \frac{2}{2m+1}\delta_{mn}, \tag{3}$$

where δ_{mn} is the KRONECKER DELTA, so

$$\int_{-1}^1 P_m(x)f(x)\,dx = \sum_{n=0}^\infty a_n \frac{2}{2m+1}\delta_{mn} = \frac{2}{2m+1}a_m \tag{4}$$

and

$$a_m = \frac{2m+1}{2}\int_{-1}^1 P_m(x)f(x)\,dx. \tag{5}$$

see also FOURIER SERIES, JACKSON'S THEOREM, LEGENDRE POLYNOMIAL, MACLAURIN SERIES, PICONE'S THEOREM, TAYLOR SERIES

Legendre Sum

see LEGENDRE'S FORMULA

Legendre Symbol

$$\left(\frac{m}{n}\right) = (m|n) \equiv \begin{cases} 0 & \text{if } m|n \\ 1 & \text{if } n \text{ is a quadratic residue modulo } m \\ -1 & \text{if } n \text{ is a quadratic nonresidue modulo } m. \end{cases}$$

If m is an ODD PRIME, then the JACOBI SYMBOL reduces to the Legendre symbol. The Legendre symbol obeys $(ab|p) = (a|p)(b|p)$.

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{12} \\ -1 & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

see also JACOBI SYMBOL, KRONECKER SYMBOL, QUADRATIC RECIPROCITY THEOREM

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Legendre Transformation

Given a function of two variables

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \equiv u\,dx + v\,dy, \tag{1}$$

change the differentials from dx and dy to du and dy with the transformation

$$g \equiv f - ux \tag{2}$$

$$\begin{aligned} dg &= df - u\,dx - x\,du = u\,dx + v\,dy - u\,dx - x\,du \\ &= v\,dy - x\,du. \end{aligned} \tag{3}$$

Then

$$x \equiv -\frac{\partial g}{\partial u} \tag{4}$$

$$v \equiv \frac{\partial g}{\partial y}. \tag{5}$$

Lehmer's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Lehmer (1938) showed that every POSITIVE IRRATIONAL NUMBER x has a unique infinite continued cotangent representation of the form

$$x = \cot\left[\sum_{k=0}^{\infty}(-1)^k \cot^{-1} b_k\right],$$

where the b_ks are NONNEGATIVE and

$$b_k \geq (b_{k-1})^2 + b_{k-1} + 1.$$

The case for which the convergence is slowest occurs when the inequality is replaced by equality, giving $c_0 = 0$ and

$$c_k = (c_{k-1})^2 + c_{k-1} + 1$$

for $k \geq 1$. The first few values are c_k are 0, 1, 3, 13, 183, 33673, ... (Sloane's A024556), resulting in the constant

$$\begin{aligned} \xi &= \cot(\cot^{-1} 0 - \cot^{-1} 1 + \cot^{-1} 3 - \cot^{-1} 13 \\ &\quad + \cot^{-1} 183 - \cot^{-1} 33673 + \cot^{-1} 1133904603 \\ &\quad - \cot^{-1} 1285739649838492213 + \ldots + (-1)^k c_k + \ldots) \\ &= \cot(\tfrac{1}{4}\pi + \cot^{-1} 3 - \cot^{-1} 13 \\ &\quad + \cot^{-1} 183 - \cot^{-1} 33673 + \cot^{-1} 1133904603 \\ &\quad - \cot^{-1} 1285739649838492213 + \ldots + (-1)^k c_k + \ldots) \\ &= 0.59263271\ldots \end{aligned}$$

(Sloane's A030125). ξ is not an ALGEBRAIC NUMBER of degree less than 4, but Lehmer's approach cannot show whether or not ξ is TRANSCENDENTAL.

see also ALGEBRAIC NUMBER, TRANSCENDENTAL NUMBER

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Lehmer's Formula

A FORMULA related to MEISSEL'S FORMULA.

$$\begin{aligned} \pi(x) &= \lfloor x \rfloor - \sum_{i=1}^{a}\left\lfloor \frac{x}{p_i} \right\rfloor + \sum_{1\leq i\leq j\leq a}\left\lfloor \frac{x}{p_ip_j} \right\rfloor - \ldots \\ &\quad + \tfrac{1}{2}(b+a-2)(b-a+1) - \sum_{a<i\leq b}\pi\left(\frac{x}{p_i}\right) \\ &\quad - \sum_{i=a+1}^{c}\sum_{j=i}^{b_i}\left[\pi\left(\frac{x}{p_ip_j}\right) - (j-1)\right], \end{aligned}$$

where

$$\begin{aligned} a &\equiv \pi(x^{1/4}) \\ b &\equiv \pi(x^{1/2}) \\ b_i &\equiv \pi(\sqrt{x/p_i}) \\ c &\equiv \pi(x^{1/3}), \end{aligned}$$

and $\pi(n)$ is the PRIME COUNTING FUNCTION.

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Lehmer Method

see LEHMER-SCHUR METHOD

Lehmer Number

A number generated by a generalization of a LUCAS SEQUENCE. Let α and β be COMPLEX NUMBERS with

$$\alpha + \beta = \sqrt{R} \tag{1}$$

$$\alpha\beta = Q, \tag{2}$$

where Q and R are RELATIVELY PRIME NONZERO INTEGERS and α/β is a ROOT OF UNITY. Then the Lehmer numbers are

$$U_n(\sqrt{R}, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{3}$$

and the companion numbers

$$V_n(\sqrt{R}, Q) = \begin{cases} \frac{\alpha^n + \beta^n}{\alpha + \beta} & \text{for } n \text{ odd} \\ \alpha^n + \beta^n & \text{for } n \text{ even.} \end{cases} \tag{4}$$

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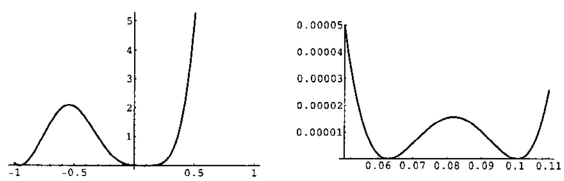
Lehmer's Phenomenon



The appearance of nontrivial zeros (i.e., those along the CRITICAL STRIP with $\Re[z] = 1/2$) of the RIEMANN ZETA FUNCTION $\zeta(z)$ very close together. An example is the pair of zeros $\zeta(\frac{1}{2} + (7005 + t)i)$ given by $t_1 \approx 0.0606918$ and $t_2 \approx 0.100055$, illustrated above in the plot of $|\zeta(\frac{1}{2} + (7005 + t)i)|^2$.

see also CRITICAL STRIP, RIEMANN ZETA FUNCTION

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Lehmer's Problem

Do there exist any COMPOSITE NUMBERS n such that $\phi(n)|(n-1)$? No such numbers are known. In 1932, Lehmer showed that such an n must be ODD and SQUAREFREE, and that the number of distinct PRIME factors $d(7) \geq 7$. This was subsequently extended to $d(n) \geq 11$. The best current results are $n > 10^{20}$ and $d(n) \geq 14$ (Cohen and Hagis 1980), if $30 \nmid n$, then $d(n) \geq 26$ (Wall 1980), and if $3|n$ then $d(n) \geq 213$ and 5.5×10^{570} (Lieuwens 1970).

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Lehmer-Schur Method

An ALGORITHM which isolates ROOTS in the COMPLEX PLANE by generalizing 1-D bracketing.

References

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Lehmer's Theorem

see FERMAT'S LITTLE THEOREM CONVERSE

Lehmus' Theorem

see STEINER-LEHMUS THEOREM

Leibniz Criterion

Also known as the ALTERNATING SERIES TEST. Given a SERIES

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

with $a_n > 0$, if a_n is monotonic decreasing as $n \to \infty$ and

$$\lim_{n \to \infty} a_n = 0,$$

then the series CONVERGES.

Leibniz Harmonic Triangle

$$\begin{array}{ccccccccc} & & & & \frac{1}{1} & & & & \\ & & & \frac{1}{2} & & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\ & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\ \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} \end{array}$$

In the Leibniz harmonic triangle, each FRACTION is the sum of numbers below it, with the initial and final entry on each row one over the corresponding entry in

PASCAL'S TRIANGLE. The DENOMINATORS in the second diagonals are 6, 12, 20, 30, 42, 56, ... (Sloane's A007622).

see also CATALAN'S TRIANGLE, CLARK'S TRIANGLE, EULER'S TRIANGLE, NUMBER TRIANGLE, PASCAL'S TRIANGLE, SEIDEL-ENTRINGER-ARNOLD TRIANGLE

References

Sloane, N. J. A. Sequence A007622/M4096 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Leibniz Identity

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + \binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \ldots + \binom{n}{r}\frac{d^{n-r}u}{dx^{n-r}}\frac{d^r v}{dx^r} + ud^n v + dx^n. \quad (1)$$

Therefore,

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \quad (2)$$

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3} \quad (3)$$

$$\frac{d^3x}{dy^3} = \left[3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{d^3y}{dx^3}\frac{dy}{dx}\right]\left(\frac{dy}{dx}\right)^{-5}. \quad (4)$$

References

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Leibniz Integral Rule

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)\,dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}\,dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}.$$

References

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Leibniz Series

The SERIES for the INVERSE TANGENT,

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots.$$

Lemarié's Wavelet

A wavelet used in multiresolution representation to analyze the information content of images. The WAVELET is defined by

$$H(\omega) = \left[2(1-u)^4 \frac{315 - 420u + 126u^2 - 4u^3}{315 - 420v + 126v^2 - 4v^3}\right]^{1/2},$$

where

$$u \equiv \sin^2(\tfrac{1}{2}\omega)$$

$$v \equiv \sin^2 \omega$$

(Mallat 1989).

see also WAVELET

References

Mallat, S. G. "A Theory for Multiresolution Signal Decomposition: The Wavelet Representation." *IEEE Trans. Pattern Analysis Machine Intel.* **11**, 674–693, 1989.

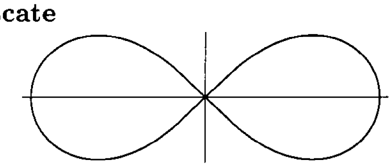
Mallat, S. G. "Multiresolution Approximation and Wavelet Orthonormal Bases of $L^2(\mathbb{R})$." *Trans. Amer. Math. Soc.* **315**, 69–87, 1989.

Lemma

A short THEOREM used in proving a larger THEOREM. Related concepts are the AXIOM, PORISM, POSTULATE, PRINCIPLE, and THEOREM.

see also ABEL'S LEMMA, ARCHIMEDES' LEMMA, BARNES' LEMMA, BLICHFELDT'S LEMMA, BOREL-CANTELLI LEMMA, BURNSIDE'S LEMMA, DANIELSON-LANCZOS LEMMA, DEHN'S LEMMA, DILWORTH'S LEMMA, DIRICHLET'S LEMMA, DIVISION LEMMA, FARKAS'S LEMMA, FATOU'S LEMMA, FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS, GAUSS'S LEMMA, HENSEL'S LEMMA, ITÔ'S LEMMA, JORDAN'S LEMMA, LAGRANGE'S LEMMA, NEYMAN-PEARSON LEMMA, POINCARÉ'S HOLOMORPHIC LEMMA, POINCARÉ'S LEMMA, PÓLYA-BURNSIDE LEMMA, RIEMANN-LEBESGUE LEMMA, SCHUR'S LEMMA, SCHUR'S REPRESENTATION LEMMA, SCHWARZ-PICK LEMMA, SPIJKER'S LEMMA, ZORN'S LEMMA

Lemniscate



A polar curve also called LEMNISCATE OF BERNOULLI which is the LOCUS of points the product of whose distances from two points (called the FOCI) is a constant. Letting the FOCI be located at $(\pm a, 0)$, the Cartesian equation is

$$[(x-a)^2 + y^2][(x+a)^2 + y^2] = a^4, \quad (1)$$

which can be rewritten

$$x^4 + y^4 + 2x^2y^2 = 2a^2(x^2 - y^2). \quad (2)$$

Letting $a' \equiv \sqrt{2}a$, the POLAR COORDINATES are given by

$$r^2 = a^2\cos(2\theta). \tag{3}$$

An alternate form is

$$r^2 = a^2\sin(2\theta). \tag{4}$$

The parametric equations for the lemniscate are

$$x = \frac{a\cos t}{1+\sin^2 t} \tag{5}$$

$$y = \frac{a\sin t\cos t}{1+\sin^2 t}. \tag{6}$$

The bipolar equation of the lemniscate is

$$rr' = \tfrac{1}{2}a^2, \tag{7}$$

and in PEDAL COORDINATES with the PEDAL POINT at the center, the equation is

$$pa^2 = r^3. \tag{8}$$

The two-center BIPOLAR COORDINATES equation with origin at a FOCUS is

$$r_1r_2 = c^2. \tag{9}$$

Jakob Bernoulli published an article in *Acta Eruditorum* in 1694 in which he called this curve the lemniscus ("a pendant ribbon"). Jakob Bernoulli was not aware that the curve he was describing was a special case of CASSINI OVALS which had been described by Cassini in 1680. The general properties of the lemniscate were discovered by G. Fagnano in 1750 (MacTutor Archive). Gauss's and Euler's investigations of the ARC LENGTH of the curve led to later work on ELLIPTIC FUNCTIONS.

The CURVATURE of the lemniscate is

$$\kappa = \frac{3\sqrt{2}\cos t}{\sqrt{3-\cos(2t)}}. \tag{10}$$

The ARC LENGTH is more problematic. Using the polar form,

$$ds^2 = dr^2 + r^2\,d\theta^2 \tag{11}$$

so

$$ds = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,dr. \tag{12}$$

But we have

$$2r\,dr = 2a^2\sin(2\theta)\,d\theta \tag{13}$$

$$r\frac{dr}{d\theta} = \frac{r^2}{a^2\sin(2\theta)} \tag{14}$$

$$\left(r\frac{d\theta}{dr}\right)^2 = \frac{r^4}{a^4\sin^2(2\theta)} = \frac{r^4}{a^4[1-\cos^2(2\theta)]} = \frac{r^4}{a^4-r^4}, \tag{15}$$

so

$$ds = \sqrt{1+\frac{r^4}{a^4-r^4}}\,dr = \sqrt{\frac{a^4}{a^4-r^4}}\,dr = \frac{a^2}{\sqrt{a^4-r^4}}\,dr = \frac{dr}{\sqrt{1-\left(\frac{r}{a}\right)^4}}, \tag{16}$$

and

$$L = \int_0^a ds = 2\int_0^a \frac{ds}{dr}\,dr = 2\int_0^a \frac{dr}{\sqrt{1-\left(\frac{r}{a}\right)^4}}. \tag{17}$$

Let $t \equiv r/a$, so $dt = dr/a$, and

$$L = 2a\int_0^1 (1-t^4)^{-1/2}\,dt, \tag{18}$$

which, as shown in LEMNISCATE FUNCTION, is given analytically by

$$L = \sqrt{2}\,aK\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})}{2^{3/2}\sqrt{\pi}}\,a. \tag{19}$$

If $a = 1$, then

$$L = 5.2441151086\ldots, \tag{20}$$

which is related to GAUSS'S CONSTANT M by

$$L = \frac{2\pi}{M}. \tag{21}$$

The quantity $L/2$ or $L/4$ is called the LEMNISCATE CONSTANT and plays a role for the lemniscate analogous to that of π for the CIRCLE.

The AREA of one loop of the lemniscate is

$$A = \tfrac{1}{2}\int r^2\,d\theta = \tfrac{1}{2}a^2\int_{-\pi/4}^{\pi/4}\cos(2\theta)\,d\theta = \tfrac{1}{4}a^2[\sin(2\theta)]_{-\pi/4}^{\pi/4} = \tfrac{1}{2}a^2[\sin(2\theta)]_0^{\pi/4} = \tfrac{1}{2}a^2[\sin(\tfrac{\pi}{2})-\sin 0] = \tfrac{1}{2}a^2. \tag{22}$$

see also LEMNISCATE FUNCTION

References

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Lemniscate of Bernoulli

see LEMNISCATE

Lemniscate Case

The case of the WEIERSTRAß ELLIPTIC FUNCTION with invariants $g_2 = 1$ and $g_3 = 0$.

see also EQUIANHARMONIC CASE, WEIERSTRAß ELLIPTIC FUNCTION, PSEUDOLEMNISCATE CASE

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Lemniscate Case ($g_2 = 1$, $g_3 = 0$)." §18.14 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 658–662, 1972.

Lemniscate Constant

Let

$$L = \frac{1}{\sqrt{2\pi}}[\Gamma(\tfrac{1}{4})]^2 = 5.2441151086\ldots$$

be the ARC LENGTH of a LEMNISCATE with $a = 1$. Then the lemniscate constant is the quantity $L/2$ (Abramowitz and Stegun 1972), or $L/4 = 1.311028777\ldots$ (Todd 1975, Le Lionnais 1983). Todd (1975) cites T. Schneider (1937) as proving L to be a TRANSCENDENTAL NUMBER.

see also LEMNISCATE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, 1972.

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Todd, J. "The Lemniscate Constant." *Comm. ACM* **18**, 14–19 and 462, 1975.

Lemniscate Function

The lemniscate functions arise in rectifying the ARC LENGTH of the LEMNISCATE. The lemniscate functions were first studied by Jakob Bernoulli and G. Fagnano. A historical account is given by Ayoub (1984), and an extensive discussion by Siegel (1969). The lemniscate functions were the first functions defined by inversion of an integral, which was first done by Gauss.

$$L = 2a\int_0^1 (1-t^4)^{-1/2}\,dt. \qquad (1)$$

Define the functions

$$\phi(x) \equiv \operatorname{arcsinlemn} x = \int_0^x (1-t^4)^{-1/2}\,dt \qquad (2)$$

$$\phi'(x) \equiv \operatorname{arccoslemn} x = \int_x^1 (1-t^4)^{-1/2}\,dt, \qquad (3)$$

where

$$\varpi \equiv \frac{L}{a}, \qquad (4)$$

and write

$$x = \operatorname{sinlemn}\phi \qquad (5)$$

$$x = \operatorname{coslemn}\phi'. \qquad (6)$$

There is an identity connecting ϕ and ϕ' since

$$\phi(x) + \phi'(x) = \frac{L}{2a} = \tfrac{1}{2}\varpi, \qquad (7)$$

so

$$\operatorname{sinlemn}\phi = \operatorname{coslemn}(\tfrac{1}{2}\varpi - \phi). \qquad (8)$$

These functions can be written in terms of JACOBI ELLIPTIC FUNCTIONS,

$$u = \int_0^{\operatorname{sd}(u,k)} [(1-k'^2y^2)(1+k^2y^2)]^{-1/2}\,dy. \qquad (9)$$

Now, if $k = k' = 1/\sqrt{2}$, then

$$\begin{aligned} u &= \int_0^{\operatorname{sd}(u,1/\sqrt{2})} [(1-\tfrac{1}{2}y^2)(1+\tfrac{1}{2}y^2)]^{-1/2}\,dy \\ &= \int_0^{\operatorname{sd}(u,1/\sqrt{2})} (1-\tfrac{1}{4}y^4)^{-1/2}\,dy. \end{aligned} \qquad (10)$$

Let $t \equiv y/\sqrt{2}$ so $dy = \sqrt{2}\,dt$,

$$u = \sqrt{2}\int_0^{\operatorname{sd}(u,1/\sqrt{2})/\sqrt{2}} (1-t^4)^{-1/2}\,dt \qquad (11)$$

$$\frac{u}{\sqrt{2}} = \int_0^{\operatorname{sd}(u,1/\sqrt{2})/\sqrt{2}} (1-t^4)^{-1/2}\,dt \qquad (12)$$

$$u = \int_0^{\operatorname{sd}(u\sqrt{2},1/\sqrt{2})/\sqrt{2}} (1-t^4)^{-1/2}\,dt, \qquad (13)$$

and

$$\operatorname{sinlemn}\phi = \frac{1}{\sqrt{2}}\operatorname{sd}\left(\phi\sqrt{2}, \frac{1}{\sqrt{2}}\right). \qquad (14)$$

Similarly,

$$\begin{aligned} u &= \int_{\operatorname{cn}(u,k)}^1 (1-t^2)^{-1/2}(k'^2+k^2t^2)^{-1/2}\,dt \\ &= \int_{\operatorname{cn}(u,1/\sqrt{2})}^1 (1-t^2)^{-1/2}\left(\tfrac{1}{2}+\tfrac{1}{2}t^2\right)^{-1/2}\,dt \\ &= \sqrt{2}\int_{\operatorname{cn}(u,1/\sqrt{2})}^1 (1-t^4)^{-1/2}\,dt \end{aligned} \qquad (15)$$

$$\frac{u}{\sqrt{2}} = \int_{\operatorname{cn}(u,1/\sqrt{2})}^1 (1-t^4)^{-1/2}\,dt \qquad (16)$$

$$u = \int_{\operatorname{cn}(u\sqrt{2},1/\sqrt{2})}^{1} (1-t^4)^{-1/2}\,dt, \qquad (17)$$

and

$$\operatorname{coslemn}\phi = \operatorname{cn}\left(\phi\sqrt{2}, \frac{1}{\sqrt{2}}\right). \qquad (18)$$

We know

$$\operatorname{coslemn}(\tfrac{1}{2}\varpi) = \operatorname{cn}\left(\tfrac{1}{2}\varpi\sqrt{2}, \frac{1}{\sqrt{2}}\right) = 0. \qquad (19)$$

But it is true that

$$\operatorname{cn}(K,k) = 0, \qquad (20)$$

so

$$K\left(\frac{1}{\sqrt{2}}\right) = \tfrac{1}{2}\sqrt{2}\,\varpi = \frac{1}{\sqrt{2}}\varpi \qquad (21)$$

$$\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}} = \frac{1}{\sqrt{2}}\varpi \qquad (22)$$

$$L = a\varpi = a\sqrt{2}\,\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}} = \frac{\Gamma^2(\frac{1}{4})}{2^{3/2}\sqrt{\pi}}\,a. \qquad (23)$$

By expanding $(1-t^4)^{-1/2}$ in a BINOMIAL SERIES and integrating term by term, the arcsinlemn function can be written

$$\phi(x) = \int_0^v \frac{dt}{\sqrt{1-t^4}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n x^{4n+1}}{n!(4n+1)}, \qquad (24)$$

where $(a)_n$ is the RISING FACTORIAL (Berndt 1994). Ramanujan gave the following inversion FORMULA for $\phi(x)$. If

$$\frac{\theta\mu}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n x^{4n+1}}{n!(4n+1)}, \qquad (25)$$

where

$$\mu = \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}} \qquad (26)$$

is the constant obtained by letting $x=1$ and $\theta = \pi/2$, and

$$v = 2^{-1/2}\operatorname{sd}(\mu\theta), \qquad (27)$$

then

$$\frac{\mu^2}{2x^2} = \csc^2\theta - \frac{1}{\pi} - 8\sum_{n=1}^{\infty}\frac{n\cos(2n\theta)}{e^{2\pi n}-1} \qquad (28)$$

(Berndt 1994). Ramanujan also showed that if $0 < \theta < \pi/2$, then

$$-\frac{\mu}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n v^{4n-1}}{n!(4n-1)} = \cot\theta + \frac{\theta}{\pi} + 4\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{2^{2\pi n}-1}, \qquad (29)$$

$$\ln v + \tfrac{1}{6}\pi - \tfrac{1}{2}\ln 2 + \sum_{n=1}^{\infty}\frac{(\frac{1}{4})_n v^{4n}}{(\frac{3}{4})_n 4n} = \ln(\sin\theta) + \frac{\theta^2}{2\pi} - 2\sum_{n=1}^{\infty}\frac{\cos(2n\theta)}{n(e^{2\pi n}-1)}, \qquad (30)$$

$$\tfrac{1}{2}\tan^{-1} v = \sum_{n=0}^{\infty}\frac{\sin[(2n+1)\theta]}{(2n+1)\cosh[\frac{1}{2}(2n+1)\pi]}, \qquad (31)$$

$$\tfrac{1}{4}\cos^{-1}(v^2) = \sum_{n=0}^{\infty}\frac{(-1)^n\cos[(2n+1)\theta]}{(2n+1)\cosh[\frac{1}{2}(2n+1)\pi]}, \qquad (32)$$

and

$$\frac{\sqrt{2}}{4\mu}\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!(4n+3)}v^{4n+3} = \frac{\pi\theta}{8} - \sum_{n=0}^{\infty}\frac{(-1)^n\sin[(2n+1)\theta]}{(2n+1)^2\cosh[\frac{1}{2}(2n+1)\pi]} \qquad (33)$$

(Berndt 1994).

A generalized version of the lemniscate function can be defined by letting $0 \le \theta \le \pi/2$ and $0 \le v \le 1$. Write

$$\tfrac{2}{3}\theta\mu = \int_0^v \frac{dt}{\sqrt{1-t^6}}, \qquad (34)$$

where μ is the constant obtained by setting $\theta = \pi/2$ and $v = 1$. Then

$$\mu = \frac{\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}, \qquad (35)$$

and Ramanujan showed

$$\frac{4\mu^2}{9v^2} = \csc^2\theta - \frac{2}{\pi\sqrt{3}} + 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\cos(2n\theta)}{e^{\pi n\sqrt{3}}-(-1)^n} \qquad (36)$$

(Berndt 1994).

see also HYPERBOLIC LEMNISCATE FUNCTION

References

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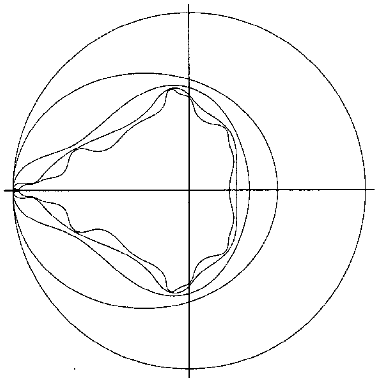
Lemniscate of Gerono

see EIGHT CURVE

Lemniscate Inverse Curve

The INVERSE CURVE of a LEMNISCATE in a CIRCLE centered at the origin and touching the LEMNISCATE where it crosses the x-AXIS produces a RECTANGULAR HYPERBOLA.

Lemniscate (Mandelbrot Set)



A curve on which points of a MAP z_n (such as the MANDELBROT SET) diverge to a given value $r_{\max}$ at the same rate. A common method of obtaining lemniscates is to define an INTEGER called the COUNT which is the largest n such that $|z_n| < r$ where r is usually taken as $r = 2$. Successive COUNTS then define a series of lemniscates, which are called EQUIPOTENTIAL CURVES by Peitgen and Saupe (1988).

see also COUNT, MANDELBROT SET

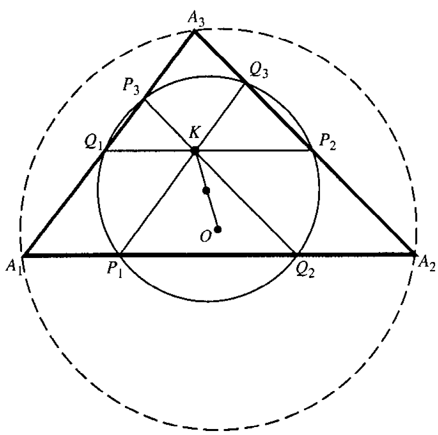
References

Peitgen, H.-O. and Saupe, D. (Eds.). *The Science of Fractal Images.* New York: Springer-Verlag, pp. 178–179, 1988.

Lemoine Axis

see LEMOINE LINE

Lemoine Circle



Also called the TRIPLICATE-RATIO CIRCLE. Draw lines through the LEMOINE POINT K and parallel to the sides of the triangle. The points where the parallel lines intersect the sides then lie on a CIRCLE known as the Lemoine circle. This circle has center at the MIDPOINT of OK, where O is the CIRCUMCENTER. The circle has radius

$$\tfrac{1}{2}\sqrt{R^2 + r^2} = \tfrac{1}{2}R\sec\omega,$$

where R is the CIRCUMRADIUS, r is the INRADIUS, and ω is the BROCARD ANGLE. The Lemoine circle divides any side into segments proportional to the squares of the sides

$$\overline{A_2P_2} : \overline{P_2Q_3} : \overline{Q_3A_3} = {a_3}^2 : {a_1}^2 : {a_2}^2.$$

Furthermore, the chords cut from the sides by the Lemoine circle are proportional to the squares of the sides.

The COSINE CIRCLE is sometimes called the second Lemoine circle.

see also COSINE CIRCLE, LEMOINE LINE, LEMOINE POINT, TUCKER CIRCLES

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 273–275, 1929.

Lemoine Line

The Lemoine line, also called the LEMOINE AXIS, is the perspectivity axis of a TRIANGLE and its TANGENTIAL TRIANGLE, and also the TRILINEAR POLAR of the CENTROID of the triangle vertices. It is also the POLAR of K with regard to its CIRCUMCIRCLE, and is PERPENDICULAR to the BROCARD AXIS.

The centers of the APOLLONIUS CIRCLES L_1, L_2, and L_3 are COLLINEAR on the LEMOINE LINE. This line is PERPENDICULAR to the BROCARD AXIS OK and is the RADICAL AXIS of the CIRCUMCIRCLE and the BROCARD CIRCLE. It has equation

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}$$

in terms of TRILINEAR COORDINATES (Oldknow 1996).

see also APOLLONIUS CIRCLES, BROCARD AXIS, CENTROID (TRIANGLE), CIRCUMCIRCLE, COLLINEAR, LEMOINE CIRCLE, LEMOINE POINT, POLAR, RADICAL AXIS, TANGENTIAL TRIANGLE, TRILINEAR POLAR

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 295, 1929.
Oldknow, A. "The Euler-Gergonne-Soddy Triangle of a Triangle." *Amer. Math. Monthly* **103**, 319–329, 1996.

Lemoine Point

The point of concurrence K of the SYMMEDIAN LINES, sometimes also called the SYMMEDIAN POINT and GREBE POINT.

Let G be the CENTROID of a TRIANGLE ΔABC, L_A, L_B, and L_C the ANGLE BISECTORS of ANGLES A, B, C, and G_A, G_B, and G_C the reflections of AG, BG, and CG about L_A, L_B, and L_C. Then K is the point of concurrence of the lines G_A, G_B, and G_C. It is the perspectivity center of a TRIANGLE and its TANGENTIAL TRIANGLE.

In AREAL COORDINATES (actual TRILINEAR COORDINATES), the Lemoine point is the point for which $\alpha^2+\beta^2+\gamma^2$ is a minimum. A center X is the CENTROID of its own PEDAL TRIANGLE IFF it is the Lemoine point.

The Lemoine point lies on the BROCARD AXIS, and its distances from the Lemoine point K to the sides of the TRIANGLE are

$$\overline{KK_i} = \tfrac{1}{2}a_i \tan\omega,$$

where ω is the BROCARD ANGLE. A BROCARD LINE, MEDIAN, and Lemoine point are concurrent, with $A_1\Omega_1$, A_2K, and A_3M meeting at a point. Similarly, $A_1\Omega'$, A_2M, and A_3K meet at a point which is the ISOGONAL CONJUGATE of the first (Johnson 1929, pp. 268–269). The line joining the MIDPOINT of any side to the midpoint of the ALTITUDE on that side passes through the Lemoine point K. The Lemoine point K is the STEINER POINT of the first BROCARD TRIANGLE.

see also ANGLE BISECTOR, BROCARD ANGLE, BROCARD AXIS, BROCARD DIAMETER, CENTROID (TRIANGLE), COSYMMEDIAN TRIANGLES, GREBE POINT, ISOGONAL CONJUGATE, LEMOINE CIRCLE, LEMOINE LINE, LINE AT INFINITY, MITTENPUNKT, PEDAL TRIANGLE, STEINER POINTS, SYMMEDIAN LINE, TANGENTIAL TRIANGLE

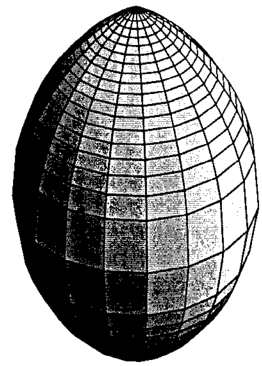
References
Gallatly, W. *The Modern Geometry of the Triangle, 2nd ed.* London: Hodgson, p. 86, 1913.
Honsberger, R. *Episodes in Nineteenth and Twentieth Century Euclidean Geometry.* Washington, DC: Math. Assoc. Amer., 1995.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 217, 268–269, and 271–272, 1929.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. "Symmedian Point." `http://www.evansville.edu/~ck6/tcenters/class/sympt.html`.
Mackay, J. S. "Early History of the Symmedian Point." *Proc. Edinburgh Math. Soc.* **11**, 92–103, 1892–1893.

Lemoine's Problem

Given the vertices of the three EQUILATERAL TRIANGLES placed on the sides of a TRIANGLE T, construct T. The solution can be given using KIEPERT'S HYPERBOLA.

see also KIEPERT'S HYPERBOLA

Lemon



A SURFACE OF REVOLUTION defined by Kepler. It consists of less than half of a circular ARC rotated about an axis passing through the endpoints of the ARC. The equations of the upper and lower boundaries in the xz plane are

$$z_\pm = \pm\sqrt{R^2-(x+r)^2}$$

for $R > r$ and $x \in [-(R-r), R-r]$. The CROSS-SECTION of a lemon is a LENS. The lemon is the inside surface of a SPINDLE TORUS.

see also APPLE, LENS, SPINDLE TORUS

Length (Curve)

Let $\gamma(t)$ be a smooth curve in a MANIFOLD M from x to y with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma'(t) \in T_{\gamma(t)}$, where T_x is the TANGENT SPACE of M at x. The length of γ with respect to the Riemannian structure is given by

$$\int_0^1 ||\gamma'(t)||_{\gamma(t)}\,dt.$$

see also ARC LENGTH, DISTANCE

Length Distribution Function

A function giving the distribution of the interpoint distances of a curve. It is defined by

$$p(r) = \frac{1}{N}\sum_{ij}\delta_{r_{ij}=r}.$$

see also RADIUS OF GYRATION

References
Pickover, C. A. *Keys to Infinity.* New York: W. H. Freeman, pp. 204–206, 1995.

Length (Number)

The length of a number n in base b is the number of DIGITS in the base-b numeral for n, given by the formula

$$L(n,b) = \lfloor \log_b(n) \rfloor + 1,$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION.

The MULTIPLICATIVE PERSISTENCE of an n-DIGIT is sometimes also called its length.

see also CONCATENATION, DIGIT, FIGURES, MULTIPLICATIVE PERSISTENCE

Length (Partial Order)

For a PARTIAL ORDER, the size of the longest CHAIN is called the length.

see also WIDTH (PARTIAL ORDER)

Length (Size)

The longest dimension of a 3-D object.

see also HEIGHT, WIDTH (SIZE)

Lengyel's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let L denote the partition lattice of the SET $\{1, 2, \ldots, n\}$. The MAXIMUM element of L is

$$M = \{\{1, 2, \ldots, n\}\} \tag{1}$$

and the MINIMUM element is

$$m = \{\{1\}, \{2\}, \ldots, \{n\}\}. \tag{2}$$

Let Z_n denote that number of chains of any length in L containing both M and m. Then Z_n satisfies the RECURRENCE RELATION

$$Z_n = \sum_{k=1}^{n-1} s(n,k) Z_k, \tag{3}$$

where $s(n,k)$ is a STIRLING NUMBER OF THE SECOND KIND. Lengyel (1984) proved that the QUOTIENT

$$r(n) = \frac{Z_n}{(n!)^2 (2\ln 2)^{-n} n^{1-(\ln 2)/3}} \tag{4}$$

is bounded between two constants as $n \to \infty$, and Flajolet and Salvy (1990) improved the result of Babai and Lengyel (1992) to show that

$$\Lambda \equiv \lim_{n\to\infty} r(n) = 1.0986858055\ldots. \tag{5}$$

References

Babai, L. and Lengyel, T. "A Convergence Criterion for Recurrent Sequences with Application to the Partition Lattice." *Analysis* **12**, 109–119, 1992.

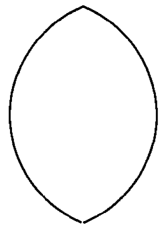
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/lngy/lngy.html`.

Flajolet, P. and Salvy, B. "Hierarchal Set Partitions and Analytic Iterates of the Exponential Function." Unpublished manuscript, 1990.

Lengyel, T. "On a Recurrence Involving Stirling Numbers." *Europ. J. Comb.* **5**, 313–321, 1984.

Plouffe, S. "The Lengyel Constant." `http://lacim.uqam.ca/piDATA/lengyel.txt`.

Lens



A figure composed of two equal and symmetrically placed circular ARCS. It is also known as the FISH BLADDER (Pedoe 1995, p. xii) or VESICA PISCIS. The latter term is often used for the particular lens formed by the intersection of two unit CIRCLES whose centers are offset by a unit distance (Rawles 1997). In this case, the height of the lens is given by letting $d = r = R = 1$ in the equation for a CIRCLE-CIRCLE INTERSECTION

$$a = \frac{1}{d}\sqrt{4d^2R^2 - (d^2 - r^2 + R^2)^2}, \tag{1}$$

giving $a = \sqrt{3}$. The AREA of the VESICA PISCIS is given by plugging $d = R$ into the CIRCLE-CIRCLE INTERSECTION area equation with $r = R$,

$$A = 2R^2 \cos^{-1}\left(\frac{d}{2R}\right) - \tfrac{1}{2} d\sqrt{4R^2 - d^2}, \tag{2}$$

giving

$$A = \tfrac{1}{6}(4\pi - 3\sqrt{3}) \approx 1.22837. \tag{3}$$

Renaissance artists frequently surrounded images of Jesus with the vesica piscis (Rawles 1997). An asymmetrical lens is produced by a CIRCLE-CIRCLE INTERSECTION for unequal CIRCLES.

see also CIRCLE, CIRCLE-CIRCLE INTERSECTION, FLOWER OF LIFE, LEMON, LUNE (PLANE), REULEAUX TRIANGLE, SECTOR, SEED OF LIFE, SEGMENT, VENN DIAGRAM

References

Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., 1995.

Rawles, B. *Sacred Geometry Design Sourcebook: Universal Dimensional Patterns.* Nevada City, CA: Elysian Pub., p. 11, 1997.

Lens Space

A lens space $L(p,q)$ is the 3-MANIFOLD obtained by gluing the boundaries of two solid TORI together such that the meridian of the first goes to a (p,q)-curve on the second, where a (p,q)-curve has p meridians and q longitudes.

References

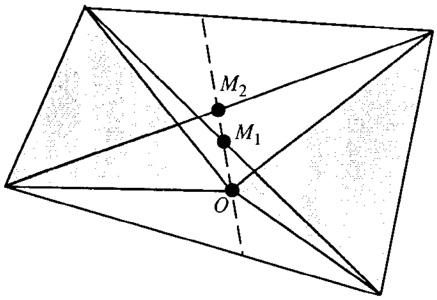
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, 1976.

Lenstra Elliptic Curve Method

A method of factoring INTEGERS using ELLIPTIC CURVES.

References
Montgomery, P. L. "Speeding up the Pollard and Elliptic Curve Methods of Factorization." *Math. Comput.* **48**, 243–264, 1987.

Léon Anne's Theorem



Pick a point O in the interior of a QUADRILATERAL which is not a PARALLELOGRAM. Join this point to each of the four VERTICES, then the LOCUS of points O for which the sum of opposite TRIANGLE areas is half the QUADRILATERAL AREA is the line joining the MIDPOINTS M_1 and M_2 of the DIAGONALS.

see also DIAGONAL (POLYGON), MIDPOINT, QUADRILATERAL

References
Honsberger, R. *More Mathematical Morsels.* Washington, DC: Math. Assoc. Amer., pp. 174–175, 1991.

Leonardo's Paradox

In the depiction of a row of identical columns parallel to the plane of a PERSPECTIVE drawing, the outer columns should appear wider even though they are farther away.

see also PERSPECTIVE, VANISHING POINT, ZEEMAN'S PARADOX

References
Dixon, R. *Mathographics.* New York: Dover, p. 82, 1991.

Leptokurtic

A distribution with a high peak so that the KURTOSIS satisfies $\gamma_2 > 0$.

see also KURTOSIS

Lerch's Theorem

If there are two functions $F_1(t)$ and $F_2(t)$ with the same integral transform

$$\mathcal{T}[F_1(t)] = \mathcal{T}[F_2(t)] \equiv f(s), \tag{1}$$

then a NULL FUNCTION can be defined by

$$\delta_0(t) \equiv F_1(t) - F_2(t) \tag{2}$$

so that the integral

$$\int_0^a \delta_0(t)\,dt = 0 \tag{3}$$

vanishes for all $a > 0$.

see also NULL FUNCTION

Lerch Transcendent

A generalization of the HURWITZ ZETA FUNCTION and POLYLOGARITHM function. Many sums of reciprocal POWERS can be expressed in terms of it. It is defined by

$$\Phi(z,s,a) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}, \tag{1}$$

where any term with $a + k = 0$ is excluded.

The Lerch transcendent can be used to express the DIRICHLET BETA FUNCTION

$$\beta(s) \equiv \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-s} 2^{-s} \Phi(-1, s, \tfrac{1}{2}), \tag{2}$$

the integral of the FERMI-DIRAC DISTRIBUTION

$$\int_0^\infty \frac{k^s}{e^{k-\mu}+1}\,dk = e^\mu \Gamma(s+1)\Phi(-e^\mu, s+1, 1), \tag{3}$$

where $\Gamma(z)$ is the GAMMA FUNCTION, and to evaluate the DIRICHLET L-SERIES.

see also DIRICHLET BETA FUNCTION, DIRICHLET L-SERIES, FERMI-DIRAC DISTRIBUTION, HURWITZ ZETA FUNCTION, POLYLOGARITHM

Less

A quantity a is said to be less than b if a is smaller than b, written $a < b$. If a is less than or EQUAL to b, the relationship is written $a \leq b$. If a is MUCH LESS than b, this is written $a \ll b$. Statements involving GREATER than and less than symbols are called INEQUALITIES.

see also EQUAL, GREATER, INEQUALITY, MUCH GREATER, MUCH LESS

Letter-Value Display

A method of displaying simple statistical parameters including HINGES, MEDIAN, and upper and lower values.

References
Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, p. 33, 1977.

Leudesdorf Theorem

Let $t(m)$ denote the set of the $\phi(m)$ numbers less than and RELATIVELY PRIME to m, where $\phi(n)$ is the TOTIENT FUNCTION. Then if

$$S_m \equiv \sum_{t(m)} \frac{1}{t},$$

then

$$\begin{cases} S_m \equiv 0 \pmod{m^2} & \text{if } 2\nmid m,\ 3\nmid m \\ S_m \equiv 0 \pmod{\tfrac{1}{3}m^2} & \text{if } 2\nmid m,\ 3|m \\ S_m \equiv 0 \pmod{\tfrac{1}{2}m^2} & 2|m,\ \nmid m,\ m \text{ not a power of 2} \\ S_m \equiv 0 \pmod{\tfrac{1}{6}m^2} & \text{if } 2|m,\ 3|m \\ S_m \equiv 0 \pmod{\tfrac{1}{4}m^2} & \text{if } m = 2^a. \end{cases}$$

see also BAUER'S IDENTICAL CONGRUENCE, TOTIENT FUNCTION

References

Hardy, G. H. and Wright, E. M. "A Theorem of Leudesdorf." §8.7 in *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, pp. 100–102, 1979.

Level Curve

A LEVEL SET in 2-D.

Level Set

The level set of c is the SET of points

$$\{(x_1, \ldots, x_n) \in U : f(x_1, \ldots, x_n) = c\} \in \mathbb{R}^n,$$

and is in the DOMAIN of the function. If $n = 2$, the level set is a plane curve (a level curve). If $n = 3$, the level set is a surface (a level surface).

References

Gray, A. "Level Surfaces in $\mathbb{R}^3$." §10.7 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 204–207, 1993.

Level Surface

A LEVEL SET in 3-D.

Levi-Civita Density

see PERMUTATION SYMBOL

Levi-Civita Symbol

see PERMUTATION SYMBOL

Levi-Civita Tensor

see PERMUTATION TENSOR

Leviathan Number

The number $(10^{666})!$, where 666 is the BEAST NUMBER and $n!$ denotes a FACTORIAL. The number of trailing zeros in the Leviathan number is $25 \times 10^{664} - 143$ (Pickover 1995).

see also 666, APOCALYPSE NUMBER, APOCALYPTIC NUMBER, BEAST NUMBER

References

Pickover, C. A. *Keys to Infinity.* New York: Wiley, pp. 97–102, 1995.

Levine-O'Sullivan Greedy Algorithm

For a sequence $\{\chi_i\}$, the Levine-O'Sullivan greedy algorithm is given by

$$\begin{aligned} \chi_1 &= 1 \\ \chi_i &= \max_{1\leq j\leq i-1} (j+1)(i-\chi_j) \end{aligned}$$

for $i > 1$.

see also GREEDY ALGORITHM, LEVINE-O'SULLIVAN SEQUENCE

References

Levine, E. and O'Sullivan, J. "An Upper Estimate for the Reciprocal Sum of a Sum-Free Sequence." *Acta Arith.* **34**, 9–24, 1977.

Levine-O'Sullivan Sequence

The sequence generated by the LEVINE-O'SULLIVAN GREEDY ALGORITHM: 1, 2, 4, 6, 9, 12, 15, 18, 21, 24, 28, 32, 36, 40, 45, 50, 55, 60, 65, ... (Sloane's A014011). The reciprocal sum of this sequence is conjectured to bound the reciprocal sum of all A-SEQUENCES.

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/erdos/erdos.html`.

Levine, E. and O'Sullivan, J. "An Upper Estimate for the Reciprocal Sum of a Sum-Free Sequence." *Acta Arith.* **34**, 9–24, 1977.

Sloane, N. J. A. Sequence A014011 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Lévy Constant

Let p_n/q_n be the nth CONVERGENT of a REAL NUMBER x. Then almost all REAL NUMBERS satisfy

$$L \equiv \lim_{n\to\infty} (q_n)^{1/n} = e^{\pi^2/(12\ln 2)} = 3.27582291872\ldots.$$

see also KHINTCHINE'S CONSTANT, KHINTCHINE-LÉVY CONSTANT

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 51, 1983.

Lévy Distribution

$$\mathcal{F}[P_N(k)] = \exp(-N|k|^\beta),$$

where $\mathcal{F}$ is the FOURIER TRANSFORM of the probability $P_N(k)$ for N-step addition of random variables. Lévy showed that $\beta \in (0,2)$ for $P(x)$ to be NONNEGATIVE. The Lévy distribution has infinite variance and sometimes infinite mean. The case $\beta = 1$ gives a CAUCHY DISTRIBUTION, while $\beta = 2$ gives a GAUSSIAN DISTRIBUTION.

see also CAUCHY DISTRIBUTION, GAUSSIAN DISTRIBUTION

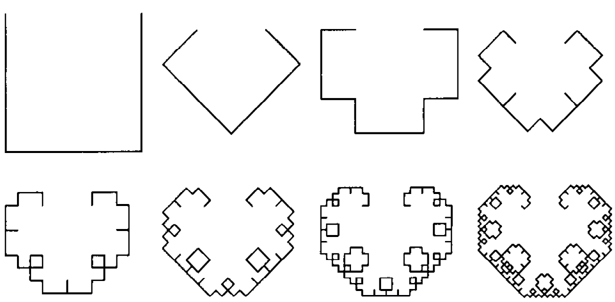
Lévy Flight

RANDOM WALK trajectories which are composed of self-similar jumps. They are described by the LÉVY DISTRIBUTION.

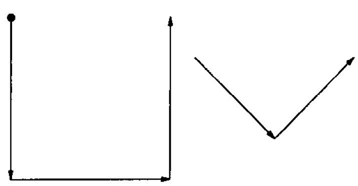
see also LÉVY DISTRIBUTION

References
Shlesinger, M.; Zaslavsky, G. M.; and Frisch, U. (Eds.). *Lévy Flights and Related Topics in Physics.* New York: Springer-Verlag, 1995.

Lévy Fractal



A FRACTAL curve, also called the C-CURVE (Beeler *et al.* 1972, Item 135). The base curve and motif are illustrated below.



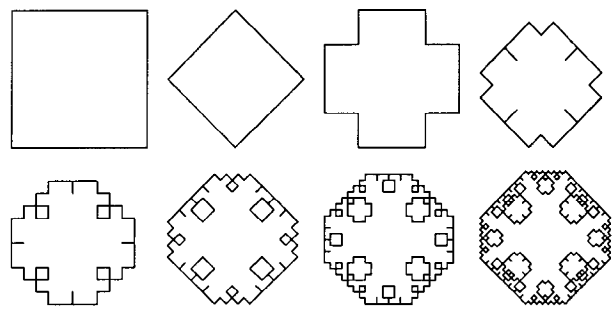
see also LÉVY TAPESTRY

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.
Dixon, R. *Mathographics.* New York: Dover, pp. 182–183, 1991.
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 45–48, 1991.
Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

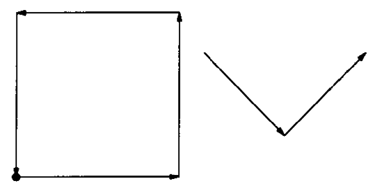
Lévy Function

see BROWN FUNCTION

Lévy Tapestry



The FRACTAL curve illustrated above, with base curve and motif illustrated below.



see also LÉVY FRACTAL

References
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 45–48, 1991.
Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Lew k-gram

Diagrams invented by Lewis Carroll which can be used to determine the number of minimal MINIMAL COVERS of n numbers with k members.

References
Macula, A. J. "Lewis Carroll and the Enumeration of Minimal Covers." *Math. Mag.* **68**, 269–274, 1995.

Lexicographic Order

An ordering of PERMUTATIONS in which they are listed in increasing numerical order. For example, the PERMUTATIONS of $\{1,2,3\}$ in lexicographic order are 123, 132, 213, 231, 312, and 321.

see also TRANSPOSITION ORDER

References
Ruskey, F. "Information on Combinations of a Set." `http://sue.csc.uvic.ca/~cos/inf/comb/CombinationsInfo.html`.

Lexis Ratio

$$L \equiv \frac{\sigma}{\sigma_B},$$

where σ is the VARIANCE in a set of s LEXIS TRIALS and σ_B is the VARIANCE assuming BERNOULLI TRIALS.

If $L < 1$, the trials are said to be SUBNORMAL, and if $L > 1$, the trials are said to be SUPERNORMAL.

see also BERNOULLI TRIAL, LEXIS TRIALS, SUBNORMAL, SUPERNORMAL

Lexis Trials

n sets of s trials each, with the probability of success p constant in each set.

$$\operatorname{var}\left(\frac{x}{n}\right) = spq + s(s-1){\sigma_p}^2,$$

where ${\sigma_p}^2$ is the VARIANCE of p_i.

see also BERNOULLI TRIAL, LEXIS RATIO

Lg

The LOGARITHM to BASE 2 is denoted lg, i.e.,

$$\lg x \equiv \log_2 x.$$

see also BASE (LOGARITHM), E, LN, LOGARITHM, NAPIERIAN LOGARITHM, NATURAL LOGARITHM

Liar's Paradox

see EPIMENIDES PARADOX

Lichnerowicz Conditions

Second and higher derivatives of the METRIC TENSOR g_{ab} need not be continuous across a surface of discontinuity, but g_{ab} and $g_{ab,c}$ must be continuous across it.

Lichnerowicz Formula

$$D^*D\psi = \nabla^*\nabla\psi + \tfrac{1}{4}R\psi - \tfrac{1}{2}F_L^+(\psi),$$

where D is the Dirac operator $D : \Gamma(W^+) \to \Gamma(W^-)$, ∇ is the COVARIANT DERIVATIVE on SPINORS, R is the CURVATURE SCALAR, and F_L^+ is the self-dual part of the curvature of L.

see also LICHNEROWICZ-WEITZENBOCK FORMULA

References

Donaldson, S. K. "The Seiberg-Witten Equations and 4-Manifold Topology." *Bull. Amer. Math. Soc.* **33**, 45–70, 1996.

Lichnerowicz-Weitzenbock Formula

$$D^*D\psi = \nabla^*\nabla\psi + \tfrac{1}{4}R\psi,$$

where D is the Dirac operator $D : \Gamma(S^+) \to \Gamma(S^-)$, ∇ is the COVARIANT DERIVATIVE on SPINORS, and R is the CURVATURE SCALAR.

see also LICHNEROWICZ FORMULA

References

Donaldson, S. K. "The Seiberg-Witten Equations and 4-Manifold Topology." *Bull. Amer. Math. Soc.* **33**, 45–70, 1996.

Lichtenfels Surface

A MINIMAL SURFACE given by the parametric equation

$$x = \Re\left[\sqrt{2}\cos(\tfrac{1}{3}\zeta)\sqrt{\cos(\tfrac{2}{3}\zeta)}\right]$$
$$y = \Re\left[-\sqrt{2}\cos(\tfrac{1}{3}\zeta)\sqrt{\cos(\tfrac{2}{3}\zeta)}\right]$$
$$z = \Re\left[-\tfrac{1}{3}\sqrt{2}i\int_0^t \frac{d\zeta}{\sqrt{\cos(\tfrac{2}{3}\zeta)}}\right].$$

References

do Carmo, M. P. "The Helicoid." §3.5F in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 47, 1986.

Lichtenfels, O. von. "Notiz über eine transcendente Minimalfläche." *Sitzungsber. Kaiserl. Akad. Wiss. Wien* **94**, 41–54, 1889.

Lie Algebra

A NONASSOCIATIVE ALGEBRA obeyed by objects such as the LIE BRACKET and POISSON BRACKET. Elements f, g, and h of a Lie algebra satisfy

$$[f,g] = -[g,f], \tag{1}$$

$$[f+g,h] = [f,h] + [g,h], \tag{2}$$

and

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0 \tag{3}$$

(the JACOBI IDENTITY), and are *not* ASSOCIATIVE. The binary operation of a Lie algebra is the bracket

$$[fg,h] = f[g,h] + g[f,h]. \tag{4}$$

see also JACOBI IDENTITIES, LIE ALGEBROID, LIE BRACKET, IWASAWA'S THEOREM, POISSON BRACKET

References

Jacobson, N. *Lie Algebras.* New York: Dover, 1979.

Lie Algebroid

The infinitesimal algebraic object associated with a LIE GROUPOID. A Lie algebroid over a MANIFOLD B is a VECTOR BUNDLE A over B with a LIE ALGEBRA structure [,] (LIE BRACKET) on its SPACE of smooth sections together with its ANCHOR ρ.

see also LIE ALGEBRA

References

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Lie Bracket

The commutation operation

$$[a,b] = ab - ba$$

corresponding to the LIE PRODUCT.

see also LAGRANGE BRACKET, POISSON BRACKET

Lie Commutator

see LIE PRODUCT

Lie Derivative

$$\mathcal{L}_x T^{ab} \equiv \lim_{\delta u \to 0} \frac{T^{ab}(x') - T'^{ab}(x)}{\delta u}.$$

Lie Group

A continuous GROUP with an infinite number of elements such that the parameters of a product element are ANALYTIC FUNCTIONS. Lie groups are also C^∞ MANIFOLDS with the restriction that the group operation maps a C^∞ map of the MANIFOLD into itself. Examples include O_3, $SU(n)$, and the LORENTZ GROUP.

see also COMPACT GROUP, LIE ALGEBRA, LIE GROUPOID, LIE-TYPE GROUP, NIL GEOMETRY, SOL GEOMETRY

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Lie Groupoid

A GROUPOID G over B for which G and B are differentiable manifolds and α, β, and multiplication are differentiable maps. Furthermore, the derivatives of α and β are required to have maximal RANK everywhere. Here, α and β are maps from G onto $\mathbb{R}^2$ with $\alpha : (x, \gamma, y) \mapsto x$ and $\beta : (x, \gamma, y) \mapsto y$.

see also LIE ALGEBROID, NILPOTENT LIE GROUP, SEMISIMPLE LIE GROUP, SOLVABLE LIE GROUP

References

Weinstein, A. "Groupoids: Unifying Internal and External Symmetry." *Not. Amer. Math. Soc.* **43**, 744–752, 1996.

Lie Product

The multiplication operation corresponding to the LIE BRACKET.

Lie-Type Group

A finite analog of LIE GROUPS. The Lie-type groups include the CHEVALLEY GROUPS [$PSL(n, q)$, $PSU(n, q)$, $PSp(2n, q)$, $P\Omega^\epsilon(n, q)$], TWISTED CHEVALLEY GROUPS, and the TITS GROUP.

see also CHEVALLEY GROUPS, FINITE GROUP, LIE GROUP, LINEAR GROUP, ORTHOGONAL GROUP, SIMPLE GROUP, SYMPLECTIC GROUP, TITS GROUP, TWISTED CHEVALLEY GROUPS, UNITARY GROUP

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#lie`.

Liebmann's Theorem

A SPHERE is RIGID.

see also RIGID

References

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O'Neill, B. *Elementary Differential Geometry, 2nd ed.* New York: Academic Press, p. 262, 1997.

Life

The most well-known CELLULAR AUTOMATON, invented by John Conway and popularized in Martin Gardner's *Scientific American* column starting in October 1970. The game was originally played (i.e., successive generations were produced) by hand with counters, but implementation on a computer greatly increased the ease of exploring patterns.

The Life AUTOMATON is run by placing a number of filled cells on a 2-D grid. Each generation then switches cells on or off depending on the state of the cells that surround it. The rules are defined as follows. All eight of the cells surrounding the current one are checked to see if they are on or not. Any cells that are on are counted, and this count is then used to determine what will happen to the current cell.

1. Death: if the count is less than 2 or greater than 3, the current cell is switched off.
2. Survival: if (a) the count is exactly 2, or (b) the count is exactly 3 and the current cell is on, the current cell is left unchanged.
3. Birth: if the current cell is off and the count is exactly 3, the current cell is switched on.

Hensel gives a Java applet (`http://www.mindspring.com/~alanh/life/`) implementing the Game of Life on his web page.

A pattern which does not change from one generation to the next is known as a Still Life, and is said to have period 1. Conway originally believed that no pattern could produce an infinite number of cells, and offered a \$50 prize to anyone who could find a counterexample before the end of 1970 (Gardner 1983, p. 216). Many counterexamples were subsequently found, including Guns and Puffer Trains.

A Life pattern which has no Father Pattern is known as a Garden of Eden (for obvious biblical reasons). The first such pattern was not found until 1971, and at least 3 are now known. It is not, however, known if a pattern exists which has a Father Pattern, but no Grandfather Pattern (Gardner 1983, p. 249).

Rather surprisingly, Gosper and J. H. Conway independently showed that Life can be used to generate a UNIVERSAL TURING MACHINE (Berlekamp *et al.* 1982, Gardner 1983, pp. 250–253).

Similar CELLULAR AUTOMATON games with different rules are HASHLIFE, HEXLIFE, and HIGHLIFE.

see also CELLULAR AUTOMATON, HASHLIFE, HEXLIFE, HIGHLIFE

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Life Expectancy

An l_x table is a tabulation of numbers which is used to calculate life expectancies.

x	n_x	d_x	l_x	q_x	L_x	T_x	e_x
0	1000	200	1.00	0.20	0.90	2.70	2.70
1	800	100	0.80	0.12	0.75	1.80	2.25
2	700	200	0.70	0.29	0.60	1.05	1.50
3	500	300	0.50	0.60	0.35	0.45	0.90
4	200	200	0.20	1.00	0.10	0.10	0.50
5	0	0	0.00	—	0.00	0.00	—
$\sum$		1000	2.70				

x : Age category ($x = 0, 1, \ldots, k$). These values can be in any convenient units, but must be chosen so that no observed lifespan extends past category $k-1$.

n_x : Census size, defined as the number of individuals in the study population who survive to the beginning of age category x. Therefore, $n_0 = N$ (the total population size) and $n_k = 0$.

d_x : $= n_x - n_{x+1}$; $\sum_{i=0}^k d_i = n_0$. Crude death rate, which measures the number of individuals who die within age category x.

l_x : $= n_x/n_0$. Survivorship, which measures the *proportion* of individuals who survive to the beginning of age category x.

q_x : $= d_x/n_x$; $q_{k-1} = 1$. Proportional death rate, or "risk," which measures the proportion of individuals surviving to the beginning of age category x who die within that category.

L_x : $= (l_x + l_{x+1})/2$. Midpoint survivorship, which measures the proportion of individuals surviving to the *midpoint* of age category x. Note that the simple averaging formula must be replaced by a more complicated expression if survivorship is nonlinear within age categories. The sum $\sum_{i=0}^k L_x$ gives the total number of age categories lived by the entire study population.

T_x : $= T_{x-1} - L_{x-1}$; $T_0 = \sum_{i=0}^k L_x$. Measures the total number of age categories left to be lived by all individuals who survive to the beginning of age category x.

e_x : $= T_x/l_x$; $e_{k-1} = 1/2$. Life expectancy, which is the mean number of age categories remaining until death for individuals surviving to the beginning of age category x.

For all x, $e_{x+1} + 1 > e_x$. This means that the total expected lifespan increases monotonically. For instance, in the table above, the one-year-olds have an average age at death of $2.25 + 1 = 3.25$, compared to 2.70 for newborns. In effect, the age of death of older individuals is a distribution conditioned on the fact that they have survived to their present age.

It is common to study survivorship as a semilog plot of l_x vs. x, known as a SURVIVORSHIP CURVE. A so-called $l_x m_x$ table can be used to calculate the mean generation time of a population. Two $l_x m_x$ tables are illustrated below.

Population 1

x	l_x	m_x	$l_x m_x$	$x l_x m_x$
0	1.00	0.00	0.00	0.00
1	0.70	0.50	0.35	0.35
2	0.50	1.50	0.75	1.50
3	0.20	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00
			$R_0 = 1.10$	$\sum = 1.85$

$$T = \frac{\sum x l_x m_x}{\sum l_x m_x} = \frac{1.85}{1.10} = 1.68$$

$$r = \frac{\ln R_0}{T} = \frac{\ln 1.10}{1.68} = 0.057.$$

Population 2

x	l_x	m_x	$l_x m_x$	$x l_x m_x$
0	1.00	0.00	0.00	0.00
1	0.70	0.00	0.00	0.00
2	0.50	2.00	1.00	2.00
3	0.20	0.50	0.10	0.30
4	0.00	0.00	0.00	0.00
			$R_0 = 1.10$	$\sum = 2.30$

$$T = \frac{\sum x l_x m_x}{\sum l_x m_x} = \frac{2.30}{1.10} = 2.09$$
$$r = \frac{\ln R_0}{T} = \frac{\ln 1.10}{2.09} = 0.046.$$

x : Age category ($x = 0, 1, \ldots, k$). These values can be in any convenient units, but must be chosen so that no observed lifespan extends past category $k - 1$ (as in an l_x table).

l_x : $= n_x/n_0$. Survivorship, which measures the *proportion* of individuals who survive to the beginning of age category x (as in an l_x table).

m_x : The average number of offspring produced by an individual in age category x *while in that age category.* $\sum_{i=0}^{k} m_x$ therefore represents the average lifetime number of offspring produced by an individual of maximum lifespan.

$l_x m_x$: The average number of offspring produced by an individual within age category x weighted by the probability of surviving to the beginning of that age category. $\sum_{i=0}^{k} l_x m_x$ therefore represents the average lifetime number of offspring produced by a member of the study population. It is called the net reproductive rate per generation and is often denoted R_0.

$x l_x m_x$: A column weighting the offspring counted in the previous column by their parents' age when they were born. Therefore, the ratio $T = \sum(x l_x m_x)/\sum(l_x m_x)$ is the mean generation time of the population.

The MALTHUSIAN PARAMETER r measures the reproductive rate per unit time and can be calculated as $r = (\ln R_0)/T$. For an exponentially increasing population, the population size $N(t)$ at time t is then given by

$$N(t) = N_0 e^{rt}.$$

In the above two tables, the populations have identical reproductive rates of $R_0 = 1.10$. However, the shift toward later reproduction in population 2 increases the generation time, thus slowing the rate of POPULATION GROWTH. Often, a slight delay of reproduction decreases POPULATION GROWTH more strongly than does even a fairly large reduction in reproductive rate.

see also GOMPERTZ CURVE, LOGISTIC GROWTH CURVE, MAKEHAM CURVE, MALTHUSIAN PARAMETER, POPULATION GROWTH, SURVIVORSHIP CURVE

Lift

Given a MAP f from a SPACE X to a SPACE Y and another MAP g from a SPACE Z to a SPACE Y, a lift is a MAP h from X to Z such that $gh = f$. In other words, a lift of f is a MAP h such that the diagram (shown below) commutes.

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[ur, dashed, "h"] \arrow[r, "f"] & Y \end{tikzcd}$$

If f is the identity from Y to Y, a MANIFOLD, and if g is the bundle projection from the TANGENT BUNDLE to Y, the lifts are precisely VECTOR FIELDS. If g is a bundle projection from any FIBER BUNDLE to Y, then lifts are precisely sections. If f is the identity from Y to Y, a MANIFOLD, and g a projection from the orientation double cover of Y, then lifts exist IFF Y is an orientable MANIFOLD.

If f is a MAP from a CIRCLE to Y, an n-MANIFOLD, and g the bundle projection from the FIBER BUNDLE of alternating n-FORMS on Y, then lifts always exist IFF Y is orientable. If f is a MAP from a region in the COMPLEX PLANE to the COMPLEX PLANE (complex analytic), and if g is the exponential MAP, lifts of f are precisely LOGARITHMS of f.

see also LIFTING PROBLEM

Lifting Problem

Given a MAP f from a SPACE X to a SPACE Y and another MAP g from a SPACE Z to a SPACE Y, does there exist a MAP h from X to Z such that $gh = f$? If such a map h exists, then h is called a LIFT of f.

see also EXTENSION PROBLEM, LIFT

Ligancy

see KISSING NUMBER

Likelihood

The hypothetical PROBABILITY that an event which has already occurred would yield a specific outcome. The concept differs from that of a probability in that a probability refers to the occurrence of future events, while a likelihood refers to past events with known outcomes.

see also LIKELIHOOD RATIO, MAXIMUM LIKELIHOOD, NEGATIVE LIKELIHOOD RATIO, PROBABILITY

Likelihood Ratio

A quantity used to test NESTED HYPOTHESES. Let H' be a NESTED HYPOTHESIS with n' DEGREES OF FREEDOM within H (which has n DEGREES OF FREEDOM), then calculate the MAXIMUM LIKELIHOOD of a given outcome, first given H', then given H. Then

$$\mathrm{LR} = \frac{[\text{likelihood H'}]}{[\text{likelihood H}]}.$$

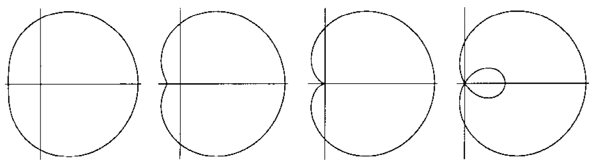
Comparison of this ratio to the critical value of the CHI-SQUARED DISTRIBUTION with $n - n'$ DEGREES OF FREEDOM then gives the SIGNIFICANCE of the increase in LIKELIHOOD.

The term likelihood ratio is also used (especially in medicine) to test nonnested complementary hypotheses as follows,

$$\text{LR} = \frac{[\text{true positive rate}]}{[\text{false positive rate}]} = \frac{[\text{sensitivity}]}{1 - [\text{specificity}]}.$$

see also NEGATIVE LIKELIHOOD RATIO, SENSITIVITY, SPECIFICITY

Limaçon



The limaçon is a polar curve of the form

$$r = b + a\cos\theta$$

also called the LIMAÇON OF PASCAL. It was first investigated by Dürer, who gave a method for drawing it in *Underweysung der Messung* (1525). It was rediscovered by Étienne Pascal, father of Blaise Pascal, and named by Gilles-Personne Roberval in 1650 (MacTutor Archive). The word "limaçon" comes from the Latin *limax*, meaning "snail."

If $b \geq 2a$, we have a convex limaçon. If $2a > b > a$, we have a dimpled limaçon. If $b = a$, the limaçon degenerates to a CARDIOID. If $b < a$, we have limaçon with an inner loop. If $b = a/2$, it is a TRISECTRIX (but *not* the MACLAURIN TRISECTRIX) with inner loop of AREA

$$A_{\text{inner loop}} = \tfrac{1}{4}a^2\left(\pi - 3\sqrt{\frac{3}{2}}\right),$$

and AREA between the loops of

$$A_{\text{between loops}} = \tfrac{1}{4}a^2(\pi + 3\sqrt{3})$$

(MacTutor Archive). The limaçon is an ANALLAGMATIC CURVE, and is also the CATACAUSTIC of a CIRCLE when the RADIANT POINT is a finite (NONZERO) distance from the CIRCUMFERENCE, as shown by Thomas de St. Laurent in 1826 (MacTutor Archive).

see also CARDIOID

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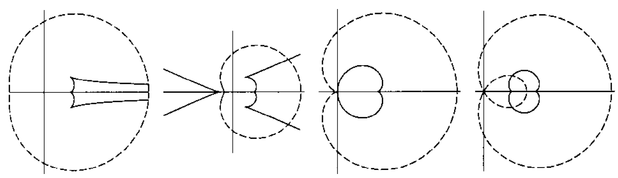
Limaçon Evolute



The CATACAUSTIC of a CIRCLE for a RADIANT POINT is the limaçon evolute. It has parametric equations

$$x = \frac{a[4a^2 + 4b^2 + 9ab\cos t - ab\cos(3t)]}{4(2a^2 + b^2 + 3ab\cos t)}$$

$$y = \frac{a^2 b\sin^3 t}{2a^2 + b^2 + 3ab\cos t}.$$

Limaçon of Pascal

see LIMAÇON

Limit

A function $f(z)$ is said to have a limit $\lim_{z\to a} f(z) = c$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - c| < \epsilon$ whenever $0 < |z - a| < \delta$.

A LOWER LIMIT

$$\text{lower} \lim_{n\to\infty} S_n = \underline{\lim}_{n\to\infty} S_n = h$$

is said to exist if, for every $\epsilon > 0$, $|S_n - h| < \epsilon$ for infinitely many values of n and if no number less than h has this property.

An UPPER LIMIT

$$\text{upper} \lim_{n\to\infty} S_n = \overline{\lim_{n\to\infty}} S_n = k$$

is said to exist if, for every $\epsilon > 0$, $|S_n - k| < \epsilon$ for infinitely many values of n and if no number larger than k has this property.

Indeterminate limit forms of types ∞/∞ and $0/0$ can be computed with L'HOSPITAL'S RULE. Types $0 \cdot \infty$ can be converted to the form $0i/0$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}.$$

Types 0^0, ∞^0, and 1^∞ are treated by introducing a dependent variable $y = f(x)g(x)$, then calculating $\lim \ln y$. The original limit then equals $e^{\lim \ln y}$.

see also CENTRAL LIMIT THEOREM, CONTINUOUS, DISCONTINUITY, L'HOSPITAL'S RULE, LOWER LIMIT, UPPER LIMIT

References
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Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be two SERIES with POSITIVE terms and suppose

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \rho.$$

If ρ is finite and $\rho > 0$, then the two SERIES both CONVERGE or DIVERGE.

see also CONVERGENCE TESTS

Limit Cycle

An attracting set to which orbits or trajectories converge and upon which trajectories are periodic.

see also HOPF BIFURCATION

Limit Point

A number x such that for all $\epsilon > 0$, there exists a member of the SET y different from x such that $|y - x| < \epsilon$. The topological definition of limit point P of A is that P is a point such that every OPEN SET around it intersects A.

see also CLOSED SET, OPEN SET

References
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 25–26, 1991.

Lin's Method

An ALGORITHM for finding ROOTS for QUARTIC EQUATIONS with COMPLEX ROOTS.

References
Acton, F. S. *Numerical Methods That Work, 2nd printing.* Washington, DC: Math. Assoc. Amer., pp. 198–199, 1990.

Lindeberg Condition

A SUFFICIENT condition on the LINDEBERG-FELLER CENTRAL LIMIT THEOREM. Given random variates X_1, X_2, ..., let $\langle X_i \rangle = 0$, the VARIANCE ${\sigma_i}^2$ of X_i be finite, and VARIANCE of the distribution consisting of a sum of X_is

$$S_n \equiv X_1 + X_2 + \ldots + X_n \tag{1}$$

be

$${s_n}^2 \equiv \sum_{i=1}^{n} {\sigma_i}^2. \tag{2}$$

Let

$$\Lambda_n(\epsilon) \equiv \sum_{k=1}^{n} \left\langle \left(\frac{X_k}{s_n}\right)^2 \middle| \frac{|X_k|}{s_n} \geq \epsilon \right\rangle, \tag{3}$$

then the Lindeberg condition is

$$\lim_{n\to\infty} \Lambda_n(\epsilon) = 0 \tag{4}$$

for all $\epsilon > 0$.

see also FELLER-LÉVY CONDITION

References
Zabell, S. L. "Alan Turing and the Central Limit Theorem." *Amer. Math. Monthly* **102**, 483–494, 1995.

Lindeberg-Feller Central Limit Theorem

If the random variates X_1, X_2, ... satisfy the LINDEBERG CONDITION, then for all $a < b$,

$$\lim_{n\to\infty} P\left(a < \frac{S_n}{s_n} < b\right) = \Phi(b) - \Phi(a),$$

where Φ is the NORMAL DISTRIBUTION FUNCTION.

see also CENTRAL LIMIT THEOREM, FELLER-LÉVY CONDITION, NORMAL DISTRIBUTION FUNCTION

References
Zabell, S. L. "Alan Turing and the Central Limit Theorem." *Amer. Math. Monthly* **102**, 483–494, 1995.

Lindelof's Theorem

The SURFACE OF REVOLUTION generated by the external CATENARY between a fixed point a and its conjugate on the ENVELOPE of the CATENARY through the fixed point is equal in AREA to the surface of revolution generated by its two Lindelof TANGENTS, which cross the axis of rotation at the point a and are calculable from the position of the points and CATENARY.

see also CATENARY, ENVELOPE, SURFACE OF REVOLUTION

Lindemann-Weierstraß Theorem

If $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, then e^{α_1}, ..., e^{α_n} are algebraically independent over $\mathbb{Q}$.

see also HERMITE-LINDEMANN THEOREM

Lindenmayer System

A STRING REWRITING system which can be used to generate FRACTALS with DIMENSION between 1 and 2. The term L-SYSTEM is often used as an abbreviation.

see also ARROWHEAD CURVE, DRAGON CURVE EXTERIOR SNOWFLAKE, FRACTAL, HILBERT CURVE, KOCH SNOWFLAKE, PEANO CURVE, PEANO-GOSPER CURVE, SIERPIŃSKI CURVE, STRING REWRITING

References
Dickau, R. M. "Two-dimensional L-systems." `http://forum.swarthmore.edu/advanced/robertd/lsys2d.html`.
Prusinkiewicz, P. and Hanan, J. *Lindenmayer Systems, Fractal, and Plants.* New York: Springer-Verlag, 1989.

Prusinkiewicz, P. and Lindenmayer, A. *The Algorithmic Beauty of Plants.* New York: Springer-Verlag, 1990.
Stevens, R. T. *Fractal Programming in C.* New York: Holt, 1989.
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Line

Euclid defined a line as a "breadthless length," and a straight line as a line which "lies evenly with the points on itself" (Kline 1956, Dunham 1990). Lines are intrinsically 1-dimensional objects, but may be embedded in higher dimensional SPACES. An infinite line passing through points A and B is denoted $\overleftrightarrow{AB}$. A LINE SEGMENT terminating at these points is denoted $\overline{AB}$. A line is sometimes called a STRAIGHT LINE or, more archaically, a RIGHT LINE (Casey 1893), to emphasize that it has no curves anywhere along its length.

Consider first lines in a 2-D PLANE. The line with x-INTERCEPT a and y-INTERCEPT b is given by the *intercept form*

$$\frac{x}{a}+\frac{y}{b}=1. \qquad (1)$$

The line through (x_1, y_1) with SLOPE m is given by the *point-slope form*

$$y-y_1=m(x-x_1). \qquad (2)$$

The line with y-intercept b and slope m is given by the *slope-intercept form*

$$y=mx+b. \qquad (3)$$

The line through (x_1, y_1) and (x_2, y_2) is given by the *two point form*

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \qquad (4)$$

Other forms are

$$a(x-x_1)+b(y-y_1)=0 \qquad (5)$$

$$ax+by+c=0 \qquad (6)$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0. \qquad (7)$$

A line in 2-D can also be represented as a VECTOR. The VECTOR along the line

$$ax+by=0 \qquad (8)$$

is given by

$$t\begin{bmatrix} -b \\ a \end{bmatrix}, \qquad (9)$$

where $t \in \mathbb{R}$. Similarly, VECTORS of the form

$$t\begin{bmatrix} a \\ b \end{bmatrix} \qquad (10)$$

are PERPENDICULAR to the line. Three points lie on a line if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0. \qquad (11)$$

The ANGLE between lines

$$A_1x+B_1y+C_1=0 \qquad (12)$$

$$A_2x+B_2y+C_2=0 \qquad (13)$$

is

$$\tan\theta=\frac{A_1B_2-A_2B_1}{A_1A_2+B_1B_2}. \qquad (14)$$

The line joining points with TRILINEAR COORDINATES $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$ is the set of point $\alpha : \beta : \gamma$ satisfying

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}=0 \qquad (15)$$

$$(\beta_1\gamma_2-\gamma_1\beta_2)\alpha+(\gamma_1\alpha_2-\alpha_1\gamma_2)\beta+(\alpha_1\beta_2-\beta_1\alpha_2)\gamma=0. \qquad (16)$$

Three lines CONCUR if their TRILINEAR COORDINATES satisfy

$$l_1\alpha+m_1\beta+n_1\gamma=0 \qquad (17)$$

$$l_2\alpha+m_2\beta+n_2\gamma=0 \qquad (18)$$

$$l_3\alpha+m_3\beta+n_3\gamma=0, \qquad (19)$$

in which case the point is

$$m_2n_3-n_2m_3 : n_2l_3-l_2n_3 : l_2m_3-m_2l_3, \qquad (20)$$

or if the COEFFICIENTS of the lines

$$A_1x+B_1y+C_1=0 \qquad (21)$$

$$A_2x+B_2y+C_2=0 \qquad (22)$$

$$A_3x+B_3y+C_3=0 \qquad (23)$$

satisfy

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=0. \qquad (24)$$

Two lines CONCUR if their TRILINEAR COORDINATES satisfy

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0. \qquad (25)$$

The line through P_1 is the direction (a_1, b_1, c_1) and the line through P_2 in direction (a_2, b_2, c_2) intersect IFF

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0. \tag{26}$$

The line through a point $\alpha' : \beta' : \gamma'$ PARALLEL to

$$l\alpha + m\beta + n\gamma = 0 \tag{27}$$

is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ bn - cm & cl - an & am - bl \end{vmatrix} = 0. \tag{28}$$

The lines

$$l\alpha + m\beta + n\gamma = 0 \tag{29}$$

$$l'\alpha + m'\beta + n'\gamma = 0 \tag{30}$$

are PARALLEL if

$$a(mn' - nm') + b(nl' - ln') + c(lm' - ml') = 0 \tag{31}$$

for all (a, b, c), and PERPENDICULAR if

$$\begin{aligned} 2abc(ll' + mm' + nn') - (mn' + m'm)\cos A \\ -(nl' + n'l)\cos B - (lm' + l'm)\cos C = 0 \end{aligned} \tag{32}$$

for all (a, b, c) (Sommerville 1924). The line through a point $\alpha' : \beta' : \gamma'$ PERPENDICULAR to (32) is given by

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ l - m\cos C & m - n\cos A & n - l\cos B \\ -n\cos B & -l\cos C & -m\cos A \end{vmatrix} = 0. \tag{33}$$

In 3-D SPACE, the line passing through the point (x_0, y_0, z_0) and PARALLEL to the NONZERO VECTOR

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{34}$$

has parametric equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases} \tag{35}$$

see also ASYMPTOTE, BROCARD LINE, COLLINEAR, CONCUR, CRITICAL LINE, DESARGUES' THEOREM, ERDŐS-ANNING THEOREM, LINE SEGMENT, ORDINARY LINE, PENCIL, POINT, POINT-LINE DISTANCE—2-D, POINT-LINE DISTANCE—3-D, PLANE, RANGE (LINE SEGMENT), RAY, SOLOMON'S SEAL LINES, STEINER SET, STEINER'S THEOREM, SYLVESTER'S LINE PROBLEM

References

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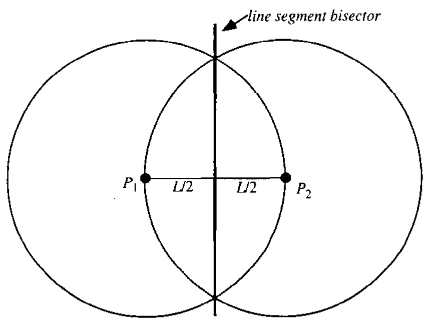
Line Bisector



The line bisecting a given LINE SEGMENT P_1P_2 can be constructed geometrically, as illustrated above.

References

Courant, R. and Robbins, H. "How to Bisect a Segment and Find the Center of a Circle with the Compass Alone." §3.4.4 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 145–146, 1996.

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Line of Curvature

A curve on a surface whose tangents are always in the direction of PRINCIPAL CURVATURE. The equation of the lines of curvature can be written

$$\begin{vmatrix} g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \\ du^2 & -du\,dv & dv^2 \end{vmatrix} = 0,$$

where g and b are the COEFFICIENTS of the first and second FUNDAMENTAL FORMS.

see also DUPIN'S THEOREM, FUNDAMENTAL FORMS, PRINCIPAL CURVATURES

Line Element

Also known as the first FUNDAMENTAL FORM

$$ds^2 = g_{ab}\,dx^a\,dx^b.$$

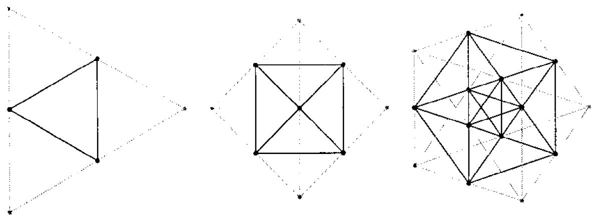
In the principal axis frame for 3-D,

$$ds^2 = g_{aa}(dx^a)^2 + g_{bb}(dx^b)^2 + g_{cc}(dx^c)^2.$$

At ORDINARY POINTS on a surface, the line element is positive definite.

see also AREA ELEMENT, FUNDAMENTAL FORMS, VOLUME ELEMENT

Line Graph



A LINE GRAPH $L(G)$ (also called an INTERCHANGE GRAPH) of a graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge IFF the corresponding edges of G meet at one or both endpoints. In the three examples above, the original graphs are the COMPLETE GRAPHS K_3, K_4, and K_5 shown in gray, and their line graphs are shown in black.

References

Saaty, T. L. and Kainen, P. C. "Line Graphs." §4-3 in *The Four-Color Problem: Assaults and Conquest.* New York: Dover, pp. 108–112, 1986.

Line at Infinity

The straight line on which all POINTS AT INFINITY lie. The line at infinity is given in terms of TRILINEAR COORDINATES by

$$a\alpha + b\beta + c\gamma = 0,$$

which follows from the fact that a REAL TRIANGLE will have POSITIVE AREA, and therefore that

$$2\Delta = a\alpha + b\beta + c\gamma > 0.$$

Instead of the three reflected segments concurring for the ISOGONAL CONJUGATE of a point X on the CIRCUMCIRCLE of a TRIANGLE, they become parallel (and can be considered to meet at infinity). As X varies around the CIRCUMCIRCLE, X^{-1} varies through a line called the line at infinity. Every line is PERPENDICULAR to the line at infinity.

see also POINT AT INFINITY

Line Integral

The line integral on a curve $\boldsymbol{\sigma}$ is defined by

$$\int_{\boldsymbol{\sigma}} \mathbf{F}\cdot d\mathbf{s} = \int_a^b \mathbf{F}(\boldsymbol{\sigma}(t))\cdot\boldsymbol{\sigma}'(t)\,dt \tag{1}$$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz, \tag{2}$$

where

$$\mathbf{F} \equiv \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}. \tag{3}$$

If $\nabla\cdot\mathbf{F} = 0$ (i.e., it is a DIVERGENCELESS FIELD), then the line integral is path independent and

$$\int_{(a,b,c)}^{(x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int_{(a,b,c)}^{(x,b,c)} F_1\,dx + \int_{(x,b,c)}^{(x,y,c)} F_2\,dy + \int_{(x,y,c)}^{(x,y,z)} F_3\,dz. \tag{4}$$

For z COMPLEX, $\gamma : z = z(t)$, and $t \in [a,b]$,

$$\int_\gamma f\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{5}$$

see also CONTOUR INTEGRAL, PATH INTEGRAL

Line Segment

A closed interval corresponding to a FINITE portion of an infinite LINE. Line segments are generally labelled with two letters corresponding to their endpoints, say A and B, and then written AB. The length of the line segment is indicated with an overbar, so the length of the line segment AB would be written $\overline{AB}$.

Curiously, the number of points in a line segment (ALEPH-1; $\aleph_1$) is equal to that in an entire 1-D SPACE (a LINE), and also to the number of points in an n-D SPACE, as first recognized by Georg Cantor.

see also ALEPH-1 ($\aleph_1$), COLLINEAR, CONTINUUM, LINE, RAY

Line Space

see LIOUVILLE SPACE

Linear Algebra

The study of linear sets of equations and their transformation properties. Linear algebra allows the analysis of ROTATIONS in space, LEAST SQUARES FITTING, solution of coupled differential equations, determination of a circle passing through three given points, as well as many other other problems in mathematics, physics, and engineering.

The MATRIX and DETERMINANT are extremely useful tools of linear algebra. One central problem of linear algebra is the solution of the matrix equation

$$\mathsf{A}\mathbf{x} = \mathbf{b}$$

for $\mathbf{x}$. While this can, in theory, be solved using a MATRIX INVERSE

$$\mathbf{x} = \mathsf{A}^{-1}\mathbf{b},$$

other techniques such as GAUSSIAN ELIMINATION are numerically more robust.

see also CONTROL THEORY, CRAMER'S RULE, DETERMINANT, GAUSSIAN ELIMINATION, LINEAR TRANSFORMATION, MATRIX, VECTOR

References

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Linear Approximation

A linear approximation to a function $f(x)$ at a point x_0 can be computed by taking the first term in the TAYLOR SERIES

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \ldots.$$

see also MACLAURIN SERIES, TAYLOR SERIES

Linear Code

A linear code over a FINITE FIELD with q elements F_q is a linear SUBSPACE $C \subset {F_q}^n$. The vectors forming the SUBSPACE are called code words. When code words are chosen such that the distance between them is maximized, the code is called error-correcting since slightly garbled vectors can be recovered by choosing the nearest code word.

see also CODE, CODING THEORY, ERROR-CORRECTING CODE, GRAY CODE, HUFFMAN CODING, ISBN

Linear Congruence

A linear congruence

$$ax \equiv b \pmod{m}$$

is solvable IFF the CONGRUENCE

$$b \equiv 0 \pmod{(a, m)}$$

is solvable, where $d \equiv (a, m)$ is the GREATEST COMMON DIVISOR, in which case the solutions are x_0, $x_0 + m/d$, $x_0 + 2m/d$, ..., $x_0 + (d-1)m/d$, where $x_0 < m/d$. If $d = 1$, then there is only one solution.

see also CONGRUENCE, QUADRATIC CONGRUENCE

Linear Congruence Method

A METHOD for generating RANDOM (PSEUDORANDOM) numbers using the linear RECURRENCE RELATION

$$X_{n+1} = aX_n + c \pmod{m},$$

where a and c must assume certain fixed values and X_0 is an initial number known as the SEED.

see also PSEUDORANDOM NUMBER, RANDOM NUMBER, SEED

References

Pickover, C. A. "Computers, Randomness, Mind, and Infinity." Ch. 31 in *Keys to Infinity.* New York: W. H. Freeman, pp. 233–247, 1995.

Linear Equation

An algebraic equation of the form

$$y = ax + b$$

involving only a constant and a first-order (linear) term.

see also LINE, POLYNOMIAL, QUADRATIC EQUATION

Linear Equation System

When solving a system of n linear equations with $k > n$ unknowns, use MATRIX operations to solve the system as far as possible. Then solve for the first $(k - n)$ components in terms of the last n components to find the solution space.

Linear Extension

A linear extension of a PARTIALLY ORDERED SET P is a PERMUTATION of the elements p_1, p_2, ... of P such that $i < j$ IMPLIES $p_i < p_j$. For example, the linear extensions of the PARTIALLY ORDERED SET $((1, 2), (3, 4))$ are 1234, 1324, 1342, 3124, 3142, and 3412, all of which have 1 before 2 and 3 before 4.

References

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Linear Fractional Transformation

see MÖBIUS TRANSFORMATION

Linear Group

see GENERAL LINEAR GROUP, LIE-TYPE GROUP, PROJECTIVE GENERAL LINEAR GROUP, PROJECTIVE SPECIAL LINEAR GROUP, SPECIAL LINEAR GROUP

References
Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#lin`.

Linear Group Theorem

Any linear system of point-groups on a curve with only ordinary singularities may be cut by ADJOINT CURVES.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 122 and 251, 1959.

Linear Operator

An operator $\tilde{L}$ is said to be linear if, for every pair of functions f and g and SCALAR t,

$$\tilde{L}(f+g) = \tilde{L}f + \tilde{L}g$$

and

$$\tilde{L}(tf) = t\tilde{L}f.$$

see also LINEAR TRANSFORMATION, OPERATOR

Linear Ordinary Differential Equation

see ORDINARY DIFFERENTIAL EQUATION—FIRST-ORDER, ORDINARY DIFFERENTIAL EQUATION—SECOND-ORDER

Linear Programming

The problem of maximizing a linear function over a convex polyhedron, also known as OPERATIONS RESEARCH, OPTIMIZATION THEORY, or CONVEX OPTIMIZATION THEORY. It can be solved using the SIMPLEX METHOD (Wood and Dantzig 1949, Dantzig 1949) which runs along EDGES of the visualization solid to find the best answer.

In 1979, L. G. Khachian found a $\mathcal{O}(x^5)$ POLYNOMIAL-time ALGORITHM. A much more efficient POLYNOMIAL-time ALGORITHM was found by Karmarkar (1984). This method goes through the middle of the solid and then transforms and warps, and offers many advantages over the simplex method.

see also CRISS-CROSS METHOD, ELLIPSOIDAL CALCULUS, KUHN-TUCKER THEOREM, LAGRANGE MULTIPLIER, VERTEX ENUMERATION

References
Bellman, R. and Kalaba, R. *Dynamic Programming and Modern Control Theory.* New York: Academic Press, 1965.
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Yudin, D. B. and Nemirovsky, A. S. *Problem Complexity and Method Efficiency in Optimization.* New York: Wiley, 1983.

Linear Recurrence Sequence

see RECURRENCE SEQUENCE

Linear Regression

The fitting of a straight LINE through a given set of points according to some specified goodness-of-fit criterion. The most common form of linear regression is LEAST SQUARES FITTING.

see also LEAST SQUARES FITTING, MULTIPLE REGRESSION, NONLINEAR LEAST SQUARES FITTING

References
Edwards, A. L. *An Introduction to Linear Regression and Correlation.* San Francisco, CA: W. H. Freeman, 1976.
Edwards, A. L. *Multiple Regression and the Analysis of Variance and Covariance.* San Francisco, CA: W. H. Freeman, 1979.

Linear Space

see VECTOR SPACE

Linear Stability

Consider the general system of two first-order ORDINARY DIFFERENTIAL EQUATIONS

$$\dot{x} = f(x,y) \tag{1}$$
$$\dot{y} = g(x,y). \tag{2}$$

Let x_0 and y_0 denote FIXED POINTS with $\dot{x} = \dot{y} = 0$, so

$$f(x_0,y_0) = 0 \tag{3}$$
$$g(x_0,y_0) = 0. \tag{4}$$

Then expand about (x_0, y_0) so

$$\delta\dot{x} = f_x(x_0,y_0)\delta x + f_y(x_0,y_0)\delta y + f_{xy}(x_0,y_0)\delta x \delta y + \ldots \tag{5}$$

$$\delta\dot{y} = g_x(x_0,y_0)\delta x + g_y(x_0,y_0)\delta y + g_{xy}(x_0,y_0)\delta x \delta y + \ldots . \tag{6}$$

To first-order, this gives

$$\frac{d}{dt}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0)\\ g_x(x_0,y_0) & g_y(x_0,y_0)\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix}, \tag{7}$$

where the 2×2 MATRIX is called the STABILITY MATRIX.

In general, given an n-D MAP $\mathbf{x}' = T(\mathbf{x})$, let $\mathbf{x}_0$ be a FIXED POINT, so that

$$T(\mathbf{x}_0) = \mathbf{x}_0. \tag{8}$$

Expand about the fixed point,

$$\begin{aligned} T(\mathbf{x}_0 + \delta\mathbf{x}) &= T(\mathbf{x}_0) + \frac{\partial T}{\partial \mathbf{x}}\delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x})^2\\ &\equiv T(\mathbf{x}_0) + \delta T, \end{aligned} \tag{9}$$

so

$$\delta T = \frac{\partial T}{\partial \mathbf{x}}\delta\mathbf{x} \equiv \mathsf{A}\,\delta\mathbf{x}. \tag{10}$$

The map can be transformed into the principal axis frame by finding the EIGENVECTORS and EIGENVALUES of the MATRIX A

$$(\mathsf{A} - \lambda\mathsf{I})\,\delta\mathbf{x} = \mathbf{0}, \tag{11}$$

so the DETERMINANT

$$|\mathsf{A} - \lambda\mathsf{I}| = 0. \tag{12}$$

The mapping is

$$\delta\mathbf{x}_{\text{princ}}{}' = \begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}. \tag{13}$$

When iterated a large number of times,

$$\delta T'_{\text{princ}} \to 0 \tag{14}$$

only if $|\Re(\lambda_i)| < 1$ for $i = 1, \ldots, n$ but $\to \infty$ if any $|\lambda_i| > 1$. Analysis of the EIGENVALUES (and EIGENVECTORS) of A therefore characterizes the type of FIXED POINT. The condition for stability is $|\Re(\lambda_i)| < 1$ for $i = 1, \ldots, n$.

see also FIXED POINT, STABILITY MATRIX

References

Tabor, M. "Linear Stability Analysis." §1.4 in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 20–31, 1989.

Linear Transformation

An $n \times n$ MATRIX A is a linear transformation (linear MAP) IFF, for every pair of n-VECTORS $\mathbf{X}$ and $\mathbf{Y}$ and every SCALAR t,

$$\mathsf{A}(\mathbf{X} + \mathbf{Y}) = \mathsf{A}(\mathbf{X}) + \mathsf{A}(\mathbf{Y}) \tag{1}$$

and

$$\mathsf{A}(t\mathbf{X}) = t\mathsf{A}(\mathbf{X}). \tag{2}$$

Consider the 2-D transformation

$$\rho x_1' = a_{11}x_1 + a_{12}x_2 \tag{3}$$

$$\rho x_2' = a_{21}x_2 + a_{22}x_2. \tag{4}$$

Rescale by defining $\lambda \equiv x_1/x_2$ and $\lambda' \equiv x_1'/x_2'$, then the above equations become

$$\lambda' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}, \tag{5}$$

where $\alpha\delta - \beta\gamma \neq 0$ and α, β, γ and δ are defined in terms of the old constants. Solving for λ gives

$$\lambda = \frac{\delta\lambda' - \beta}{-\gamma\lambda' + \alpha}, \tag{6}$$

so the transformation is ONE-TO-ONE. To find the FIXED POINTS of the transformation, set $\lambda = \lambda'$ to obtain

$$\gamma\lambda^2 + (\delta - \alpha)\lambda - \beta = 0. \tag{7}$$

This gives two fixed points which may be distinct or coincident. The fixed points are classified as follows.

variables	type
$(\delta-\alpha)^2 + 4\beta\gamma > 0$	hyperbolic fixed point
$(\delta-\alpha)^2 + 4\beta\gamma < 0$	elliptic fixed point
$(\delta-\alpha)^2 + 4\beta\gamma = 0$	parabolic fixed point

see also ELLIPTIC FIXED POINT (MAP), HYPERBOLIC FIXED POINT (MAP), INVOLUNTARY, LINEAR OPERATOR, PARABOLIC FIXED POINT

References

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, pp. 13–15, 1961.

Linearly Dependent Curves

Two curves ϕ and ψ satisfying

$$\phi + \psi = 0$$

are said to be linearly dependent. Similarly, n curves ϕ_i, $i = 1, \ldots, n$ are said to be linearly dependent if

$$\sum_{i=1}^{n} \phi_i = 0.$$

see also BERTINI'S THEOREM, STUDY'S THEOREM

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 32–34, 1959.

Linearly Dependent Functions

The n functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ are linearly dependent if, for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$ not all zero,

$$c_i f_i(x) = 0 \tag{1}$$

(where EINSTEIN SUMMATION is used) for all x in some interval I. If the functions are not linearly dependent, they are said to be linearly independent. Now, if the functions $\in \mathbb{R}^{n-1}$, we can differentiate (1) up to $n-1$ times. Therefore, linear dependence also requires

$$c_i f_i' = 0 \tag{2}$$

$$c_i f_i'' = 0 \tag{3}$$

$$c_i f_i^{(n-1)} = 0, \tag{4}$$

where the sums are over $i = 1, \ldots, n$. These equations have a nontrivial solution IFF the DETERMINANT

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = 0, \tag{5}$$

where the DETERMINANT is conventionally called the WRONSKIAN and is denoted $W(f_1, f_2, \ldots, f_n)$. If the WRONSKIAN $\neq 0$ for any value c in the interval I, then the only solution possible for (2) is $c_i = 0$ ($i = 1, \ldots, n$), and the functions are linearly independent. If, on the other hand, $W = 0$ for a range, the functions are linearly dependent in the range. This is equivalent to stating that if the vectors $\mathbf{V}[f_1(c)], \ldots, \mathbf{V}[f_n(c)]$ defined by

$$\mathbf{V}[f_i(x)] = \begin{bmatrix} f_i(x) \\ f_i'(x) \\ f_i''(n) \\ \vdots \\ f_i^{(n-1)}(x) \end{bmatrix} \tag{6}$$

are linearly independent for at least one $c \in I$, then the functions f_i are linearly independent in I.

References

Sansone, G. "Linearly Independent Functions." §1.2 in *Orthogonal Functions, rev. English ed.* New York: Dover, pp. 2–3, 1991.

Linearly Dependent Vectors

n VECTORS $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are linearly dependent IFF there exist SCALARS $c_1, c_2, \ldots, c_n$ such that

$$c_i \mathbf{X}_i = 0, \tag{1}$$

where EINSTEIN SUMMATION is used and $i = 1, \ldots, n$. If no such SCALARS exist, then the vectors are said to be linearly independent. In order to satisfy the CRITERION for linear dependence,

$$c_1 \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} + c_2 \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3}$$

In order for this MATRIX equation to have a nontrivial solution, the DETERMINANT must be 0, so the VECTORS are linearly dependent if

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} = 0, \tag{4}$$

and linearly independent otherwise.

Let $\mathbf{p}$ and $\mathbf{q}$ be n-D VECTORS. Then the following three conditions are equivalent (Gray 1993).

1. $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent.
2. $\begin{vmatrix} \mathbf{p}\cdot\mathbf{p} & \mathbf{p}\cdot\mathbf{q} \\ \mathbf{q}\cdot\mathbf{p} & \mathbf{q}\cdot\mathbf{q} \end{vmatrix} = 0.$
3. The $2 \times n$ MATRIX $\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}$ has rank less than two.

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 186–187, 1993.

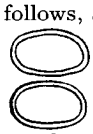
Linearly Independent

Two or more functions, equations, or vectors which are not linearly dependent are said to be linearly independent.

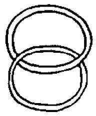
see also LINEARLY DEPENDENT CURVES, LINEARLY DEPENDENT FUNCTIONS, LINEARLY DEPENDENT VECTORS, MAXIMALLY LINEAR INDEPENDENT

Link

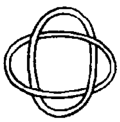
Formally, a link is one or more disjointly embedded CIRCLES in 3-space. More informally, a link is an assembly of KNOTS with mutual entanglements. Kuperberg (1994) has shown that a nontrivial KNOT or link in $\mathbb{R}^3$ has four COLLINEAR points (Eppstein). Doll and Hoste (1991) list POLYNOMIALS for oriented links of nine or fewer crossings. A listing of the first few simple links follows, arranged by CROSSING NUMBER.



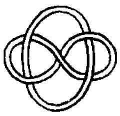
0_1^2



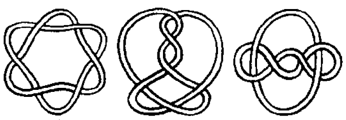
2^2_1



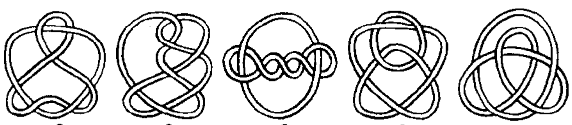
4^2_1



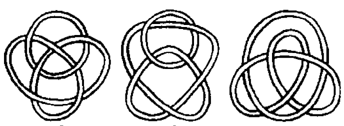
5^2_1



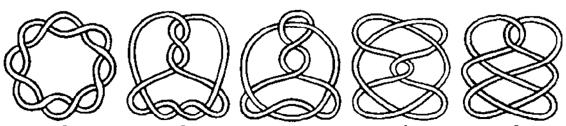
6^2_1 6^2_2 6^2_3



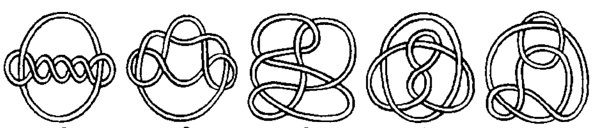
7^2_1 7^2_2 7^2_3 7^2_4 7^2_5



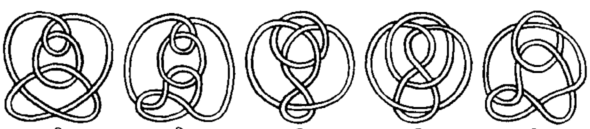
7^2_6 7^2_7 7^2_8



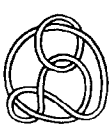
8^2_1 8^2_2 8^2_3 8^2_4 8^2_5



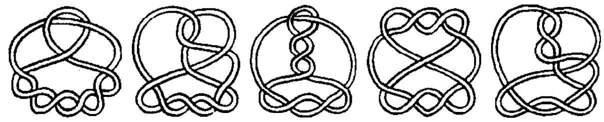
8^2_6 8^2_7 8^2_8 8^2_9 8^2_{10}



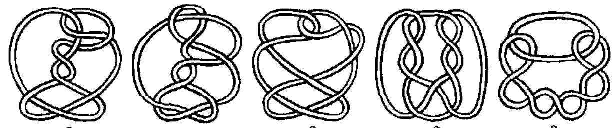
8^2_{11} 8^2_{12} 8^2_{13} 8^2_{14} 8^2_{15}



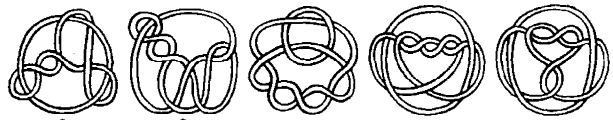
8^2_{16}



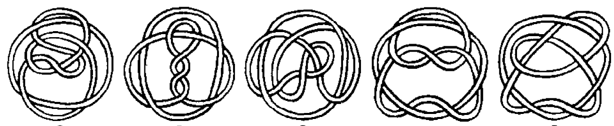
9^2_1 9^2_2 9^2_3 9^2_4 9^2_5



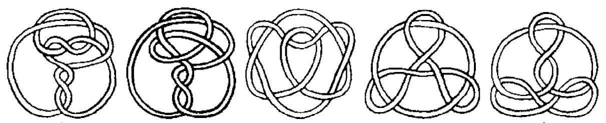
9^2_6 9^2_7 9^2_8 9^2_9 9^2_{10}



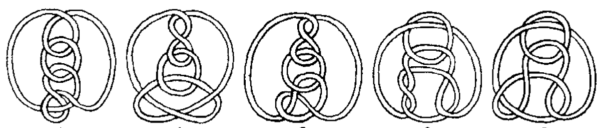
9^2_{11} 9^2_{12} 9^2_{13} 9^2_{14} 9^2_{15}



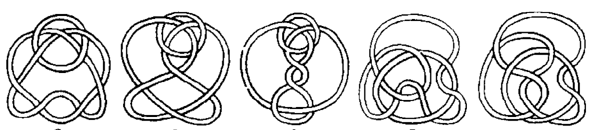
9^2_{16} 9^2_{17} 9^2_{18} 9^2_{19} 9^2_{20}



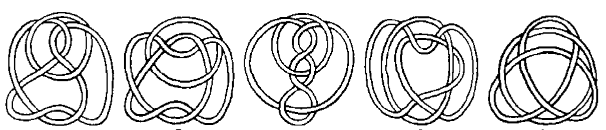
9^2_{21} 9^2_{22} 9^2_{23} 9^2_{24} 9^2_{25}



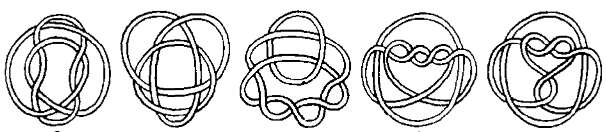
9^2_{26} 9^2_{27} 9^2_{28} 9^2_{29} 9^2_{30}



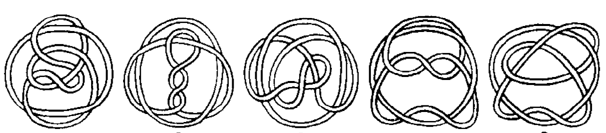
9^2_{31} 9^2_{32} 9^2_{33} 9^2_{34} 9^2_{35}



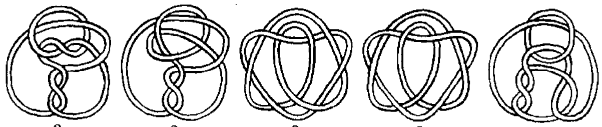
9^2_{36} 9^2_{37} 9^2_{38} 9^2_{39} 9^2_{40}



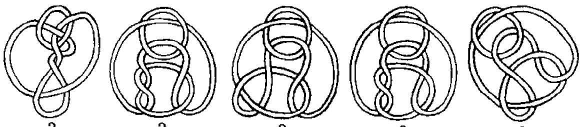
9^2_{41} 9^2_{42} 9^2_{43} 9^2_{44} 9^2_{45}



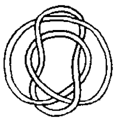
9^2_{46} 9^2_{47} 9^2_{48} 9^2_{49} 9^2_{50}



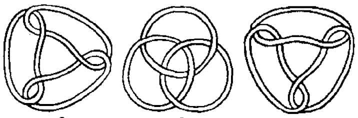
9^2_{51} 9^2_{52} 9^2_{53} 9^2_{54} 9^2_{55}



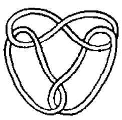
9^2_{56} 9^2_{57} 9^2_{58} 9^2_{59} 9^2_{60}



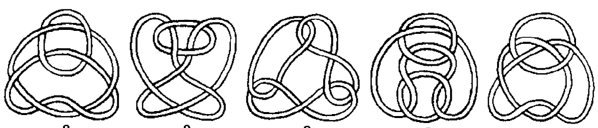
9^2_{61}



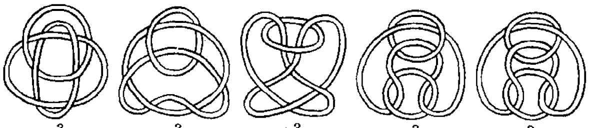
6^3_1 6^3_2 6^3_3



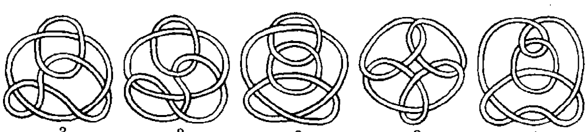
7^3_1



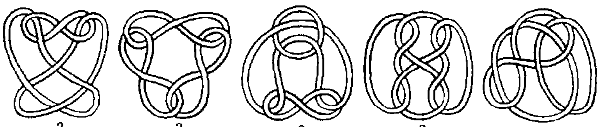
8^3_1 8^3_2 8^3_3 8^3_4 8^3_5



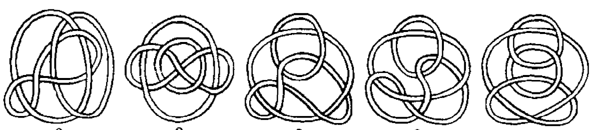
8^3_6 8^3_7 8^3_8 8^3_9 8^3_{10}



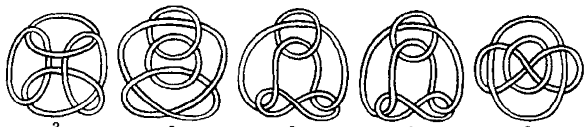
9^3_1 9^3_2 9^3_3 9^3_4 9^3_5



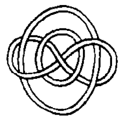
9^3_6 9^3_7 9^3_8 9^3_9 9^3_{10}



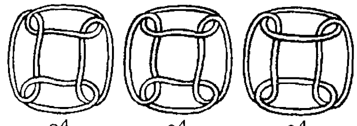
9^3_{11} 9^3_{12} 9^3_{13} 9^3_{14} 9^3_{15}



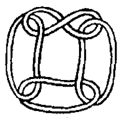
9^3_{16} 9^3_{17} 9^3_{18} 9^3_{19} 9^3_{20}



9^3_{21}



8^4_1 8^4_2 8^4_3



9^4_1

see also ANDREWS-CURTIS LINK, BORROMEAN RINGS, BRUNNIAN LINK, HOPF LINK, KNOT, WHITEHEAD LINK

References

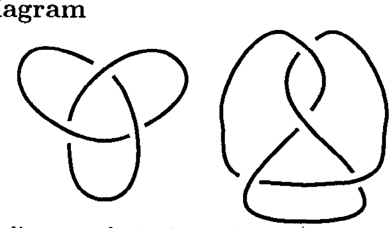
Doll, H. and Hoste, J. "A Tabulation of Oriented Links." *Math. Comput.* **57**, 747–761, 1991.

Eppstein, D. "Colinear Points on Knots." `http://www.ics.uci.edu/~eppstein/junkyard/knot-colinear.html`.

Kuperberg, G. "Quadrisecants of Knots and Links." *J. Knot Theory Ramifications* **3**, 41–50, 1994.

Weisstein, E. W. "Knots." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Knots.m`.

Link Diagram



A planar diagram depicting a LINK (or KNOT) as a sequence of segments with gaps representing undercrossings and solid lines overcrossings. In such a diagram, only two segments should ever cross at a single point. Link diagrams for the TREFOIL KNOT and FIGURE-OF-EIGHT KNOT are illustrated above.

Linkage

Sylvester, Kempe and Cayley developed the geometry associated with the theory of linkages in the 1870s. Kempe proved that every finite segment of an algebraic curve can be generated by a linkage in the manner of WATT'S CURVE.

see also HART'S INVERSOR, KEMPE LINKAGE, PANTOGRAPH, PEAUCELLIER INVERSOR, SARRUS LINKAGE, WATT'S PARALLELOGRAM

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., 1989.

Linking Number

A LINK invariant. Given a two-component oriented LINK, take the sum of $+1$ crossings and -1 crossing over all crossings between the two links and divide by 2. For components α and β,

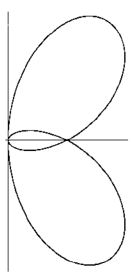
$$L(\alpha, \beta) \equiv \tfrac{1}{2} \sum_{p \in \alpha \sqcap \beta} \epsilon(p),$$

where $\alpha \sqcap \beta$ is the set of crossings of α with β and $\epsilon(p)$ is the sign of the crossing. The linking number of a splittable two-component link is always 0.

see also JONES POLYNOMIAL, LINK

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 132–133, 1976.

Links Curve



The curve given by the Cartesian equation

$$(x^2 + y^2 - 3x)^2 = 4x^2(2 - x).$$

The origin of the curve is a TACNODE.

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 72, 1989.

Linnik's Constant

The constant L in LINNIK'S THEOREM. Heath-Brown (1992) has shown that $L \leq 5.5$, and Schinzel, Sierpiński, and Kanold (Ribenboim 1989) have conjectured that $L = 2$.

References
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/linnik/linnik.html`.
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 13, 1994.
Heath-Brown, D. R. "Zero-Free Regions for Dirichlet L-Functions and the Least Prime in an Arithmetic Progression." *Proc. London Math. Soc.* **64**, 265–338, 1992.
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, 1989.

Linnik's Theorem

Let $p(d, a)$ be the smallest PRIME in the arithmetic progression $\{a + kd\}$ for k an INTEGER > 0. Let

$$p(d) \equiv \max p(d, a)$$

such that $1 \leq a < d$ and $(a, d) = 1$. Then there exists a $d_0 \geq 2$ and an $L > 1$ such that $p(d) < d^L$ for all $d > d_0$. L is known as LINNIK'S CONSTANT.

References
Linnik, U. V. "On the Least Prime in an Arithmetic Progression. I. The Basic Theorem." *Mat. Sbornik N. S.* **15 (57)**, 139–178, 1944.
Linnik, U. V. "On the Least Prime in an Arithmetic Progression. II. The Deuring-Heilbronn Phenomenon" *Mat. Sbornik N. S.* **15 (57)**, 347–368, 1944.

Liouville's Boundedness Theorem

A bounded ENTIRE FUNCTION in the COMPLEX PLANE $\mathbb{C}$ is constant. The FUNDAMENTAL THEOREM OF ALGEBRA follows as a simple corollary.

see also COMPLEX PLANE, ENTIRE FUNCTION, FUNDAMENTAL THEOREM OF ALGEBRA

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 381–382, 1953.

Liouville's Conformality Theorem

In SPACE, the only CONFORMAL TRANSFORMATIONS are inversions, SIMILARITY TRANSFORMATIONS, and CONGRUENCE TRANSFORMATIONS. Or, restated, every ANGLE-preserving transformation is a SPHERE-preserving transformation.

see also CONFORMAL MAP

Liouville's Conic Theorem

The lengths of the TANGENTS from a point P to a CONIC C are proportional to the CUBE ROOTS of the RADII OF CURVATURE of C at the corresponding points of contact.

see also CONIC SECTION

Liouville's Constant

$$L \equiv \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001\ldots$$

(Sloane's A012245). Liouville's constant is a decimal fraction with a 1 in each decimal place corresponding to a FACTORIAL $n!$, and ZEROS everywhere else. This number was among the first to be proven to be TRANSCENDENTAL. It nearly satisfies

$$10x^6 - 75x^3 - 190x + 21 = 0,$$

but with $x = L$, this equation gives $-0.0000000059\ldots$.

see also LIOUVILLE NUMBER

References

Conway, J. H. and Guy, R. K. "Liouville's Number." In *The Book of Numbers.* New York: Springer-Verlag, pp. 239–241, 1996.

Courant, R. and Robbins, H. "Liouville's Theorem and the Construction of Transcendental Numbers." §2.6.2 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 104–107, 1996.

Liouville, J. "Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même reductible à des irrationelles algébriques." *C. R. Acad. Sci. Paris* **18**, 883–885 and 993–995, 1844.

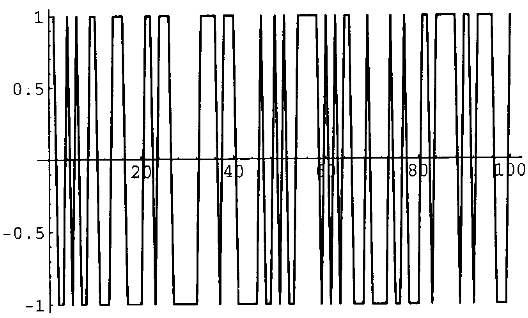
Liouville, J. "Sur des classes très-étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationelles algébriques." *J. Math. pures appl.* **15**, 133–142, 1850.

Sloane, N. J. A. Sequence A012245 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Liouville's Elliptic Function Theorem

An ELLIPTIC FUNCTION with no POLES in a fundamental cell is a constant.

Liouville Function



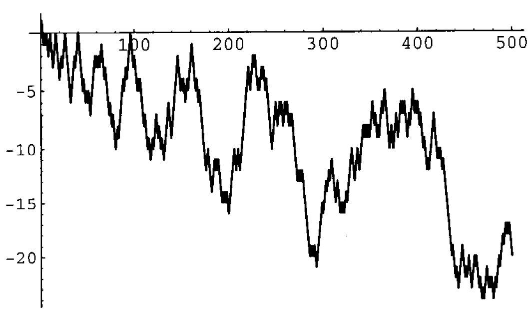
The function

$$\lambda(n) = (-1)^{r(n)}, \tag{1}$$

where $r(n)$ is the number of not necessarily distinct PRIME FACTORS of n, with $r(1) = 0$. The first few values of $\lambda(n)$ are 1, -1, -1, 1, -1, 1, -1, -1, 1, 1, -1, -1, ... The Liouville function is connected with the RIEMANN ZETA FUNCTION by the equation

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{2}$$

(Lehman 1960).



The CONJECTURE that the SUMMATORY FUNCTION

$$L(n) \equiv \sum_{k=1}^{n} \lambda(n) \tag{3}$$

satisfies $L(n) \leq 0$ for $n \geq 2$ is called the PÓLYA CONJECTURE and has been proved to be false. The first n for which $L(n) = 0$ are for $n = 2$, 4, 6, 10, 16, 26, 40, 96, 586, 906150256, ... (Sloane's A028488), and $n = 906150257$ is, in fact, the first counterexample to the PÓLYA CONJECTURE (Tanaka 1980). However, it is unknown if $L(x)$ changes sign infinitely often (Tanaka 1980). The first few values of $L(n)$ are 1, 0, -1, 0, -1, 0, -1, -2, -1, 0, -1, -2, -3, -2, -1, 0, -1, -2, -3, -4, ... (Sloane's A002819). $L(n)$ also satisfies

$$\sum_{n=1}^{x} L\left(\frac{x}{n}\right) = \lfloor\sqrt{x}\rfloor, \tag{4}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION (Lehman 1960). Lehman (1960) also gives the formulas

$$L(x) = \sum_{m=1}^{x/w} \mu(m) \left\{ \left\lfloor\sqrt{\frac{x}{m}}\right\rfloor - \sum_{k=1}^{v-1} \lambda(k) \left(\left\lfloor\frac{x}{km}\right\rfloor - \left\lfloor\frac{x}{mv}\right\rfloor \right) \right\} - \sum_{l=x/w-1}^{x/v} L\left(\frac{x}{l}\right) \sum_{\substack{m|l \\ m=1}}^{x/w} \mu(m) \tag{5}$$

and

$$L(x) = \sum_{k=1}^{g} M\left(\frac{x}{k^2}\right) + \sum_{l=1}^{x/g^2} \mu(l) \left\lfloor\sqrt{\frac{x}{l}}\right\rfloor - M\left(\frac{x}{g^2}\right) \left\lfloor\sqrt{\frac{x}{g^2}}\right\rfloor, \tag{6}$$

where k, l, and m are variables ranging over the POSITIVE integers, $\mu(n)$ is the MÖBIUS FUNCTION, $M(x)$ is MERTENS FUNCTION, and v, w, and x are POSITIVE real numbers with $v < w < x$.

see also PÓLYA CONJECTURE, PRIME FACTORS, RIEMANN ZETA FUNCTION

References

Fawaz, A. Y. "The Explicit Formula for $L_0(x)$." *Proc. London Math. Soc.* **1**, 86–103, 1951.

Lehman, R. S. "On Liouville's Function." *Math. Comput.* **14**, 311–320, 1960.

Sloane, N. J. A. Sequences A028488 and A002819/M0042 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Tanaka, M. "A Numerical Investigation on Cumulative Sum of the Liouville Function." *Tokyo J. Math.* **3**, 187–189, 1980.

Liouville Measure

$$\prod_i dp_i\, dq_i,$$

where p_i and q_i are momenta and positions of particles.

see also LIOUVILLE'S PHASE SPACE THEOREM, PHASE SPACE

Liouville Number

A Liouville number is a TRANSCENDENTAL NUMBER which is very close to a RATIONAL NUMBER. An IRRATIONAL NUMBER β is a Liouville number if, for any n, there exist an infinite number of pairs of INTEGERS p and q such that

$$0 < \left|\beta - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Mahler (1953) proved that π is not a Liouville number.

see also LIOUVILLE'S CONSTANT, LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, ROTH'S THEOREM, TRANSCENDENTAL NUMBER

References

Mahler, K. "On the Approximation of π." *Nederl. Akad. Wetensch. Proc. Ser. A.* **56**/*Indagationes Math.* **15**, 30–42, 1953.

Liouville's Phase Space Theorem

States that for a nondissipative HAMILTONIAN SYSTEM, phase space density (the AREA between phase space contours) is constant. This requires that, given a small time increment dt,

$$q_1 \equiv q(t_0 + dt) = q_0 + \frac{\partial H(q_0, p_0, t)}{\partial p_0} dt + \mathcal{O}(dt^2) \quad (1)$$

$$p_1 \equiv p(t_0 + dt) = p_0 - \frac{\partial H(q_0, p_0, t)}{\partial q_0} dt + \mathcal{O}(dt^2), \quad (2)$$

the JACOBIAN be equal to one:

$$\begin{aligned}\frac{\partial(q_1, p_1)}{\partial(q_0, p_0)} &= \begin{vmatrix} \frac{\partial q_1}{\partial q_0} & \frac{\partial p_1}{\partial q_0} \\ \frac{\partial q_1}{\partial p_0} & \frac{\partial p_1}{\partial p_0} \end{vmatrix} \\ &= \begin{vmatrix} 1 + \frac{\partial^2 H}{\partial q_0 \partial p_0} dt & -\frac{\partial^2 H}{\partial {q_0}^2} dt \\ \frac{\partial^2 H}{\partial {p_0}^2} dt & 1 - \frac{\partial^2 H}{\partial q_0 \partial p_0} dt \end{vmatrix} + \mathcal{O}(dt^2) \\ &= 1 + \mathcal{O}(dt^2). \end{aligned} \quad (3)$$

Expressed in another form, the integral of the LIOUVILLE MEASURE,

$$\prod_{i=1}^N \int dp_i\, dq_i, \quad (4)$$

is a constant of motion. SYMPLECTIC MAPS of HAMILTONIAN SYSTEMS must therefore be AREA preserving (and have DETERMINANTS equal to 1).

see also LIOUVILLE MEASURE, PHASE SPACE

References

Chavel, I. *Riemannian Geometry: A Modern Introduction.* New York: Cambridge University Press, 1994.

Liouville Polynomial Identity

$$\begin{aligned} 6({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2) &= (x_1 + x_2)^4 + (x_1 + x_3)^4 \\ &+ (x_2+x_3)^4 + (x_1+x_4)^4 + (x_2+x_4)^4 + (x_3+x_4)^4 + (x_1-x_2)^4 \\ &+ (x_1 - x_3)^4 + (x_2 - x_3)^4 + (x_1 - x_4)^4 + (x_2 - x_4)^4 \\ &+ (x_3 - x_4)^4. \end{aligned}$$

This is proven in Rademacher and Toeplitz (1957).

see also WARING'S PROBLEM

References

Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 55–56, 1957.

Liouville's Rational Approximation Theorem

For any ALGEBRAIC NUMBER x of degree $n > 1$, a RATIONAL approximation $x = p/q$ must satisfy

$$\left|x - \frac{p}{q}\right| > \frac{1}{q^{n+1}}$$

for sufficiently large q. Writing $r \equiv n + 1$ leads to the definition of the LIOUVILLE-ROTH CONSTANT of a given number.

see also LAGRANGE NUMBER (RATIONAL APPROXIMATION), LIOUVILLE'S CONSTANT, LIOUVILLE NUMBER, LIOUVILLE-ROTH CONSTANT, MARKOV NUMBER, ROTH'S THEOREM, THUE-SIEGEL-ROTH THEOREM

References

Courant, R. and Robbins, H. "Liouville's Theorem and the Construction of Transcendental Numbers." §2.6.2 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 104–107, 1996.

Liouville-Roth Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let x be a REAL NUMBER, and let R be the SET of POSITIVE REAL NUMBERS for which

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^r} \quad (1)$$

has (at most) finitely many solutions p/q for p and q INTEGERS. Then the Liouville-Roth constant (or IRRATIONALITY MEASURE) is defined as the threshold at which LIOUVILLE'S RATIONAL APPROXIMATION THEOREM kicks in and x is no longer approximable by RATIONAL NUMBERS,

$$r(x) \equiv \inf_{r \in R} r. \quad (2)$$

There are three regimes:

$$\begin{cases} r(x) = 1 & x \text{ is rational} \\ r(x) = 2 & x \text{ is algebraic irrational} \\ r(x) \geq 2 & x \text{ is transcendental.} \end{cases} \tag{3}$$

The best known upper bounds for common constants are

$$r(L) = \infty \tag{4}$$
$$r(e) = 2 \tag{5}$$
$$r(\pi) < 8.0161 \tag{6}$$
$$r(\ln 2) < 4.13 \tag{7}$$
$$r(\pi^2) < 6.3489 \tag{8}$$
$$r(\zeta(3)) < 13.42, \tag{9}$$

where L is LIOUVILLE'S CONSTANT, $\zeta(3)$ is APÉRY'S CONSTANT, and the lower bounds are 2 for the inequalities.

see also LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, ROTH'S THEOREM, THUE-SIEGEL-ROTH THEOREM

References
Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, 1987.
Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/lvlrth/lvlrth.html.
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford: Clarendon Press, 1979.
Hata, M. "Improvement in the Irrationality Measures of π and π^2." *Proc. Japan. Acad. Ser. A Math. Sci.* **68**, 283–286, 1992.
Hata, M. "Rational Approximations to π and Some Other Numbers." *Acta Arith.* **63** 335–349, 1993.
Hata, M. "A Note on Beuker's Integral." *J. Austral. Math. Soc.* **58**, 143–153, 1995.
Stark, H. M. *An Introduction to Number Theory.* Cambridge, MA: MIT Press, 1978.

Liouville Space

Also known as LINE SPACE or "extended" HILBERT SPACE, it is the DIRECT PRODUCT of two HILBERT SPACES.

see also DIRECT PRODUCT (SET), HILBERT SPACE

Liouville's Sphere-Preserving Theorem

see LIOUVILLE'S CONFORMALITY THEOREM

Lipschitz Condition

A function $f(x)$ satisfies the Lipschitz condition of order α at $x = 0$ if

$$|f(h) - f(0)| \leq B|h|^\beta$$

for all $|h| < \epsilon$, where B and β are independent of h, $\beta > 0$, and α is an UPPER BOUND for all β for which a finite B exists.

see also HILLAM'S THEOREM, LIPSCHITZ FUNCTION

Lipschitz Function

A function f such that

$$|f(x) - f(y)| \leq C|x - y|$$

is called a Lipschitz function.

see also LIPSCHITZ CONDITION

References
Morgan, F. "What Is a Surface?" *Amer. Math. Monthly* **103**, 369–376, 1996.

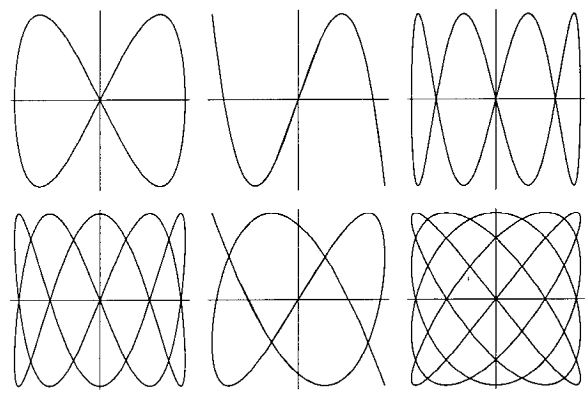
Lipschitz's Integral

$$\int_0^\infty e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{a^2 + b^2}},$$

where $J_0(z)$ is the zeroth order BESSEL FUNCTION OF THE FIRST KIND.

References
Bowman, F. *Introduction to Bessel Functions.* New York: Dover, p. 58, 1958.

Lissajous Curve



Lissajous curves are the family of curves described by the parametric equations

$$x(t) = A\cos(\omega_x t - \delta_x) \tag{1}$$
$$y(t) = B\cos(\omega_y t - \delta_y), \tag{2}$$

sometimes also written in the form

$$x(t) = a\sin(nt + c) \tag{3}$$
$$y(t) = b\sin t. \tag{4}$$

They are sometimes known as BOWDITCH CURVES after Nathaniel Bowditch, who studied them in 1815. They were studied in more detail (independently) by Jules-Antoine Lissajous in 1857 (MacTutor Archive). Lissajous curves have applications in physics, astronomy, and other sciences. The curves close IFF ω_x/ω_y is RATIONAL.

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 53–54, 1993.
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 178–179 and 181–183, 1972.
MacTutor History of Mathematics Archive. "Lissajous Curves." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Lissajous.html.

Lissajous Figure

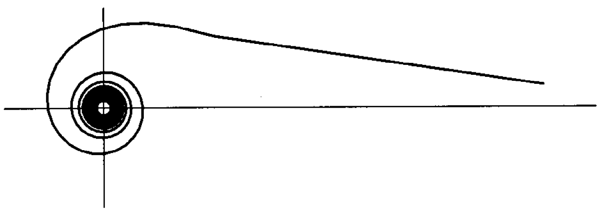
see LISSAJOUS CURVE

List

A DATA STRUCTURE consisting of an order SET of elements, each of which may be a number, another list, etc. A list is usually denoted $(a_1, a_2, \ldots, a_n)$ or $\{a_1, a_2, \ldots, a_n\}$.

see also QUEUE, STACK

Lituus



An ARCHIMEDEAN SPIRAL with $m = -2$, having polar equation

$$r^2\theta = a^2.$$

Lituus means a "crook," in the sense of a bishop's crosier. The lituus curve originated with Cotes in 1722. Maclaurin used the term lituus in his book *Harmonia Mensurarum* in 1722 (MacTutor Archive). The lituus is the locus of the point P moving such that the AREA of a circular SECTOR remains constant.

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 69–70, 1993.

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 186 and 188, 1972.

Lockwood, E. H. *A Book of Curves.* Cambridge, England: Cambridge University Press, p. 175, 1967.

MacTutor History of Mathematics Archive. "Lituus." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Lituus.html`.

Lituus Inverse Curve

The INVERSE CURVE of the LITUUS is an ARCHIMEDEAN SPIRAL with $m = 2$, which is FERMAT'S SPIRAL.

see also ARCHIMEDEAN SPIRAL, FERMAT'S SPIRAL, LITUUS

LLL Algorithm

An INTEGER RELATION algorithm.

see also FERGUSON-FORCADE ALGORITHM, HJLS ALGORITHM, INTEGER RELATION, PSLQ ALGORITHM, PSOS ALGORITHM

References

Lenstra, A. K.; Lenstra, H. W.; and Lovasz, L. "Factoring Polynomials with Rational Coefficients." *Math. Ann.* **261**, 515–534, 1982.

Ln

The LOGARITHM to BASE e, also called the NATURAL LOGARITHM, is denoted ln, i.e.,

$$\ln x \equiv \log_e x.$$

see also BASE (LOGARITHM), E, LG, LOGARITHM, NAPIERIAN LOGARITHM, NATURAL LOGARITHM

Lo Shu

8	1	6
3	5	7
4	9	2

The unique MAGIC SQUARE of order three. The Lo Shu is an ASSOCIATIVE MAGIC SQUARE, but not a PANMAGIC SQUARE.

see also ASSOCIATIVE MAGIC SQUARE, MAGIC SQUARE, PANMAGIC SQUARE

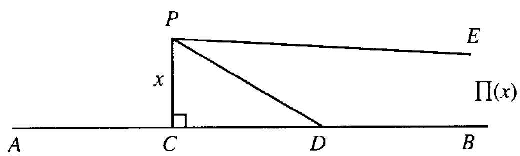
References

Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, pp. 23–24, 1975.

Lobachevsky-Bolyai-Gauss Geometry

see HYPERBOLIC GEOMETRY

Lobachevsky's Formula



Given a point P and a LINE AB, draw the PERPENDICULAR through P and call it PC. Let PD be any other line from P which meets CB in D. In a HYPERBOLIC GEOMETRY, as D moves off to infinity along CB, then the line PD approaches the limiting line PE, which is said to be parallel to CB at P. The angle $\angle CPE$ which PE makes with PC is then called the ANGLE OF PARALLELISM for perpendicular distance x, and is given by

$$\Pi(x) = 2\tan^{-1}(e^{-x}),$$

which is called Lobachevsky's formula.

see also ANGLE OF PARALLELISM, HYPERBOLIC GEOMETRY

References

Manning, H. P. *Introductory Non-Euclidean Geometry.* New York: Dover, p. 58, 1963.

Lobatto Quadrature

Also called RADAU QUADRATURE (Chandrasekhar 1960). A GAUSSIAN QUADRATURE with WEIGHTING FUNCTION $W(x) = 1$ in which the endpoints of the interval $[-1, 1]$ are included in a total of n ABSCISSAS, giving $r = n - 2$ free abscissas. ABSCISSAS are symmetrical about the origin, and the general FORMULA is

$$\int_{-1}^{1} f(x)\,dx = w_1 f(-1) + w_n f(1) + \sum_{i=2}^{n-1} w_i f(x_i). \quad (1)$$

The free ABSCISSAS x_i for $i = 2, \ldots, n-1$ are the roots of the POLYNOMIAL $P'_{n-1}(x)$, where $P(x)$ is a LEGENDRE POLYNOMIAL. The weights of the free abscissas are

$$w_i = -\frac{2n}{(1 - x_i{}^2)P''_{n-1}(x_i)P'_m(x_i)} \quad (2)$$

$$= \frac{2}{n(n-1)[P_{n-1}(x_i)]^2}, \quad (3)$$

and of the endpoints are

$$w_{1,n} = \frac{2}{n(n-1)}. \quad (4)$$

The error term is given by

$$E = -\frac{n(n-1)^3 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\xi), \quad (5)$$

for $\xi \in (-1, 1)$. Beyer (1987) gives a table of parameters up to n=11 and Chandrasekhar (1960) up to n=9 (although Chandrasekhar's $\mu_{3,4}$ for $m = 5$ is incorrect).

n	x_i	w_i
3	0	1.33333
	± 1	0.333333
4	± 0.447214	0.833333
	± 1	0.166667
5	0	0.711111
	± 0.654654	0.544444
	± 1	0.100000
6	± 0.285232	0.554858
	± 0.765055	0.378475
	± 1	0.0666667

The ABSCISSAS and weights can be computed analytically for small n.

n	x_i	w_i
3	0	$\frac{4}{3}$
	± 1	$\frac{1}{3}$
4	$\pm \frac{1}{5}\sqrt{5}$	$\frac{1}{6}$
	± 1	$\frac{5}{6}$
5	0	$\frac{32}{45}$
	$\pm \frac{1}{7}\sqrt{21}$	$\frac{49}{90}$
	± 1	$\frac{1}{10}$

see also CHEBYSHEV QUADRATURE, RADAU QUADRATURE

References

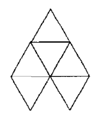
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 888–890, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 465, 1987.

Chandrasekhar, S. *Radiative Transfer.* New York: Dover, pp. 63–64, 1960.

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 343–345, 1956.

Lobster



A 6-POLYIAMOND.

References

Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings, 2nd ed.* Princeton, NJ: Princeton University Press, p. 92, 1994.

Local Cell

The POLYHEDRON resulting from letting each SPHERE in a SPHERE PACKING expand uniformly until it touches its neighbors on flat faces.

see also LOCAL DENSITY

Local Degree

The degree of a VERTEX of a GRAPH is the number of EDGES which touch the VERTEX, also called the LOCAL DEGREE. The VERTEX degree of a point A in a GRAPH, denoted $\rho(A)$, satisfies

$$\sum_{i=1}^{n} \rho(A_i) = 2E,$$

where E is the total number of EDGES. Directed graphs have two types of degrees, known as the INDEGREE and OUTDEGREE.

see also INDEGREE, OUTDEGREE

Local Density

Let each SPHERE in a SPHERE PACKING expand uniformly until it touches its neighbors on flat faces. Call the resulting POLYHEDRON the LOCAL CELL. Then the local density is given by

$$\rho \equiv \frac{V_{\text{sphere}}}{V_{\text{local cell}}}.$$

When the LOCAL CELL is a regular DODECAHEDRON, then

$$\rho_{\text{dodecahedron}} = \frac{\pi\sqrt{5+\sqrt{5}}}{5\sqrt{10}\,(\sqrt{5}-2)} = 0.7547\ldots.$$

see also LOCAL DENSITY CONJECTURE

Local Density Conjecture
The CONJECTURE that the maximum LOCAL DENSITY is given by $\rho_{\text{dodecahedron}}$.

see also LOCAL DENSITY

Local Extremum
A LOCAL MINIMUM or LOCAL MAXIMUM.

see also EXTREMUM, GLOBAL EXTREMUM

Local Field
A FIELD which is complete with respect to a discrete VALUATION is called a local field if its FIELD of RESIDUE CLASSES is FINITE. The HASSE PRINCIPLE is one of the chief applications of local field theory.

see also HASSE PRINCIPLE, VALUATION

References

Iyanaga, S. and Kawada, Y. (Eds.). "Local Fields." §257 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 811–815, 1980.

Local-Global Principle
see HASSE PRINCIPLE

Local Group Theory
The study of a FINITE GROUP G using the LOCAL SUBGROUPS of G. Local group theory plays a critical role in the CLASSIFICATION THEOREM.

see also SYLOW THEOREMS

Local Maximum
The largest value of a set, function, etc., within some local neighborhood.

see also GLOBAL MAXIMUM, LOCAL MINIMUM, MAXIMUM, PEANO SURFACE

Local Minimum
The smallest value of a set, function, etc., within some local neighborhood.

see also GLOBAL MINIMUM, LOCAL MAXIMUM, MINIMUM

Local Ring
A NOETHERIAN RING R with a Jacobson radical which has only a single maximal ideal.

References

Iyanaga, S. and Kawada, Y. (Eds.). "Local Rings." §281D in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 890–891, 1980.

Local Subgroup
A normalizer of a nontrivial SYLOW p-SUBGROUP of a GROUP G.

see also LOCAL GROUP THEORY

Local Surface
see PATCH

Locally Convex Space
see LOCALLY PATHWISE-CONNECTED SPACE

Locally Finite Space
A locally finite SPACE is one for which every point of a given space has a NEIGHBORHOOD that meets only finitely many elements of the COVER.

Locally Pathwise-Connected Space
A SPACE X is locally pathwise-connected if for every NEIGHBORHOOD around every point in X, there is a smaller, PATHWISE-CONNECTED NEIGHBORHOOD.

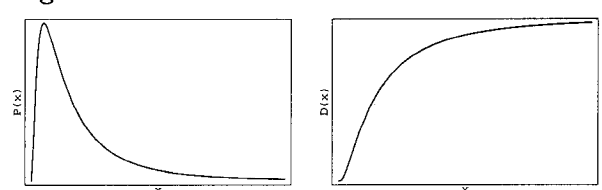
Loculus of Archimedes
see STOMACHION

Locus
The set of all points (usually forming a curve or surface) satisfying some condition. For example, the locus of points in the plane equidistant from a given point is a CIRCLE, and the set of points in 3-space equidistant from a given point is a SPHERE.

Log
The symbol $\log x$ is used by physicists, engineers, and calculator keypads to denote the BASE 10 LOGARITHM. However, mathematicians generally use the same symbol to mean the NATURAL LOGARITHM LN, $\ln x$. In this work, $\log x = \log_{10} x$, and $\ln x = \log_e x$ is used for the NATURAL LOGARITHM.

see also LG, LN, LOGARITHM, NATURAL LOGARITHM

Log Likelihood Procedure
A method for testing NESTED HYPOTHESES. To apply the procedure, given a specific model, calculate the LIKELIHOOD of observing the actual data. Then compare this likelihood to a nested model (i.e., one in which fewer parameters are allowed to vary independently).

Log Normal Distribution


A CONTINUOUS DISTRIBUTION in which the LOGARITHM of a variable has a NORMAL DISTRIBUTION. It is a general case of GILBRAT'S DISTRIBUTION, to which the log normal distribution reduces with $S = 1$ and

$M = 0$. The probability density and cumulative distribution functions are log normal distribution

$$P(x) = \frac{1}{Sx\sqrt{2\pi}} e^{-(\ln x - M)^2/(2S^2)} \tag{1}$$

$$D(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{\ln x - M}{S\sqrt{2}}\right)\right], \tag{2}$$

where $\operatorname{erf}(x)$ is the ERF function. This distribution is normalized, since letting $y \equiv \ln x$ gives $dy = dx/x$ and $x = e^y$, so

$$\int_0^\infty P(x)\,dx = \frac{1}{S\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(y-M)^2/2S^2}\,dy = 1. \tag{3}$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are given by

$$\mu = e^{M+S^2/2} \tag{4}$$

$$\sigma^2 = e^{S^2+2M}(e^{S^2} - 1) \tag{5}$$

$$\gamma_1 = \sqrt{e^S - 1}\,(2 + e^{S^2}) \tag{6}$$

$$\gamma_2 = e^{2S^2}(3 + 2e^{S^2} + e^{2s^2}) - 3. \tag{7}$$

These can be found by direct integration

$$\begin{aligned}
\mu &= \frac{1}{S\sqrt{2\pi}} \int_0^\infty e^{-(\ln x - M)^2/2S^2}\,dx \\
&= \frac{1}{S\sqrt{2\pi}} \int_{-\infty}^\infty e^{(y-M)^2/2S^2} e^y\,dy \\
&= \frac{1}{S\sqrt{2\pi}} \int_{-\infty}^\infty e^{-[-y+(y-M)^2/2S^2]}\,dy \\
&= \frac{1}{S\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(-2S^2y+y^2-2yM+M^2)/2S^2}\,dy \\
&= \frac{1}{S\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\{[y-(S^2+M)]^2+S^2(S^2+2M)\}/2S^2}\,dy \\
&= \frac{1}{S\sqrt{2\pi}} e^{M+S^2/2} \int_{-\infty}^\infty e^{-[y-(S^2+M)^2]/2S^2}\,dy \\
&= e^{M+S^2/2},
\end{aligned} \tag{8}$$

and similarly for σ^2. Examples of variates which have approximately log normal distributions include the size of silver particles in a photographic emulsion, the survival time of bacteria in disinfectants, the weight and blood pressure of humans, and the number of words written in sentences by George Bernard Shaw.

see also GILBRAT'S DISTRIBUTION, WEIBULL DISTRIBUTION

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Log-Series Distribution

The terms in the series expansion of $\ln(1-\theta)$ about $\theta = 0$ are proportional to this distribution.

$$P(n) = -\frac{\theta^n}{n\ln(1-\theta)} \tag{1}$$

$$D(n) \equiv \sum_{i=1}^n P(i) = \frac{\theta^{1+n}\Phi(\theta, 1, 1+n) + \ln(1-\theta)}{\ln(1-\theta)}, \tag{2}$$

where Φ is the LERCH TRANSCENDENT. The MEAN, VARIANCE, SKEWNESS, and KURTOSIS

$$\mu = \frac{\theta}{(\theta - 1)\ln(1-\theta)} \tag{3}$$

$$\sigma^2 = -\frac{\theta[\theta + \ln(1-\theta)]}{(\theta-1)^2[\ln(1-\theta)]^2} \tag{4}$$

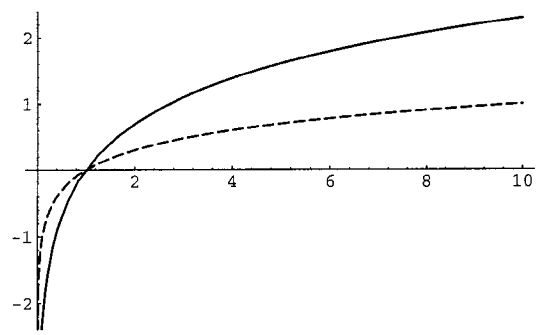
$$\gamma_1 = \frac{2\theta^2 + 3\theta\ln(1-\theta) + (1+\theta)\ln^2(1-\theta)}{\ln(1-\theta)[\theta + \ln(1-\theta)]\sqrt{-\theta[\theta+\ln(1-\theta)]}} \ln(1-\theta) \tag{5}$$

$$\gamma_2 = \frac{6\theta^3 + 12\theta^2\ln(1-\theta) + \theta(7+4\theta)\ln^2(1-\theta)}{\theta[\theta+\ln(1-\theta)]^2} + \frac{(1+4\theta+\theta^2)\ln^3(1-\theta)}{\theta[\theta+\ln(1-\theta)]^2}. \tag{6}$$

Log-Weibull Distribution

see FISHER-TIPPETT DISTRIBUTION

Logarithm



The logarithm is defined to be the INVERSE FUNCTION of taking a number to a given POWER. Therefore, for any x and b,

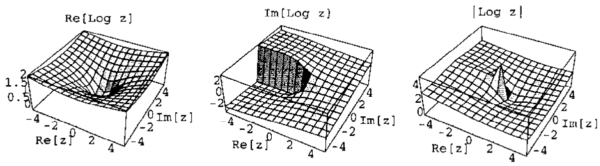
$$x = b^{\log_b x}, \tag{1}$$

or equivalently,

$$x = \log_b(b^x). \tag{2}$$

Here, the POWER b is known as the BASE of the logarithm. For any BASE, the logarithm function has a SINGULARITY at $x = 0$. In the above plot, the solid curve is the logarithm to BASE e (the NATURAL LOGARITHM), and the dotted curve is the logarithm to BASE 10 (LOG).

Logarithms are used in many areas of science and engineering in which quantities vary over a large range. For example, the decibel scale for the loudness of sound, the Richter scale of earthquake magnitudes, and the astronomical scale of stellar brightnesses are all logarithmic scales.



The logarithm can also be defined for COMPLEX arguments, as shown above. If the logarithm is taken as the forward function, the function taking the BASE to a given POWER is then called the ANTILOGARITHM.

For $x = \log N$, $\lfloor x \rfloor$ is called the CHARACTERISTIC and $x - \lfloor x \rfloor$ is called the MANTISSA. Division and multiplication identities follow from these

$$xy = b^{\log_b x} b^{\log_b y} = b^{\log_b x + \log_b y}, \tag{3}$$

from which it follows that

$$\log_b(xy) = \log_b x + \log_b y \tag{4}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \tag{5}$$

$$\log_b x^n = n \log_b x. \tag{6}$$

There are a number of properties which can be used to change from one BASE to another

$$a = a^{\log_a b/\log_a b} = (a^{\log_a b})^{1/\log_a b} = b^{1/\log_a b} \tag{7}$$

$$\log_b a = \frac{1}{\log_a b} \tag{8}$$

$$\log_x z = \log_x(y^{\log_y z}) = \log_y z \log_x y \tag{9}$$

$$\log_y z = \frac{\log_x z}{\log_x y} \tag{10}$$

$$a^x = b^{x/\log_a b} = b^{x \log_b a}. \tag{11}$$

The logarithm BASE e is called the NATURAL LOGARITHM and is denoted $\ln x$ (LN). The logarithm BASE 10 is denoted $\log x$ (LOG), (although mathematics texts often use $\log x$ to mean $\ln x$). The logarithm BASE 2 is denoted $\lg x$ (LG).

An interesting property of logarithms follows from looking for a number y such that

$$\log_b(x+y) = -\log_b(x-y) \tag{12}$$

$$x + y = \frac{1}{x-y} \tag{13}$$

$$x^2 - y^2 = 1 \tag{14}$$

$$y = \sqrt{x^2 - 1}, \tag{15}$$

so

$$\log_b(x + \sqrt{x^2-1}) = -\log_b(x - \sqrt{x^2-1}). \tag{16}$$

Numbers of the form $\log_a b$ are IRRATIONAL if a and b are INTEGERS, one of which has a PRIME factor which the other lacks. A. Baker made a major step forward in TRANSCENDENTAL NUMBER theory by proving the transcendence of sums of numbers of the form $\alpha \ln \beta$ for α and β ALGEBRAIC NUMBERS.

see also ANTILOGARITHM, COLOGARITHM, e, EXPONENTIAL FUNCTION, HARMONIC LOGARITHM, LG, LN, LOG, LOGARITHMIC NUMBER, NAPIERIAN LOGARITHM, NATURAL LOGARITHM, POWER

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Logarithmic Binomial Formula

see LOGARITHMIC BINOMIAL THEOREM

Logarithmic Binomial Theorem

For all integers n and $|x| < a$,

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(a) x^k,$$

where $\lambda_n^{(t)}$ is the HARMONIC LOGARITHM and $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil$ is a ROMAN COEFFICIENT. For $t = 0$, the logarithmic binomial theorem reduces to the classical BINOMIAL THEOREM for POSITIVE n, since $\lambda_{n-k}^{(0)}(a) = a^{n-k}$ for $n \geq k$, $\lambda_{n-k}^{(0)}(a) = 0$ for $n < k$, and $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \binom{n}{k}$ when $n \geq k \geq 0$.

Similarly, taking $t = 1$ and $n < 0$ gives the NEGATIVE BINOMIAL SERIES. Roman (1992) gives expressions obtained for the case $t = 1$ and $n \geq 0$ which are not obtainable from the BINOMIAL THEOREM.

see also HARMONIC LOGARITHM, ROMAN COEFFICIENT

References

Roman, S. "The Logarithmic Binomial Formula." *Amer. Math. Monthly* **99**, 641–648, 1992.

Logarithmic Distribution

A CONTINUOUS DISTRIBUTION for a variate with probability function

$$P(x) = \frac{\log x}{b(\log b - 1) - a(\log a - 1)}$$

and distribution function

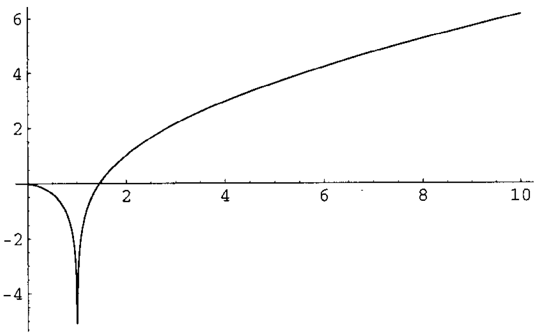
$$D(x) = \frac{a(1 - \log a) - x(1 - \log x)}{a(1 - \log a) - b(1 - \log b)}.$$

The MEAN is

$$\mu = \frac{a^2(1 - 2\log a) - b^2(1 - 2\log b)}{4[a(1 - \log a) - b(1 - \log b)]},$$

but higher order moments are rather messy.

Logarithmic Integral



The logarithmic integral is defined by

$$\mathrm{li}(x) \equiv \int_0^x \frac{du}{\ln u}. \tag{1}$$

The offset form appearing in the PRIME NUMBER THEOREM is defined so that $\mathrm{Li}(2) = 0$:

$$\begin{aligned} \mathrm{Li}(x) &\equiv \int_2^x \frac{du}{\ln u} & (2)\\ &= \mathrm{li}(x) - \mathrm{li}(2) \approx \mathrm{li}(x) - 1.04516 & (3)\\ &= \mathrm{ei}(\ln x), & (4) \end{aligned}$$

where $\mathrm{ei}(x)$ is the EXPONENTIAL INTEGRAL. (Note that the NOTATION $\mathrm{Li}_n(z)$ is also used for the POLYLOGARITHM.) Nielsen (1965, pp. 3 and 11) showed and Ramanujan independently discovered (Berndt 1994) that

$$\int_\mu^x \frac{dt}{\ln t} = \gamma + \ln \ln x + \sum_{k=1}^\infty \frac{(\ln x)^k}{k!k}, \tag{5}$$

where γ is the EULER-MASCHERONI CONSTANT and μ is SOLDNER'S CONSTANT. Another FORMULA due to Ramanujan which converges more rapidly is

$$\int_\mu^x \frac{dt}{\ln t} = \gamma + \ln \ln x + \sqrt{x} \sum_{n=0}^\infty \frac{(-1)^{n-1}(\ln x)^n}{n!2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1} \tag{6}$$

(Berndt 1994).

see also POLYLOGARITHM, PRIME CONSTELLATION, PRIME NUMBER THEOREM, SKEWES NUMBER

References

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 126–131, 1994.
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Logarithmic Number

A COEFFICIENT of the MACLAURIN SERIES of

$$\frac{1}{\ln(1+x)} = \frac{1}{x} + \tfrac{1}{2} + \tfrac{1}{12}x^2 - \tfrac{19}{720}x^3 + \tfrac{3}{160}x^4 + \ldots$$

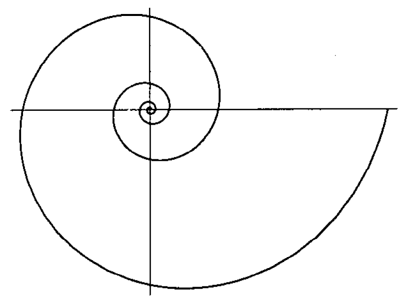
(Sloane's A002206 and A002207), the multiplicative inverse of the MERCATOR SERIES function $\ln(1+x)$.

see also MERCATOR SERIES

References

Sloane, N. J. A. Sequences A002206/M5066 and A002207/M2017 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Logarithmic Spiral



A curve whose equation in POLAR COORDINATES is given by

$$r = ae^{b\theta}, \tag{1}$$

where r is the distance from the ORIGIN, θ is the angle from the x-axis, and a and b are arbitrary constants. The logarithmic spiral is also known as the GROWTH SPIRAL, EQUIANGULAR SPIRAL, and SPIRA MIRABILIS. It can be expressed parametrically using

$$\cos\theta = \frac{1}{\sqrt{1 - \tan^2\theta}} = \frac{1}{\sqrt{1 + \frac{y^2}{x^2}}} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \tag{2}$$

which gives

$$\begin{aligned} x &= r\cos\theta = a\cos\theta e^{b\theta} & (3)\\ y &= x\tan\theta = r\sin\theta = a\sin\theta e^{b\theta}. & (4) \end{aligned}$$

The logarithmic spiral was first studied by Descartes in 1638 and Jakob Bernoulli. Bernoulli was so fascinated by the spiral that he had one engraved on his tombstone (although the engraver did not draw it true to form). Torricelli worked on it independently and found the length of the curve (MacTutor Archive).

The rate of change of RADIUS is

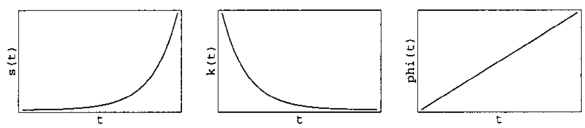
$$\frac{dr}{d\theta} = abe^{b\theta} = br, \tag{5}$$

and the ANGLE between the tangent and radial line at the point (r, θ) is

$$\psi = \tan^{-1}\left(\frac{r}{\frac{dr}{d\theta}}\right) = \tan^{-1}\left(\frac{1}{b}\right) = \cot^{-1} b. \tag{6}$$

So, as $b \to 0$, $\psi \to \pi/2$ and the spiral approaches a CIRCLE.

If P is any point on the spiral, then the length of the spiral from P to the origin is finite. In fact, from the point P which is at distance r from the origin measured along a RADIUS vector, the distance from P to the POLE along the spiral is just the ARC LENGTH. In addition, any RADIUS from the origin meets the spiral at distances which are in GEOMETRIC PROGRESSION (MacTutor Archive).



The ARC LENGTH, CURVATURE, and TANGENTIAL ANGLE of the logarithmic spiral are

$$s = \int ds = \int \sqrt{x'^2 + y'^2}\, dt = \frac{a\sqrt{1+b^2}}{b} e^{b\theta}$$
$$= \frac{r\sqrt{1+b^2}}{b} \tag{7}$$
$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} = (a\sqrt{1+b^2}e^{b\theta})^{-1} \tag{8}$$
$$\phi = \int \kappa(s)\, ds = \theta. \tag{9}$$

The CESÀRO EQUATION is

$$\kappa = \frac{1}{bs}. \tag{10}$$

On the surface of a SPHERE, the analog is a LOXODROME. This SPIRAL is related to FIBONACCI NUMBERS and the GOLDEN MEAN.

References

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MacTutor History of Mathematics Archive. "Equiangular Spiral." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Equiangular.html`.

Logarithmic Spiral Caustic Curve

The CAUSTIC of a LOGARITHMIC SPIRAL, where the pole is taken as the RADIANT POINT, is an equal LOGARITHMIC SPIRAL.

Logarithmic Spiral Evolute

$$R = \frac{(r^2 + r_\theta{}^2)^{3/2}}{r^2 + 2r^2 r_\theta{}^2 - r r_{\theta\theta}}. \tag{1}$$

Using

$$r = ae^{b\theta} \qquad r_\theta = abe^{b\theta} \qquad r_{\theta\theta} = ab^2 e^{b\theta} \tag{2}$$

gives

$$\begin{aligned} R &= \frac{(a^2 e^{2b\theta} + a^2 b^2 e^{2b\theta})^{3/2}}{(ae^{b\theta})^2 + 2(abe^{b\theta})^2 - (ab^{b\theta})(ab^2 e^{b\theta})} \\ &= \frac{(1+b^2)^{3/2} a^3 e^{3b\theta}}{2a^2 b^2 e^{2b\theta} + a^2 e^{2b\theta} - a^2 b^2 e^{2b\theta}} \\ &= \frac{(1+b^2)^{3/2} a^3 e^{3b\theta}}{a^2 b^2 e^{2b\theta} + a^2 e^{2b\theta}} = \frac{(1+b^2)^{3/2} a^3 e^{3b\theta}}{a^2 (1+b^2) e^{2b\theta}} \\ &= a\sqrt{1+b^2}\, e^{b\theta} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} ae^{b\theta}\cos\theta \\ ae^{b\theta}\sin\theta \end{bmatrix} \\ \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} abe^{b\theta}\cos\theta - ae^{b\theta}\sin\theta \\ abe^{b\theta}\sin\theta + ae^{b\theta}\cos\theta \end{bmatrix} \\ &= ae^{b\theta} \begin{bmatrix} b\cos\theta - \sin\theta \\ b\sin\theta + \cos\theta \end{bmatrix}, \end{aligned} \tag{4}$$

so

$$\begin{aligned} |\mathbf{r}'| &= ae^{b\theta}\sqrt{(b\cos\theta - \sin\theta)^2 + (b\sin\theta + \cos\theta)^2} \\ &= ae^{b\theta}\sqrt{1+b^2}, \end{aligned} \tag{5}$$

and the TANGENT VECTOR is given by

$$\begin{aligned} \hat{\mathbf{T}} &= \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{ae^{b\theta}\sqrt{1+b^2}} \begin{bmatrix} ae^{b\theta}\cos\theta \\ ae^{b\theta}\sin\theta \end{bmatrix} \\ &= \frac{1}{\sqrt{1+b^2}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \end{aligned} \tag{6}$$

The coordinates of the EVOLUTE are therefore

$$\xi = -abe^{b\theta}\sin\theta \tag{7}$$
$$\eta = abe^{b\theta}\cos\theta. \tag{8}$$

So the EVOLUTE is another logarithmic spiral with $a' \equiv ab$, as first shown by Johann Bernoulli. However, in some cases, the EVOLUTE is identical to the original, as can be demonstrated by making the substitution to the new variable

$$\theta \equiv \phi - \tfrac{1}{2}\pi \pm 2n\pi. \tag{9}$$

Then the above equations become

$$\begin{aligned}\xi &= -abe^{b(\phi-\pi/2\pm 2n\pi)}\sin(\phi-\pi/2\pm 2n\pi)\\ &= abe^{b\phi}e^{b(-\pi/2\pm 2n\pi)}\cos\phi \qquad (10)\\ \eta &= abe^{b(\phi-\pi/2\pm 2n\pi)}\cos(\phi-\pi/2\pm 2n\pi)\\ &= abe^{b\phi}e^{b(-\pi/2\pm 2n\pi)}\sin\phi, \qquad (11)\end{aligned}$$

which are equivalent to the form of the original equation if

$$be^{b(-\frac{1}{2}\pi\pm 2n\pi)} = 1 \qquad (12)$$

$$\ln b + b(-\tfrac{1}{2}\pi \pm 2n\pi) = 0 \qquad (13)$$

$$\frac{\ln b}{b} = \tfrac{1}{2}\pi \mp 2n\pi = -(2n-\tfrac{1}{2})\pi, \qquad (14)$$

where only solutions with the minus sign in $\mp$ exist. Solving gives the values summarized in the following table.

n	b_n	$\psi = \cot^{-1} b_n$
1	0.2744106319...	74°39′18.53″
2	0.1642700512...	80°40′16.80″
3	0.1218322508...	83°03′13.53″
4	0.0984064967...	84°22′47.53″
5	0.0832810611...	85°14′21.60″
6	0.0725974881...	85°50′51.92″
7	0.0645958183...	86°18′14.64″
8	0.0583494073...	86°39′38.20″
9	0.0533203211...	86°56′52.30″
10	0.0491732529...	87°11′05.45″

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Logarithmic Spiral Inverse Curve

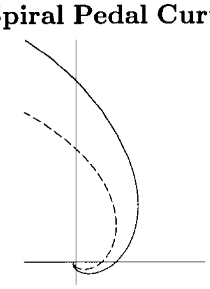
The INVERSE CURVE of the LOGARITHMIC SPIRAL

$$r = e^{a\theta}$$

with INVERSION CENTER at the origin and inversion radius k is the LOGARITHMIC SPIRAL

$$r = ke^{-a\theta}.$$

Logarithmic Spiral Pedal Curve



The PEDAL CURVE of a LOGARITHMIC SPIRAL with parametric equation

$$f = e^{at}\cos t \qquad (1)$$

$$g = e^{at}\sin t \qquad (2)$$

for a PEDAL POINT at the pole is an identical LOGARITHMIC SPIRAL

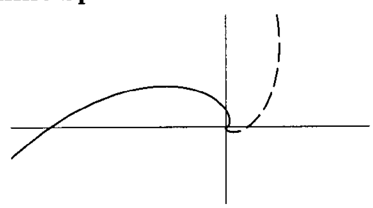
$$x = \frac{(a\sin t + \cos t)e^{at}}{1+a^2} \qquad (3)$$

$$y = \frac{(\sin t - a\cos t)e^{at}}{1+a^2}, \qquad (4)$$

so

$$r = \sqrt{x^2+y^2} = \frac{e^{at}}{\sqrt{1+a^2}}. \qquad (5)$$

Logarithmic Spiral Radial Curve



The RADIAL CURVE of the LOGARITHMIC SPIRAL is another LOGARITHMIC SPIRAL.

Logarithmically Convex Function

A function $f(x)$ is logarithmically convex on the interval $[a,b]$ if $f > 0$ and $\ln f(x)$ is CONCAVE on $[a,b]$. If $f(x)$ and $g(x)$ are logarithmically convex on the interval $[a,b]$, then the functions $f(x)+g(x)$ and $f(x)g(x)$ are also logarithmically convex on $[a,b]$.

see also CONVEX FUNCTION

References

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Logic

The formal mathematical study of the methods, structure, and validity of mathematical deduction and proof. Formal logic seeks to devise a complete, consistent formulation of mathematics such that propositions can be formally stated and proved using a small number of symbols with well-defined meanings. While this sounds like an admirable pursuit in principle, in practice the study of mathematical logic can rapidly become bogged down in pages of dense and unilluminating mathematical symbols, of which Whitehead and Russell's *Principia Mathematica* (1925) is perhaps the best (or worst) example.

A very simple form of logic is the study of "TRUTH TABLES" and digital logic circuits in which one or more outputs depend on a combination of circuit elements (AND, NAND, OR, XOR, etc.; "gates") and the input values. In such a circuit, values at each point can take on values of only TRUE (1) or FALSE (0). DE MORGAN'S DUALITY LAW is a useful principle for the analysis and simplification of such circuits.

A generalization of this simple type of logic in which possible values are TRUE, FALSE, and "undecided" is called THREE-VALUED LOGIC. A further generalization called FUZZY LOGIC treats "truth" as a continuous quantity ranging from 0 to 1.

see also ABSORPTION LAW, ALETHIC, BOOLEAN ALGEBRA, BOOLEAN CONNECTIVE, BOUND, CALIBAN PUZZLE, CONTRADICTION LAW, DE MORGAN'S DUALITY LAW, DE MORGAN'S LAWS, DEDUCIBLE, EXCLUDED MIDDLE LAW, FREE, FUZZY LOGIC, GÖDEL'S INCOMPLETENESS THEOREM, KHOVANSKI'S THEOREM, LOGICAL PARADOX, LOGOS, LÖWENHEIMER-SKOLEM THEOREM, METAMATHEMATICS, MODEL THEORY, QUANTIFIER, SENTENCE, TARSKI'S THEOREM, TAUTOLOGY, THREE-VALUED LOGIC, TOPOS, TRUTH TABLE, TURING MACHINE, UNIVERSAL STATEMENT, UNIVERSAL TURING MACHINE, VENN DIAGRAM, WILKIE'S THEOREM

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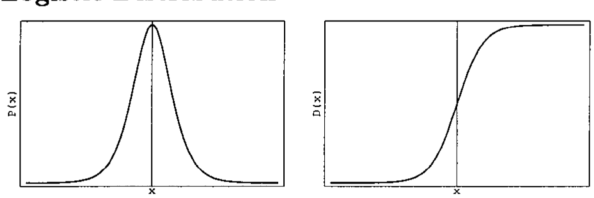
Logical Paradox

see PARADOX

Logistic Distribution



$$P(x) = \frac{e^{(x-m)/b}}{|b|[1+e^{(x-m)/b}]^2} \tag{1}$$

$$D(x) = \frac{1}{1+e^{(m-x)/|b|}}, \tag{2}$$

and the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = m \tag{3}$$
$$\sigma^2 = \tfrac{1}{3}\pi^2\beta^2 \tag{4}$$
$$\gamma_1 = 0 \tag{5}$$
$$\gamma_2 = \tfrac{6}{5}. \tag{6}$$

see also LOGISTIC EQUATION, LOGISTIC GROWTH CURVE

References
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Logistic Equation

The logistic equation (sometimes called the VERHULST MODEL since it was first published in 1845 by the Belgian P.-F. Verhulst) is defined by

$$x_{n+1} = rx_n(1-x_n), \tag{1}$$

where r (sometimes also denoted μ) is a POSITIVE constant (the "biotic potential"). We will start x_0 in the interval $[0,1]$. In order to keep points in the interval, we must find appropriate conditions on r. The maximum value x_{n+1} can take is found from

$$\frac{dx_{n+1}}{dx_n} = r(1-2x_n) = 0, \tag{2}$$

so the largest value of x_{n+1} occurs for $x_n = 1/2$. Plugging this in, $\max(x_{n+1}) = r/4$. Therefore, to keep the MAP in the desired region, we must have $r \in (0,4]$. The JACOBIAN is

$$J = \left|\frac{dx_{n+1}}{dx_n}\right| = |r(1-2x_n)|, \tag{3}$$

and the MAP is stable at a point x_0 if $J(x_0) < 1$. Now we wish to find the FIXED POINTS of the MAP, which occur when $x_{n+1} = x_n$. Drop the n subscript on x_n

$$f(x) = rx(1-x) = x \tag{4}$$

$$x[1-r(1-x)] = x(1-r+rx) = rx\left[x-\left(1-r^{-1}\right)\right] = 0, \tag{5}$$

so the FIXED POINTS are $x_1^{(1)} = 0$ and $x_2^{(1)} = 1 - r^{-1}$. An interesting thing happens if a value of r greater than 3 is chosen. The map becomes unstable and we get a PITCHFORK BIFURCATION with two stable orbits of period two corresponding to the two stable FIXED POINTS of $f^2(x)$. The fixed points of order two must satisfy $x_{n+2} = x_n$, so

$$\begin{aligned} x_{n+2} &= rx_{n+1}(1-x_{n+1}) \\ &= r[rx_n(1-x_n)][1-rx_n(1-x_n)] \\ &= r^2x_n(1-x_n)(1-rx_n+rx_n{}^2) = x_n. \end{aligned} \tag{6}$$

Now, drop the n subscripts and rewrite

$$x\{r^2[1-x(1+r)+2rx^2-rx^3]-1\}=0 \quad (7)$$

$$x[-r^3x^3+2r^3x^2-r^2(1+r)x+(r^2-1)]=0 \quad (8)$$

$$-r^3x[x-(1-r^{-1})][x^2-(1+r^{-1})x+r^{-1}(1+r^{-1})] = 0. \quad (9)$$

Notice that we have found the first-order FIXED POINTS as well, since two iterations of a first-order FIXED POINT produce a trivial second-order FIXED POINT. The true 2-CYCLES are given by solutions to the quadratic part

$$\begin{aligned} x_\pm^{(2)} &= \tfrac{1}{2}[(1+r^{-1})\pm\sqrt{(1+r^{-1})^2-4r^{-1}(1+r^{-1})}\,] \\ &= \tfrac{1}{2}[(1+r^{-1})\pm\sqrt{1+2r^{-1}+r^{-2}-4r^{-1}-4r^{-2}}\,] \\ &= \tfrac{1}{2}[(1+r^{-1})\pm\sqrt{1-2r^{-1}-3r^{-2}}\,] \\ &= \tfrac{1}{2}[(1+r^{-1})\pm r^{-1}\sqrt{(r-3)(r+1)}\,]. \end{aligned} \quad (10)$$

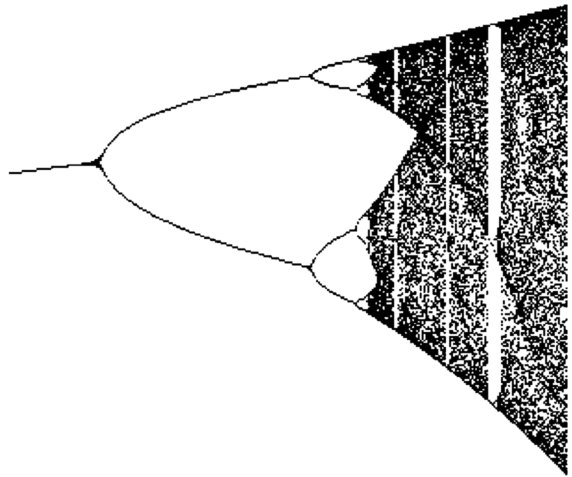
These solutions are only REAL for $r \geq 3$, so this is where the 2-CYCLE begins. Now look for the onset of the 4-CYCLE. To eliminate the 2- and 1-CYCLES, consider

$$\frac{f^4(x)-x}{f^2(x)-x}=0. \quad (11)$$

This gives

$$\begin{aligned} 1&+r^2+(-r^2-r^3-r^4-r^5)x \\ &+(2r^3+r^4+4r^5+r^6+2r^7)x^2 \\ &+(-r^3-5r^5-4r^6-5r^7-4r^8-r^9)x^3 \\ &+(2r^5+6r^6+4r^7+14r^8+5r^9+3r^{10})x^4 \\ &+(-4r^6-r^7-18r^8-12r^9-12r^{10}-3r^{11})x^5 \\ &+(r^6+10r^8+17r^9+18r^{10}+15r^{11}+r^{12})x^6 \\ &+(-2r^8-14r^9-12r^{10}-30r^{11}-6r^{12})x^7 \\ &+(6r^9+3r^{10}+30r^{11}+15r^{12})x^8 \\ &+(-r^9-15r^{11}-20r^{12})x^9+(3r^{11}+15r^{12})x^{10} \\ &-6r^{12}x^{11}+r^{12}x^{12}. \end{aligned} \quad (12)$$

The ROOTS of this equation are all IMAGINARY for $r < 1+\sqrt{6}$, but two of them convert to REAL roots at this value (although this is difficult to show except by plugging in). The 4-CYCLE therefore starts at $1+\sqrt{6}=3.449490\ldots$. The BIFURCATIONS come faster and faster (8, 16, 32, ...), then suddenly break off. Beyond a certain point known as the ACCUMULATION POINT, periodicity gives way to CHAOS.



A table of the CYCLE type and value of r_n at which the cycle 2^n appears is given below.

n	cycle (2^n)	r_n
1	2	3
2	4	3.449490
3	8	3.544090
4	16	3.564407
5	32	3.568750
6	64	3.56969
7	128	3.56989
8	256	3.569934
9	512	3.569943
10	1024	3.5699451
11	2048	3.569945557
∞	acc. pt.	3.569945672

For additional values, see Rasband (1990, p. 23). Note that the table in Tabor (1989, p. 222) is incorrect, as is the $n=2$ entry in Lauweirer 1991. In the middle of the complexity, a window suddenly appears with a regular period like 3 or 7 as a result of MODE LOCKING. The period 3 BIFURCATION occurs at $r=1+2\sqrt{2}=3.828427\ldots$, as is derived below. Following the 3-CYCLE, the PERIOD DOUBLINGS then begin again with CYCLES of 6, 12, ... and 7, 14, 28, ..., and then once again break off to CHAOS.

A set of $n+1$ equations which can be solved to give the onset of an arbitrary n-cycle (Saha and Strogatz 1995) is

$$\begin{cases} x_2 = rx_1(1-x_1) \\ x_3 = rx_2(1-x_2) \\ \vdots \\ x_n = rx_{n-1}(1-x_{n-1}) \\ x_1 = rx_n(1-x_n) \\ r^n\prod_{k=1}^n(1-2x_k)=1. \end{cases} \quad (13)$$

The first n of these give $f(x)$, $f^2(x)$, ..., $f^n(x)$, and the last uses the fact that the onset of period n occurs by a TANGENT BIFURCATION, so the nth DERIVATIVE is 1.

For $n = 2$, the solutions $(x_1, \ldots, x_n, r)$ are $(0, 0, \pm 1)$ and $(2/3, 2/3, 3)$, so the desired BIFURCATION occurs at $r_2 = 3$. Taking $n = 3$ gives

$$\begin{aligned} \frac{d[f^3(x)]}{dx} &= \frac{d[f^3(x)]}{d[f^2(x)]}\frac{d[f^2(x)]}{d[f(x)]}\frac{d[f(x)]}{dx} \\ &= \frac{d[f(z)]}{dz}\frac{d[f(y)]}{dy}\frac{d[f(x)]}{dx} \\ &= r^3(1-2z)(1-2y)(1-2x). \end{aligned} \tag{14}$$

Solving the resulting CUBIC EQUATION using computer algebra gives

$$\begin{aligned} x_1 = &-\left(\frac{2^{5^6}}{63 \cdot 7^{1/3}} + \frac{1}{63 \cdot 28^{1/3}}\right)c^2 \\ &- \frac{1}{9 \cdot 98^{1/3}}c + \frac{10+\sqrt{2}}{21} + \left(\frac{4 \cdot 2^{5/6}}{9 \cdot 7^{1/3}} - \frac{2^{1/3}}{7^{1/3}}\right)c^{-1} \\ &+ \frac{25 \cdot 28^{1/3} - 44 \cdot 2^{1/6}7^{1/3}}{9}c^{-2} \end{aligned} \tag{15}$$

$$\begin{aligned} x_2 = &\left(\frac{1}{63 \cdot 28^{1/3}} + \frac{2^{5/6}}{63 \cdot 7^{1/3}}\right)c^2 - \frac{2^{2/3}}{9 \cdot 7^{2/3}}c + \frac{10+\sqrt{2}}{21} \\ &+ \left(\frac{8 \cdot 2^{5/6}}{9 \cdot 7^{1/3}} - \frac{2 \cdot 2^{1/3}}{7^{1/3}}\right)c^{-1} \\ &+ \frac{44 \cdot 2^{1/6}7^{1/3} - 25 \cdot 28^{1/3}}{9}c^{-2} \end{aligned} \tag{16}$$

$$x_3 = \frac{1}{3 \cdot 98^{1/3}}c + \frac{10+\sqrt{2}}{21} + \frac{2^{1/3}(9-4\sqrt{2})}{3 \cdot 7^{1/3}}c^{-1} \tag{17}$$

$$r = 1 + 2\sqrt{2}, \tag{18}$$

where

$$c \equiv (-25 + 22\sqrt{2} + 3\sqrt{3}\sqrt{1100\sqrt{2} - 1593})^{1/3}. \tag{19}$$

Numerically,

$$x_1 = 0.514355\ldots \tag{20}$$

$$x_2 = 0.956318\ldots \tag{21}$$

$$x_3 = 0.159929\ldots \tag{22}$$

$$r = 3.828427\ldots. \tag{23}$$

Saha and Strogatz (1995) give a simplified algebraic treatment which involves solving

$$r^3(1 - 2\alpha + 4\beta - 8\gamma) = 1, \tag{24}$$

together with three other simultaneous equations, where

$$\alpha \equiv x_1 + x_2 + x_3 \tag{25}$$

$$\beta \equiv x_1x_2 + x_1x_3 + x_2x_3 \tag{26}$$

$$\gamma \equiv x_1x_2x_3. \tag{27}$$

Further simplifications still are provided in Bechhoeffer (1996) and Gordon (1996), but neither of these techniques generalizes easily to higher CYCLES. Bechhoeffer (1996) expresses the three additional equations as

$$2\alpha = 3 + r^{-1} \tag{28}$$

$$4\beta = \tfrac{3}{2} + 5r^{-1} + \tfrac{3}{2}r^{-2} \tag{29}$$

$$8\gamma = -\tfrac{1}{2} + \tfrac{7}{2}r^{-1} + \tfrac{5}{2}r^{-2} + \tfrac{5}{2}r^{-3}, \tag{30}$$

giving

$$r^2 - 2r - 7 = 0. \tag{31}$$

Gordon (1996) derives not only the value for the onset of the 3-CYCLE, but also an upper bound for the r-values supporting stable period 3 orbits. This value is obtained by solving the CUBIC EQUATION

$$s^3 - 11s^2 + 37s - 108 = 0 \tag{32}$$

for s, then

$$r' = 1 + \sqrt{s} \tag{33}$$

$$\begin{aligned} &= 1 + \sqrt{\tfrac{11}{3} + (\tfrac{1915}{54} + \tfrac{5}{2}\sqrt{201})^{1/3} + (\tfrac{1915}{54} - \tfrac{5}{2}\sqrt{201})^{1/3}} \\ &= 3.841499007543\ldots. \end{aligned} \tag{34}$$

The logistic equation has CORRELATION EXPONENT 0.500 ± 0.005 (Grassberger and Procaccia 1983), CAPACITY DIMENSION 0.538 (Grassberger 1981), and INFORMATION DIMENSION 0.5170976 (Grassberger and Procaccia 1983).

see also BIFURCATION, FEIGENBAUM CONSTANT, LOGISTIC DISTRIBUTION, LOGISTIC EQUATION—$r = 4$, LOGISTIC GROWTH CURVE, PERIOD THREE THEOREM, QUADRATIC MAP

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Logistic Equation—$r = 4$

With $r = 4$, the LOGISTIC EQUATION becomes

$$x_{n+1} = 4x_n(1 - x_n). \tag{1}$$

Now let

$$x \equiv \sin^2(\tfrac{1}{2}\pi y) = \tfrac{1}{2}[1 - \cos(\pi y)] \tag{2}$$

$$\sqrt{x} = \sin(\tfrac{1}{2}\pi y) \tag{3}$$

$$y = \frac{2}{\pi}\sin^{-1}(\sqrt{x}) \tag{4}$$

$$\frac{dy}{dx} = \frac{2}{\pi}\frac{1}{\sqrt{1-x}}\tfrac{1}{2}x^{-1/2} = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{5}$$

Manipulating (2) gives

$$\begin{aligned}\sin^2(\tfrac{1}{2}\pi y_{n+1})& \\ &= 4\tfrac{1}{2}[1 - \cos(\pi y_n)]\{1 - \tfrac{1}{2}[1 - \tfrac{1}{2}(1 - \cos(\pi y_n)]\} \\ &= 2[1 - \cos(\pi y = 1 - \cos^2(\pi y_n)\sin^2(\pi y_n),\end{aligned} \tag{6}$$

so

$$\tfrac{1}{2}\pi y_{n+1} = \pm y_n + s\pi \tag{7}$$

$$y_{n+1} = \pm 2y_n + \tfrac{1}{2}s. \tag{8}$$

But $y \in [0, 1]$. Taking $y_n \in [0, 1/2]$, then $s = 0$ and

$$y_{n+1} = 2y_n. \tag{9}$$

For $y \in [1/2, 1]$, $s = 1$ and

$$y_{n+1} = 2 - 2y_n. \tag{10}$$

Combining

$$y_n = \begin{cases} 2y_n & \text{for } y_n \in [0, \tfrac{1}{2}] \\ 2 - 2y_n & \text{for } y_n \in [\tfrac{1}{2}, 1], \end{cases} \tag{11}$$

which can be written

$$y_n = 1 - 2|x_n - h|, \tag{12}$$

the TENT MAP with $\mu = 1$, so the NATURAL INVARIANT in y is

$$\rho(y) = 1. \tag{13}$$

Transforming back to x gives

$$\begin{aligned}\rho(x) &= \left|\frac{dy}{dx}\right|\rho(y(x)) = \frac{2}{\pi}\frac{1}{\sqrt{1-x}}\tfrac{1}{2}x^{-1/2} \\ &= \frac{1}{\pi\sqrt{x(1-x)}}.\end{aligned} \tag{14}$$

This can also be derived from

$$\rho(x) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\delta(x_i - x) = \frac{1}{\pi\sqrt{x(1-x)}}, \tag{15}$$

where $\delta(x)$ is the DELTA FUNCTION.

see also LOGISTIC EQUATION

Logistic Growth Curve

The POPULATION GROWTH law which arises frequently in biology and is given by the differential equation

$$\frac{dN}{dt} = \frac{r(K - N)}{K}, \tag{1}$$

where r is the MALTHUSIAN PARAMETER and K is the so-called CARRYING CAPACITY (i.e., the maximum sustainable population). Rearranging and integrating both sides gives

$$\int_{N_0}^{N}\frac{dN}{K - N} = \frac{r}{K}\int_0^t dt \tag{2}$$

$$\ln\left(\frac{N_0 - K}{N - K}\right) = \frac{r}{K}t \tag{3}$$

$$N(t) = K + (N_0 - K)e^{-rt/K}. \tag{4}$$

The curve

$$y = \frac{a}{1 + bq^x} \tag{5}$$

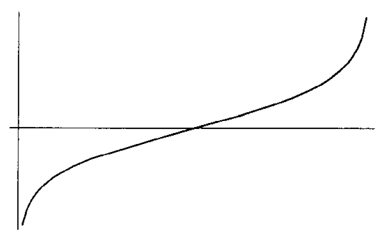
is sometimes also known as the logical curve.

see also GOMPERTZ CURVE, LIFE EXPECTANCY, LOGISTIC EQUATION, MAKEHAM CURVE, MALTHUSIAN PARAMETER, POPULATION GROWTH

Logistic Map

see LOGISTIC EQUATION

Logit Transformation



The function

$$z = f(x) = \ln\left(\frac{x}{1-x}\right).$$

This function has an inflection point at $x = 1/2$, where

$$f''(x) = \frac{2x-1}{x^2(x-1)^2} = 0.$$

Applying the logit transformation to values obtained by iterating the LOGISTIC EQUATION generates a sequence of RANDOM NUMBERS having distribution

$$P_z = \frac{1}{\pi(e^{x/2} + e^{-x/2})},$$

which is very close to a GAUSSIAN DISTRIBUTION.

References

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Logos

A generalization of a HEYTING ALGEBRA which replaces BOOLEAN ALGEBRA in "intuitionistic" LOGIC.

see also TOPOS

Lommel Differential Equation

A generalization of the BESSEL DIFFERENTIAL EQUATION (Watson 1966, p. 345),

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2+\nu^2)y = kz^{\mu+1}.$$

A further generalization gives

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2+\nu^2)y = \pm kz^{\mu+1}.$$

The solutions are LOMMEL FUNCTIONS.

see also LOMMEL FUNCTION

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1966.

Lommel Function

There are several functions called "Lommel functions." One type of Lommel function is the solution to the LOMMEL DIFFERENTIAL EQUATION with a PLUS SIGN,

$$y = ks_{\mu,\nu}(z), \tag{1}$$

where

$$s_{\mu,\nu}^{(+)}(z) \equiv \tfrac{1}{2}\pi\left[Y_\nu(z)\int_0^z z^\mu J_\nu(z)\,dz - J_\nu(z)\int_0^z z^\mu Y_\nu(z)\,dz\right]. \tag{2}$$

Here, $J_\nu(z)$ and $Y_\nu(z)$ are BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS (Watson 1966, p. 346). If a minus sign precedes k, then the solution is

$$s_{\mu,\nu}^{(-)} \equiv I_\nu(z)\int_z^{c_1} z^\mu K_\nu(z)\,dz - J_\nu(z)\int_{c_2}^z z^\mu I_\nu(z)\,dz, \tag{3}$$

where $K_\nu(z)$ and $I_\nu(z)$ are MODIFIED BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS.

Lommel functions of two variables are related to the BESSEL FUNCTION OF THE FIRST KIND and arise in the theory of diffraction and, in particular, Mie scattering (Watson 1966, p. 537),

$$U_n(w,z) = \sum_{m=0}^\infty (-1)^m \left(\frac{w}{z}\right)^{n+2m} J_{n+2m}(z) \tag{4}$$

$$V_n(w,z) = \sum_{m=0}^\infty (-1)^m \left(\frac{w}{z}\right)^{-n-2m} J_{-n-2m}(z). \tag{5}$$

see also LOMMEL DIFFERENTIAL EQUATION, LOMMEL POLYNOMIAL

References

Chandrasekhar, S. *Radiative Transfer.* New York: Dover, p. 369, 1960.

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Lommel's Integrals

$$(\beta^2-\alpha^2)\int xJ_n(\alpha x)J_n(\beta x)\,dx = x[\alpha J_n'(\alpha x)J_n(\beta x) - \beta J_n'(\beta x)J_n(\alpha x)]$$

$$\int x{J_n}^2(\alpha x)\,dx = \tfrac{1}{2}x^2[{J_n}^2(\alpha x) + J_{n-1}(\alpha x)J_{n+1}(\alpha x)],$$

where $J_n(x)$ is a BESSEL FUNCTION OF THE FIRST KIND.

References

Bowman, F. *Introduction to Bessel Functions.* New York: Dover, p. 101, 1958.

Lommel Polynomial

$$\begin{aligned}R_{m,\nu}(z) &= \frac{\Gamma(\nu+m)}{\Gamma(\nu)(z/2)^m}\,{}_2F_3(\tfrac{1}{2}(1-m), -\tfrac{1}{2}m; \nu, -m, 1-\nu-m; z^2)\\ &\quad\times\frac{\pi z}{2\sin(\nu\pi)}[J_{\nu+m}(z)J_{-\nu+1}(z) + (-1)^m J_{-\nu-m}(z)J_{\nu-1}(z)],\end{aligned}$$

where $\Gamma(z)$ is a GAMMA FUNCTION, $J_n(x)$ is a BESSEL FUNCTION OF THE FIRST KIND, and ${}_2F_3(a,b;c,d,e;z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION.

see also LOMMEL FUNCTION

References

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Long Division

```
       7                 72                  726
17 ) 123456.      17 ) 123456.       17 ) 123456.
    -119              -119               -119
      44                44                 44
                       -34                -34
                        105                105
                                          -102
                                            36

     7262.1             7262.11...
17 ) 123456.0     17 ) 123456.00
    -119              -119
      44                44
     -34               -34
      105               105
     -102              -102
        36                36
       -34               -34
        20                20
                         -17
                          30
```

Long division is an algorithm for dividing two numbers, obtaining the QUOTIENT one DIGIT at a time. The above example shows how the division of 123456/17 is performed to obtain the result 7262.11....

see also DIVISION

Long Exact Sequence of a Pair Axiom

One of the EILENBERG-STEENROD AXIOMS. It states that, for every pair (X, A), there is a natural long exact sequence

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \ldots,$$

where the MAP $H_n(A) \to H_n(X)$ is induced by the INCLUSION MAP $A \to X$, $H_n(X) \to H_n(X, A)$ is induced by the INCLUSION MAP $(X, \phi)\check{} \to (X, A)$. The MAP $H_n(X, A) \to H_{n-1}(A)$ is called the BOUNDARY MAP.

see also EILENBERG-STEENROD AXIOMS

Long Prime

see DECIMAL EXPANSION

Longitude

The azimuthal coordinate on the surface of a SPHERE (θ in SPHERICAL COORDINATES) or on a SPHEROID (in PROLATE or OBLATE SPHEROIDAL COORDINATES). Longitude is defined such that $0° = 360°$. Lines of constant longitude are generally called MERIDIANS. The other angular coordinate on the surface of a SPHERE is called the LATITUDE.

The shortest distance between any two points on a SPHERE is the so-called GREAT CIRCLE distance, which can be directly computed from the LATITUDE and longitudes of two points.

see also GREAT CIRCLE, LATITUDE, MERIDIAN, OBLATE SPHEROIDAL COORDINATES, PROLATE SPHEROIDAL COORDINATES

Look and Say Sequence

The INTEGER SEQUENCE beginning with a single digit in which the next term is obtained by describing the previous term. Starting with 1, the sequence would be defined by "one 1, two 1s, one 2 two 1s," etc., and the result is 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ... (Sloane's A005150).

Starting the sequence instead with the digit d for $2 \leq d \leq 9$ gives d, $1d$, $111d$, $311d$, $13211d$, $111312211d$, $31131122211d$, $1321132132211d$, ... The sequences for $d = 2$ and 3 are Sloane's A006715 and A006751. The number of DIGITS in the nth term of both the sequences for $1 \leq n \leq 9$ is asymptotic to $C\lambda^n$, where C is a constant and

$$\lambda = 1.3035772690342696\ldots$$

(Sloane's A014715) is CONWAY'S CONSTANT. λ is given by the largest ROOT of the POLYNOMIAL

$$\begin{aligned}0 = {} & x^{71} \\ & - x^{69} - 2x^{68} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} \\ & - x^{60} - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} \\ & - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} + x^{49} + 9x^{48} - 3x^{47} \\ & - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} - 4x^{40} \\ & - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} - 3x^{34} + x^{35} + 10x^{33} \\ & + x^{32} - 6x^{31} - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} \\ & - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} + 9x^{20} - 3x^{19} - 4x^{18} \\ & - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} \\ & - 4x^{11} - 2x^{10} - 5x^{9} + x^{7} - 7x^{6} \\ & + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6.\end{aligned}$$

In fact, the constant is even more general than this, applying to *all* starting sequences (i.e., even those starting with arbitrary starting digits), with the exception of 22, a result which follows from the COSMOLOGICAL THEOREM. Conway discovered that strings sometimes factor as a concatenation of two strings whose descendants never interfere with one another. A string with no nontrivial splittings is called an "element," and other strings are called "compounds." Every string of 1s, 2s, and 3s eventually "decays" into a compound of 92 special elements, named after the chemical elements.

see also CONWAY'S CONSTANT, COSMOLOGICAL THEOREM

References

Conway, J. H. "The Weird and Wonderful Chemistry of Audioactive Decay." *Eureka,* 5–18, 1985.

Conway, J. H. "The Weird and Wonderful Chemistry of Audioactive Decay." §5.11 in *Open Problems in Communications and Computation.* (Ed. T. M. Cover and B. Gopinath). New York: Springer-Verlag, pp. 173–188, 1987.

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Sloane, N. J. A. Sequences A005150/M4780, A006715/M2965, and A6751/M2052 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 13–14, 1991.

Loop (Algebra)

A nonassociative ALGEBRA (and QUASIGROUP) which has a single binary operation.

Loop Gain

The loop gain is usually assigned a value between 0.1 and 0.5. The CLEAN ALGORITHM performs better for extended structures if μ is set to the lower part of this range. However, the time required for the CLEAN ALGORITHM increases rapidly for small μ. From Thompson *et al.* (1986), the number of cycles needed for one point source is

$$[\text{cycles}] = -\frac{\ln(\text{SNR})}{\ln(1-\gamma)}.$$

see also CLEAN ALGORITHM

References

Thompson, A. R.; Moran, J. M.; and Swenson, G. W. Jr. *Interferometry and Synthesis in Radio Astronomy.* New York: Wiley, p. 348, 1986.

Loop (Graph)

A degenerate edge of a graph which joins a vertex to itself.

Loop (Knot)

A KNOT or HITCH which holds its form rigidly.

References

Owen, P. *Knots.* Philadelphia, PA: Courage, p. 35, 1993.

Loop Space

Let Y^X be the set of continuous mappings $f: X \to Y$. Then the TOPOLOGICAL SPACE for Y^X supplied with a compact-open topology is called a MAPPING SPACE, and if $Y = I$ is taken as the interval $(0,1)$, then $Y^I = \Omega(Y)$ is called a loop space (or SPACE OF CLOSED PATHS).

see also MACHINE, MAPPING SPACE, MAY-THOMASON UNIQUENESS THEOREM

References

Brylinski, J.-L. *Loop Spaces, Characteristic Classes and Geometric Quantization.* Boston, MA: Birkhäuser, 1993.

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 658, 1980.

Lorentz Group

The Lorentz group is the GROUP L of time-preserving linear ISOMETRIES of MINKOWSKI SPACE $\mathbb{R}^4$ with the pseudo-Riemannian metric

$$d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

It is also the GROUP of ISOMETRIES of 3-D HYPERBOLIC SPACE. It is time-preserving in the sense that the unit time VECTOR $(1,0,0,0)$ is sent to another VECTOR (t,x,y,z) such that $t > 0$.

A consequence of the definition of the Lorentz group is that the full GROUP of time-preserving isometries of MINKOWSKI $\mathbb{R}^4$ is the DIRECT PRODUCT of the group of translations of $\mathbb{R}^4$ (i.e., $\mathbb{R}^4$ itself, with addition as the group operation), with the Lorentz group, and that the full isometry group of the MINKOWSKI $\mathbb{R}^4$ is a group extension of $\mathbb{Z}_2$ by the product $L \otimes \mathbb{R}^4$.

The Lorentz group is invariant under space rotations and LORENTZ TRANSFORMATIONS.

see also LORENTZ TENSOR, LORENTZ TRANSFORMATION

References

Arfken, G. "Homogeneous Lorentz Group." §4.13 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 271–275, 1985.

Lorentz Tensor

The TENSOR in the LORENTZ TRANSFORMATION given by

$$\mathsf{L} \equiv \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

where beta and gamma are defined by

$$\beta \equiv \frac{v}{c} \tag{2}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}. \tag{3}$$

see also LORENTZ GROUP, LORENTZ TRANSFORMATION

Lorentz Transformation

A 4-D transformation satisfied by all FOUR-VECTORS a^ν,

$$a'^\mu = \Lambda^\mu_\nu a^\nu. \tag{1}$$

In the theory of special relativity, the Lorentz transformation replaces the GALILEAN TRANSFORMATION as the valid transformation law between reference frames moving with respect to one another at constant VELOCITY. Let x^ν be the POSITION FOUR-VECTOR with $x^0 = ct$, and let the relative motion be along the x^1 axis with VELOCITY v. Then (1) becomes

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{2}$$

where the LORENTZ TENSOR is given by

$$\mathsf{L} = \begin{bmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{bmatrix} \equiv \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Here,

$$\beta \equiv \frac{v}{c} \tag{4}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}. \tag{5}$$

Written explicitly, the transformation between x^ν and $x^{\nu\prime}$ coordinate is

$$x^{0\prime} = \gamma(x^0 - \beta x^1) \tag{6}$$

$$x^{1\prime} = \gamma(x^1 - \beta x^0) \tag{7}$$

$$x^{2\prime} = x^2 \tag{8}$$

$$x^{3\prime} = x^3. \tag{9}$$

The DETERMINANT of the upper left 2×2 MATRIX in (3) is

$$D = (\gamma)^2 - (-\gamma\beta)^2 = \gamma^2(1-\beta^2) = \frac{\gamma^2}{\gamma^2} = 1, \tag{10}$$

so

$$\mathsf{L}^{-1} = \begin{bmatrix} (\Lambda^{-1})^0_0 & (\Lambda^{-1})^0_1 & (\Lambda^{-1})^0_2 & (\Lambda^{-1})^0_3 \\ (\Lambda^{-1})^1_0 & (\Lambda^{-1})^1_1 & (\Lambda^{-1})^1_2 & (\Lambda^{-1})^1_3 \\ (\Lambda^{-1})^2_0 & (\Lambda^{-1})^2_1 & (\Lambda^{-1})^2_2 & (\Lambda^{-1})^2_3 \\ (\Lambda^{-1})^3_0 & (\Lambda^{-1})^3_1 & (\Lambda^{-1})^3_2 & (\Lambda^{-1})^3_3 \end{bmatrix} \equiv \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

A Lorentz transformation along the x_1-axis can also be written

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} \cosh\theta & i\sinh\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sinh\theta & \cosh\theta & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{12}$$

where θ is called the rapidity,

$$x_4 \equiv ict, \tag{13}$$

and

$$\tanh\theta \equiv \beta \equiv \frac{v}{c} \tag{14}$$

$$\cosh\theta \equiv \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{15}$$

$$\sinh\theta = \beta\gamma. \tag{16}$$

see also HYPERBOLIC ROTATION, LORENTZ GROUP, LORENTZ TENSOR

References

Fraundorf, P. "Accel-1D: Frame-Dependent Relativity at UM-StL." `http://www.umsl.edu/~fraundor/a1toc.html`.

Griffiths, D. J. *Introduction to Electrodynamics.* Englewood Cliffs, NJ: Prentice-Hall, pp. 412–414, 1981.

Morse, P. M. and Feshbach, H. "The Lorentz Transformation, Four-Vectors, Spinors." §1.7 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 93–107, 1953.

Lorentzian Distribution

see CAUCHY DISTRIBUTION

Lorentzian Function

The Lorentzian function is given by

$$L(x) = \frac{1}{\pi} \frac{\frac{1}{2}\Gamma}{(x-x_0)^2 + (\frac{1}{2}\Gamma)^2}.$$

Its FULL WIDTH AT HALF MAXIMUM is Γ. This function gives the shape of certain types of spectral lines and is the distribution function in the CAUCHY DISTRIBUTION. The Lorentzian function has FOURIER TRANSFORM

$$\mathcal{F}\left[\frac{1}{\pi} \frac{\frac{1}{2}\Gamma}{(x-x_0)^2 + (\frac{1}{2}\Gamma)^2}\right] = e^{-2\pi i k x_0 - \Gamma\pi|k|}.$$

see also DAMPED EXPONENTIAL COSINE INTEGRAL, FOURIER TRANSFORM—LORENTZIAN FUNCTION

Lorenz System

A simplified system of equations describing the 2-D flow of fluid of uniform depth H, with an imposed temperature difference ΔT, under gravity g, with buoyancy α, thermal diffusivity κ, and kinematic viscosity ν. The full equations are

$$\frac{\partial}{\partial t}(\nabla^2\phi) = \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}(\nabla^2\psi) + \nu\nabla^2(\nabla^2\psi) + g\alpha\frac{dT}{dx} \tag{1}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial z}\frac{\partial\psi}{\partial x} - \frac{\partial\theta}{\partial x}\frac{\partial\psi}{\partial z} + \kappa\nabla^2 T + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x}. \tag{2}$$

Here, ψ is the "stream function," as usual defined such that

$$u = \frac{\partial \psi}{\partial x}, \qquad v = \frac{\partial \psi}{\partial x}. \tag{3}$$

In the early 1960s, Lorenz accidentally discovered the chaotic behavior of this system when he found that, for a simplified system, periodic solutions of the form

$$\psi = \psi_0 \sin\left(\frac{\pi a x}{H}\right) \sin\left(\frac{\pi z}{H}\right) \tag{4}$$

$$\theta = \theta_0 \cos\left(\frac{\pi a x}{H}\right) \sin\left(\frac{\pi z}{H}\right) \tag{5}$$

grew for Rayleigh numbers larger than the critical value, $Ra > Ra_c$. Furthermore, vastly different results were obtained for very small changes in the initial values, representing one of the earliest discoveries of the so-called BUTTERFLY EFFECT.

Lorenz included the following terms in his system of equations,

$$X \equiv \psi_{11} \propto \text{ convective intensity} \tag{6}$$

$$Y \equiv T_{11} \propto \Delta T \text{ between descending and ascending currents} \tag{7}$$

$$Z \equiv T_{02} \propto \Delta \text{ vertical temperature profile from linearity,} \tag{8}$$

and obtained the simplified equations

$$\dot{X} = \sigma(Y - X) \tag{9}$$

$$\dot{Y} = -XZ + rX - Y \tag{10}$$

$$\dot{Z} = XY - bZ, \tag{11}$$

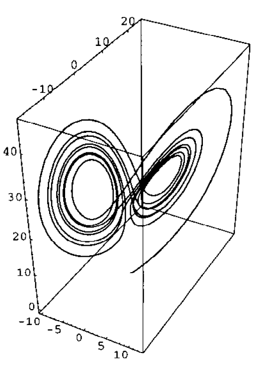
where

$$\sigma \equiv \frac{\nu}{\kappa} = \text{ Prandtl number} \tag{12}$$

$$r \equiv \frac{Ra}{Ra_c} = \text{ normalized Rayleigh number} \tag{13}$$

$$b \equiv \frac{4}{1 + a^2} = \text{ geometric factor.} \tag{14}$$

Lorenz took $b \equiv 8/3$ and $\sigma \equiv 10$.



The CRITICAL POINTS at (0, 0, 0) correspond to no convection, and the CRITICAL POINTS at

$$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \tag{15}$$

and

$$(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \tag{16}$$

correspond to steady convection. This pair is stable only if

$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}, \tag{17}$$

which can hold only for POSITIVE r if $\sigma > b + 1$. The Lorenz attractor has a CORRELATION EXPONENT of 2.05 ± 0.01 and CAPACITY DIMENSION 2.06 ± 0.01 (Grassberger and Procaccia 1983). For more details, see Lichtenberg and Lieberman (1983, p. 65) and Tabor (1989, p. 204).

see also BUTTERFLY EFFECT, RÖSSLER MODEL

References

Gleick, J. *Chaos: Making a New Science.* New York: Penguin Books, pp. 27–31, 1988.

Grassberger, P. and Procaccia, I. "Measuring the Strangeness of Strange Attractors." *Physica D* **9**, 189–208, 1983.

Lichtenberg, A. and Lieberman, M. *Regular and Stochastic Motion.* New York: Springer-Verlag, 1983.

Lorenz, E. N. "Deterministic Nonperiodic Flow." *J. Atmos. Sci.* **20**, 130–141, 1963.

Peitgen, H.-O.; Jürgens, H.; and Saupe, D. *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag, pp. 697–708, 1992.

Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, 1989.

Lorraine Cross

see GAULLIST CROSS

Lotka-Volterra Equations

An ecological model which assumes that a population x increases at a rate $dx = Ax\,dt$, but is destroyed at a rate $dx = -Bxy\,dt$. Population y decreases at a rate $dy = -Cy\,dt$, but increases at $dy = Dxy\,dt$, giving the coupled differential equations

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -Cy + Dxy.$$

Critical points occur when $dx/dt = dy/dt = 0$, so

$$A - By = 0$$

$$-C + Dx = 0.$$

The sole STATIONARY POINT is therefore located at $(x, y) = (C/D, A/B)$.

Low-Dimensional Topology

Low-dimensional topology usually deals with objects that are 2-, 3-, or 4-dimensional in nature. Properly speaking, low-dimensional topology should be part of DIFFERENTIAL TOPOLOGY, but the general machinery of ALGEBRAIC and DIFFERENTIAL TOPOLOGY gives only limited information. This fact is particularly noticeable in dimensions three and four, and so alternative specialized methods have evolved.

see also ALGEBRAIC TOPOLOGY, DIFFERENTIAL TOPOLOGY, TOPOLOGY

Löwenheimer-Skolem Theorem

A fundamental result in MODEL THEORY which states that if a countable theory has a model, then it has a countable model. Furthermore, it has a model of every CARDINALITY greater than or equal to $\aleph_0$ (ALEPH-0). This theorem established the existence of "nonstandard" models of arithmetic.

see also ALEPH-0 ($\aleph_0$), CARDINALITY, MODEL THEORY

References

Chang, C. C. and Keisler, H. J. *Model Theory, 3rd enl. ed.* New York: Elsevier, 1990.

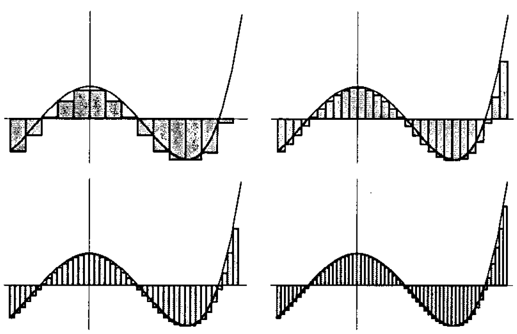
Lower Bound

see GREATEST LOWER BOUND

Lower Denjoy Sum

see LOWER SUM

Lower Integral



The limit of a LOWER SUM, when it exists, as the MESH SIZE approaches 0.

see also LOWER SUM, RIEMANN INTEGRAL, UPPER INTEGRAL

Lower Limit

Let the least term h of a SEQUENCE be a term which is smaller than all but a finite number of the terms which are equal to h. Then h is called the lower limit of the SEQUENCE.

A lower limit of a SERIES

$$\text{lower} \lim_{n\to\infty} S_n = \varliminf_{n\to\infty} S_n = h$$

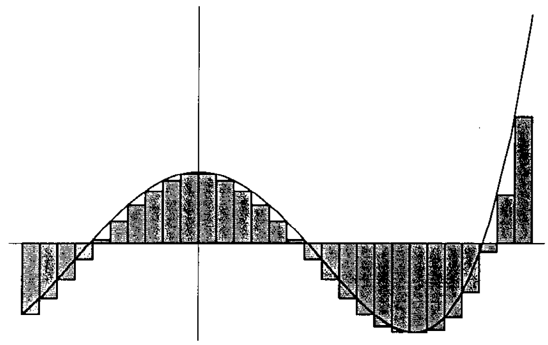
is said to exist if, for every $\epsilon > 0$, $|S_n - h| < \epsilon$ for infinitely many values of n and if no number less than h has this property.

see also LIMIT, UPPER LIMIT

References

Bromwich, T. J. I'a and MacRobert, T. M. "Upper and Lower Limits of a Sequence." §5.1 in *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 40 1991.

Lower Sum



For a given function $f(x)$ over a partition of a given interval, the lower sum is the sum of box areas $f(x_k^*)\Delta x_k$ using the smallest value of the function $f(x_k^*)$ in each subinterval Δx_k.

see also LOWER INTEGRAL, RIEMANN INTEGRAL, UPPER SUM

Lower-Trimmed Subsequence

The lower-trimmed subsequence of $x = \{x_n\}$ is the sequence $V(x)$ obtained by subtracting 1 from each x_n and then removing all 0s. If x is a FRACTAL SEQUENCE, then $V(x)$ is a FRACTAL SEQUENCE. If x is a SIGNATURE SEQUENCE, then $V(x) = x$.

see also SIGNATURE SEQUENCE, UPPER-TRIMMED SUBSEQUENCE

References

Kimberling, C. "Fractal Sequences and Interspersions." *Ars Combin.* **45**, 157–168, 1997.

Lowest Terms Fraction

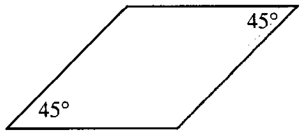
A FRACTION p/q for which $(p, q) = 1$, where (p, q) denotes the GREATEST COMMON DIVISOR.

Loxodrome

A path, also known as a RHUMB LINE, which cuts a MERIDIAN on a given surface (usually a SPHERE, in which case the loxodrome is also called a SPHERICAL HELIX) at any constant ANGLE but a RIGHT ANGLE. The loxodrome is the path taken when a compass is kept pointing in a constant direction. It is *not* the shortest distance between two points.

see also GREAT CIRCLE

Lozenge



A PARALLELOGRAM whose ACUTE ANGLES are 45°.

see also DIAMOND, PARALLELOGRAM, QUADRILATERAL, RHOMBUS

Lozenge Method

A method for constructing MAGIC SQUARES of ODD order.

see also MAGIC SQUARE

Lozi Map

A 2-D map similar to the HÉNON MAP which is given by the equations

$$x_{n+1} = 1 - \alpha|x_n| + y_n$$
$$y_{n+1} = \beta x_n.$$

see also HÉNON MAP

References

Dickau, R. M. "Lozi Attractor." `http://www.prairienet.org/~pops/lozi.html`.

Peitgen, H.-O.; Jürgens, H.; and Saupe, D. §12.1 in *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag, p. 672, 1992.

LU Decomposition

A procedure for decomposing an $N \times N$ matrix A into a product of a lower TRIANGULAR MATRIX L and an upper TRIANGULAR MATRIX U,

$$\mathsf{LU} = \mathsf{A}. \tag{1}$$

Written explicitly for a 3×3 MATRIX, the decomposition is

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{22} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{3}$$

This gives three types of equations

$$\begin{aligned} i < j \qquad & l_{i1}u_{ij} + l_{i2}u_{2j} + \ldots + l_{ii}u_{ij} = a_{ij} && (4) \\ i = j \qquad & l_{i1}u_{ij} + l_{i2}u_{2j} + \ldots + l_{ii}u_{jj} = a_{ij} && (5) \\ i > j \qquad & l_{i1}u_{1j} + l_{i2}u_{2j} + \ldots + l_{ij}u_{jj} = a_{ij}. && (6) \end{aligned}$$

This gives N^2 equations for $N^2 + N$ unknowns (the decomposition is not unique), and can be solved using CROUT'S METHOD. To solve the MATRIX equation

$$\mathsf{A}\mathbf{x} = (\mathsf{LU})\mathbf{x} = \mathsf{L}(\mathsf{U}\mathbf{x}) = \mathbf{b}, \tag{7}$$

first solve $\mathsf{L}\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. This can be done by forward substitution

$$y_1 = \frac{b_1}{l_{11}} \tag{8}$$

$$y_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}y_j\right) \tag{9}$$

for $i = 2, \ldots, N$. Then solve $\mathsf{U}\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. This can be done by back substitution

$$x_N = \frac{y_N}{u_{NN}} \tag{10}$$

$$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{N} u_{ij}x_j\right) \tag{11}$$

for $i = N-1, \ldots, 1$.

see also CHOLESKY DECOMPOSITION, QR DECOMPOSITION, TRIANGULAR MATRIX

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "LU Decomposition and Its Applications." §2.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 34–42, 1992.

Lucas Correspondence

The correspondence which relates the HANOI GRAPH to the ISOMORPHIC GRAPH of the ODD BINOMIAL COEFFICIENTS in PASCAL'S TRIANGLE, where the adjacencies are determined by adjacency (either horizontal or diagonal) in PASCAL'S TRIANGLE. The proof that the correspondence is given by the LUCAS CORRESPONDENCE THEOREM.

see also BINOMIAL COEFFICIENT, HANOI GRAPH, PASCAL'S TRIANGLE

References

Poole, David G. "The Towers and Triangles of Professor Claus (or, Pascal Knows Hanoi)." *Math. Mag.* **67**, 323–344, 1994.

Lucas Correspondence Theorem

Let p be PRIME and

$$r = r_m p^m + \ldots + r_1 p + r_0 \qquad (0 \le r_i < p) \tag{1}$$

$$k = k_m p^m + \ldots + k_1 p + k_0 \qquad (0 \le k_i < p), \tag{2}$$

then

$$\binom{r}{k} = \prod_{i=0}^{m} \binom{r_i}{k_i} \pmod{p}. \tag{3}$$

This is proved in Fine (1947).

References

Fine, N. J. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* **54**, 589–592, 1947.

Lucas-Lehmer Residue

see LUCAS-LEHMER TEST

Lucas-Lehmer Test

A MERSENNE NUMBER M_p is prime IFF M_p divides s_{p-2}, where $s_0 \equiv 4$ and

$$s_i \equiv {s_{i-1}}^2 - 2 \pmod{2^p - 1} \tag{1}$$

for $i \geq 1$. The first few terms of this series are 4, 14, 194, 37634, 1416317954, ... (Sloane's A003010). The remainder when s_{p-2} is divided by M_p is called the LUCAS-LEHMER RESIDUE for p. The LUCAS-LEHMER RESIDUE is 0 IFF M_p is PRIME. This test can also be extended to arbitrary INTEGERS.

A generalized version of the Lucas-Lehmer test lets

$$N + 1 = \prod_{j=1}^{n} {q_j}^{\beta_j}, \tag{2}$$

with q_j the distinct PRIME factors, and β_j their respective POWERS. If there exists a LUCAS SEQUENCE U_ν such that

$$\mathrm{GCD}(U_{(N+1)/q_j}, N) = 1 \tag{3}$$

for $j = 1, \ldots, n$ and

$$U_{N+1} \equiv 0 \pmod{N}, \tag{4}$$

then N is a PRIME. The test is particularly simple for MERSENNE NUMBERS, yielding the conventional Lucas-Lehmer test.

see also LUCAS SEQUENCE, MERSENNE NUMBER, RABIN-MILLER STRONG PSEUDOPRIME TEST

References

Sloane, N. J. A. Sequence A003010/M3494 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Lucas' Married Couples Problem

see MARRIED COUPLES PROBLEM

Lucas Number

The numbers produced by the V recurrence in the LUCAS SEQUENCE with $(P, Q) = (1, -1)$ are called Lucas numbers. They are the companions to the FIBONACCI NUMBERS F_n and satisfy the same recurrence

$$L_n = L_{n-1} + L_{n-2}, \tag{1}$$

where $L_1 = 1$, $L_2 = 3$. The first few are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ... (Sloane's A000204).

In terms of the FIBONACCI NUMBERS,

$$L_n = F_{n-1} + F_{n+1}. \tag{2}$$

The analog of BINET'S FORMULA for Lucas numbers is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{3}$$

Another formula is

$$L_n = [\phi^n], \tag{4}$$

where ϕ is the GOLDEN RATIO and $[x]$ denotes the NINT function. Given L_n,

$$L_{n+1} = \left\lfloor \frac{L_n(1+\sqrt{5})+1}{2} \right\rfloor, \tag{5}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION,

$${L_n}^2 - L_{n-1}L_{n+1} = 5(-1)^n, \tag{6}$$

and

$$\sum_{k=0}^{n} {L_k}^2 = L_n L_{n+1} - 2. \tag{7}$$

Let p be a PRIME > 3 and k be a POSITIVE INTEGER. Then L_{2p^k} ends in a 3 (Honsberger 1985, p. 113). Analogs of the Cesàro identities for FIBONACCI NUMBERS are

$$\sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n} \tag{8}$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n}, \tag{9}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

$L_n | F_m$ (L_n DIVIDES F_m) IFF n DIVIDES into m an EVEN number of times. $L_n | L_m$ IFF n divides into m an ODD number of times. $2^n L_n$ always ends in 2 (Honsberger 1985, p. 137).

Defining

$$D_n \equiv \begin{vmatrix} 3 & i & 0 & 0 & \cdots & 0 & 0 \\ i & 1 & i & 0 & \cdots & 0 & 0 \\ 0 & i & 1 & i & \cdots & 0 & 0 \\ 0 & 0 & i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & i \\ 0 & 0 & 0 & 0 & \cdots & i & 1 \end{vmatrix} = L_{n+1} \tag{10}$$

gives

$$D_n = D_{n-1} + D_{n-2} \tag{11}$$

(Honsberger 1985, pp. 113–114).

The number of ways of picking a set (including the EMPTY SET) from the numbers 1, 2, ..., n without picking two consecutive numbers (where 1 and n are now consecutive) is L_n (Honsberger 1985, p. 122).

The only SQUARE NUMBERS in the Lucas sequence are 1 and 4, as proved by John H. E. Cohn (Alfred 1964). The only TRIANGULAR Lucas numbers are 1, 3, and 5778 (Ming 1991). The only Lucas CUBIC NUMBER is 1. The first few Lucas PRIMES L_n occur for $n = 2$, 4, 5, 7, 8, 11, 13, 16, 17, 19, 31, 37, 41, 47, 53, 61, 71, 79, 113, 313, 353, ... (Dubner and Keller 1998, Sloane's A001606).

see also FIBONACCI NUMBER

References

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Lucas Polynomial

The w POLYNOMIALS obtained by setting $p(x) = x$ and $q(x) = 1$ in the LUCAS POLYNOMIAL SEQUENCE. The first few are

$$\begin{aligned}
F_1(x) &= x \\
F_2(x) &= x^2 + 2 \\
F_3(x) &= 3x^3 + 3x \\
F_4(x) &= x^4 + 4x^2 + 2 \\
F_5(x) &= x^5 + 5x^3 + 5x.
\end{aligned}$$

The corresponding W POLYNOMIALS are called FIBONACCI POLYNOMIALS. The Lucas polynomials satisfy

$$L_n(1) = L_n,$$

where the L_ns are LUCAS NUMBERS.

see also FIBONACCI POLYNOMIAL, LUCAS NUMBER, LUCAS POLYNOMIAL SEQUENCE

Lucas Polynomial Sequence

A pair of generalized POLYNOMIALS which generalize the LUCAS SEQUENCE to POLYNOMIALS is given by

$$W_n^k(x) = \frac{\Delta^k(x)[a^n(x) - (-1)^k b^n(x)]}{\Delta(x)} \qquad (1)$$

$$w_n^k(x) = \Delta^k(x)[a^n(x) + (-1)^k b^n(x)], \qquad (2)$$

where

$$a(x) + b(x) = p(x) \qquad (3)$$

$$a(x)b(x) = -q(x) \qquad (4)$$

$$a(x) - b(x) = \sqrt{p^2(x) + 4q(x)} \equiv \Delta(x) \qquad (5)$$

(Horadam 1996). Setting $n = 0$ gives

$$W_0^k(x) = \Delta^k(x)\frac{1 - (-1)^k}{\Delta(x)} \qquad (6)$$

$$w_0^k(x) = \Delta^k(x)[1 + (-1)^k], \qquad (7)$$

giving

$$W_0^0(x) = 0 \qquad (8)$$

$$w_0^0(x) = 2. \qquad (9)$$

When $k = 1$,

$$W_n^1(x) = w_n^0(x) = w_n(x) \qquad (10)$$

$$W_n^1(x) = \Delta^2(x)W_n^0(x) = \Delta^2(x)W_n(x). \qquad (11)$$

Special cases are given in the following table.

$p(x)$	$q(x)$	Polynomial 1	Polynomial 2
x	1	Fibonacci $F_n(x)$	Lucas $L_n(x)$
$2x$	1	Pell $P_n(x)$	Pell-Lucas $Q_n(x)$
1	$2x$	Jacobsthal $J_n(x)$	Jacobsthal-Lucas $j_n(x)$
$3x$	-2	Fermat $\mathcal{F}_n(x)$	Fermat-Lucas $f_n(x)$
$2x$	-1	Chebyshev $U_{n-1}(x)$	Chebyshev $2T_n(x)$

see also LUCAS SEQUENCE

References

Horadam, A. F. "Extension of a Synthesis for a Class of Polynomial Sequences." *Fib. Quart.* **34**, 68–74, 1996.

Lucas Pseudoprime

When P and Q are INTEGERS such that $D = P^2 - 4Q \neq 0$, define the LUCAS SEQUENCE $\{U_k\}$ by

$$U_k = \frac{a^k - b^k}{a - b}$$

for $k \geq 0$, with a and b the two ROOTS of $x^2 - Px + Q = 0$. Then define a Lucas pseudoprime as an ODD COMPOSITE number n such that $n \nmid Q$, the JACOBI SYMBOL $(D/n) = -1$, and $n | U_{n+1}$.

There are no EVEN Lucas pseudoprimes (Bruckman 1994). The first few Lucas pseudoprimes are 705, 2465, 2737, 3745, ... (Sloane's A005845).

see also EXTRA STRONG LUCAS PSEUDOPRIME, LUCAS SEQUENCE, PSEUDOPRIME, STRONG LUCAS PSEUDOPRIME

References
Bruckman, P. S. "Lucas Pseudoprimes are Odd." *Fib. Quart.* **32**, 155–157, 1994.
Ribenboim, P. "Lucas Pseudoprimes (lpsp(P, Q))." §2.X.B in *The New Book of Prime Number Records, 3rd ed.* New York: Springer-Verlag, p. 129, 1996.
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Lucas Sequence

Let P, Q be POSITIVE INTEGERS. The ROOTS of

$$x^2 - Px + Q = 0 \tag{1}$$

are

$$a \equiv \tfrac{1}{2}(P + \sqrt{D}) \tag{2}$$
$$b \equiv \tfrac{1}{2}(P - \sqrt{D}), \tag{3}$$

where

$$D \equiv P^2 - 4Q, \tag{4}$$

so

$$a + b = P \tag{5}$$
$$ab = \tfrac{1}{4}(P^2 - D) = Q \tag{6}$$
$$a - b = \sqrt{D}. \tag{7}$$

Then define

$$U_n(P, Q) \equiv \frac{a^n - b^n}{a - b} \tag{8}$$
$$V_n(P, Q) \equiv a^n + b^n. \tag{9}$$

The first few values are therefore

$$U_0(P, Q) = 0 \tag{10}$$
$$U_1(P, Q) = 1 \tag{11}$$
$$V_0(P, Q) = 2 \tag{12}$$
$$V_1(P, Q) = P. \tag{13}$$

The sequences

$$U(P, Q) = \{U_n(P, Q) : n \geq 1\} \tag{14}$$
$$V(P, Q) = \{V_n(P, Q) : n \geq 1\} \tag{15}$$

are called Lucas sequences, where the definition is usually extended to include

$$U_{-1} = \frac{a^{-1} - b^{-1}}{a - b} = \frac{-1}{ab} = -\frac{1}{Q}. \tag{16}$$

For $(P, Q) = (1, -1)$, the U_n are the FIBONACCI NUMBERS and V_n are the LUCAS NUMBERS. For $(P, Q) = (2, -1)$, the PELL NUMBERS and Pell-Lucas numbers are obtained. $(P, Q) = (1, -2)$ produces the JACOBSTHAL NUMBERS and Pell-Jacobsthal Numbers.

The Lucas sequences satisfy the general RECURRENCE RELATIONS

$$\begin{aligned} U_{m+n} &= \frac{a^{m+n} - b^{m+n}}{a - b} \\ &= \frac{(a^m - b^m)(a^n + b^n)}{a - b} - \frac{a^n b^n (a^{m-n} - b^{m-n})}{a - b} \\ &= U_m V_n - a^n b^n U_{m-n} \end{aligned} \tag{17}$$

$$\begin{aligned} V_{m+n} &= a^{m+n} + b^{m+n} \\ &= (a^m + b^m)(a^n + b^n) - a^n b^n (a^{m-n} + b^{m-n}) \\ &= V_m V_n - a^n b^n V_{m-n}. \end{aligned} \tag{18}$$

Taking $n = 1$ then gives

$$U_m(P, Q) = PU_{m-1}(P, Q) - QU_{m-2}(P, Q) \tag{19}$$
$$V_m(P, Q) = PV_{m-1}(P, Q) - QV_{m-2}(P, Q). \tag{20}$$

Other identities include

$$U_{2n} = U_n V_n \tag{21}$$
$$U_{2n+1} = U_{n+1} V_n - Q^n \tag{22}$$
$$V_{2n} = {V_n}^2 - 2(ab)^n = {V_n}^2 - 2Q^n \tag{23}$$
$$V_{2n+1} = V_{n+1} V_n - PQ^n. \tag{24}$$

These formulas allow calculations for large n to be decomposed into a chain in which only four quantities must be kept track of at a time, and the number of steps needed is $\sim \lg n$. The chain is particularly simple if n has many 2s in its factorization.

The Us in a Lucas sequence satisfy the CONGRUENCE

$$U_{p^{n-1}[p-(D/p)]} \equiv 0 \pmod{p^n} \tag{25}$$

if

$$\mathrm{GCD}(2QcD, p) = 1, \tag{26}$$

where

$$P^2 - 4Q^2 = c^2 D. \tag{27}$$

This fact is used in the proof of the general LUCAS-LEHMER TEST.

see also FIBONACCI NUMBER, JACOBSTHAL NUMBER, LUCAS-LEHMER TEST, LUCAS NUMBER, LUCAS POLYNOMIAL SEQUENCE, PELL NUMBER, RECURRENCE SEQUENCE, SYLVESTER CYCLOTOMIC NUMBER

References
Dickson, L. E. "Recurring Series; Lucas' u_n, v_n." Ch. 17 in *History of the Theory of Numbers, Vol. 1: Divisibility and Primality.* New York: Chelsea, pp. 393–411, 1952.
Ribenboim, P. *The Little Book of Big Primes.* New York: Springer-Verlag, pp. 35–53, 1991.

Lucas's Theorem

The primitive factors $Q_n(x,y)$ of x^n+y^n can be written in the form

$$Q_n(x,y) = U^2(x,y) \pm nxyV^2(x,y)$$

for SQUAREFREE n where U and V are HOMOGENEOUS POLYNOMIALS with the sign chosen according to

$$\begin{cases} + & \text{for } n = 4l+1 \\ - & \text{for } n = 4l+3 \\ \text{either} & \text{for } n = 4l+2. \end{cases}$$

Lucky Number

Write out all the ODD numbers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, ... The first ODD number > 1 is 3, so strike out every third number from the list: 1, 3, 7, 9, 13, 15, 19, ... The first ODD number greater than 3 in the list is 7, so strike out every seventh number: 1, 3, 7, 9, 13, 15, 21, 25, 31, ...

Numbers remaining after this procedure has been carried out completely are called lucky numbers. The first few are 1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, ... (Sloane's A000959). Many asymptotic properties of the PRIME NUMBERS are shared by the lucky numbers. The asymptotic density is $1/\ln N$, just as the PRIME NUMBER THEOREM, and the frequency of TWIN PRIMES and twin lucky numbers are similar. A version of the GOLDBACH CONJECTURE also seems to hold.

It therefore appears that the SIEVING process accounts for many properties of the PRIMES.

see also GOLDBACH CONJECTURE, LUCKY NUMBER OF EULER, PRIME NUMBER, PRIME NUMBER THEOREM, SIEVE

References
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Lucky Number of Euler

A number p such that the PRIME-GENERATING POLYNOMIAL

$$n^2 - n + p$$

is PRIME for $n = 0, 1, \ldots, p-2$. Such numbers are related to the COMPLEX QUADRATIC FIELD in which the RING of INTEGERS is factorable. Specifically, the Lucky numbers of Euler (excluding the trivial case $p = 3$) are those numbers p such that the QUADRATIC FIELD $\mathbb{Q}(\sqrt{1-4p})$ has CLASS NUMBER 1 (Rabinowitz 1913, Le Lionnais 1983, Conway and Guy 1996).

As established by Stark (1967), there are only nine numbers $-d$ such that $h(-d) = 1$ (the HEEGNER NUMBERS -2, -3, -7, -11, -19, -43, -67, and -163), and of these, only 7, 11, 19, 43, 67, and 163 are of the required form. Therefore, the only Lucky numbers of Euler are 2, 3, 5, 11, 17, and 41 (Le Lionnais 1983, Sloane's A014556), and there does not exist a better PRIME-GENERATING POLYNOMIAL of Euler's form.

see also CLASS NUMBER, HEEGNER NUMBER, PRIME-GENERATING POLYNOMIAL

References
Conway, J. H. and Guy, R. K. "The Nine Magic Discriminants." In *The Book of Numbers.* New York: Springer-Verlag, pp. 224–226, 1996.
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Stark, H. M. "A Complete Determination of the Complex Quadratic Fields of Class Number One." *Michigan Math. J.* **14**, 1–27, 1967.

LUCY

A nonlinear DECONVOLUTION technique used in deconvolving images from the Hubble Space Telescope before corrective optics were installed.

see also CLEAN ALGORITHM, DECONVOLUTION, MAXIMUM ENTROPY METHOD

Ludolph's Constant

see PI

Ludwig's Inversion Formula

Expresses a function in terms of its RADON TRANSFORM,

$$\begin{aligned} f(x,y) &= \mathcal{R}^{-1}(\mathcal{R}f)(x,y) \\ &= \frac{1}{\pi}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial p}(\mathcal{R}f)(p,\alpha)}{x\cos\alpha + y\sin\alpha - p}\,dp\,d\alpha. \end{aligned}$$

see also RADON TRANSFORM

Lukács Theorem

Let $\rho(x)$ be an mth degree POLYNOMIAL which is NONNEGATIVE in $[-1,1]$. Then $\rho(x)$ can be represented in the form

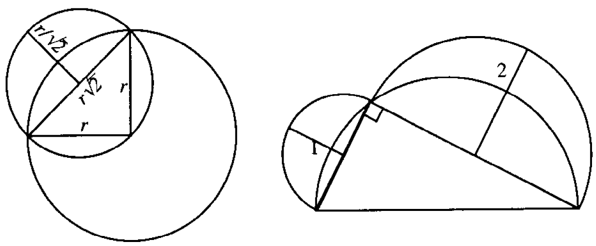
$$\begin{cases} [A(x)]^2 + (1-x^2)[B(x)]^2 & \text{for } m \text{ even} \\ (1+x)[C(x)]^2 + (1-x)[D(x)]^2 & \text{for } m \text{ odd,} \end{cases}$$

where $A(x)$, $B(x)$, $C(x)$, and $D(x)$ are REAL POLYNOMIALS whose degrees do not exceed m.

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 4, 1975.

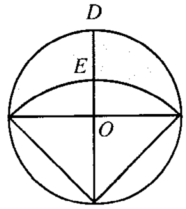
Lune (Plane)



A figure bounded by two circular ARCS of unequal RADII. Hippocrates of Chios SQUARED the above left lune, as well as two others, in the fifth century BC. Two more SQUARABLE lunes were found by T. Clausen in the 19th century (Dunham 1990 attributes these discoveries to Euler in 1771). In the 20th century, N. G. Tschebatorew and A. W. Dorodnow proved that these are the only five squarable lunes (Shenitzer and Steprans 1994). The left lune above is squared as follows,

$$\begin{aligned} A_{\text{half small circle}} &= \tfrac{1}{2}\pi\left(\frac{r}{\sqrt{2}}\right)^2 = \tfrac{1}{4}\pi r^2 \\ A_{\text{lens}} &= A_{\text{quarter big circle}} - A_{\text{triangle}} \\ &= \tfrac{1}{4}\pi r^2 - \tfrac{1}{2}r^2 \\ A_{\text{lune}} &= A_{\text{half small circle}} - A_{\text{lens}} = \tfrac{1}{2}r^2 \\ &= A_{\text{triangle}}, \end{aligned}$$

so the lune and TRIANGLE have the same AREA. In the right figure, $A_1 + A_2 = A_\Delta$.



For the above lune,

$$A_{\text{lune}} = 2A_{\Delta OBC}.$$

see also ANNULUS, ARC, CIRCLE, LUNE (SURFACE)

References

Dunham, W. "Hippocrates' Quadrature of the Lune." Ch. 1 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 1–20, 1990.

Heath, T. L. *A History of Greek Mathematics.* New York: Dover, p. 185, 1981.

Pappas, T. "Lunes." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 72–73, 1989.

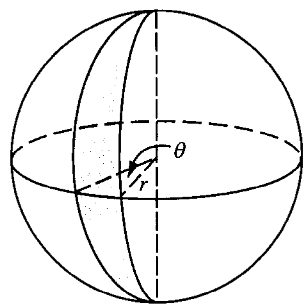
Shenitzer, A. and Steprans, J. "The Evolution of Integration." *Amer. Math. Monthly* **101**, 66–72, 1994.

Lune (Solid)

A geometric figure consisting of two TRIANGLES attached to opposite sides of a SQUARE.

see also SQUARE, TRIANGLE

Lune (Surface)



A sliver of the surface of a SPHERE of RADIUS r cut out by two planes through the azimuthal axis with DIHEDRAL ANGLE θ. The SURFACE AREA of the lune is

$$S = 2r^2\theta,$$

which is just the area of the SPHERE times $\theta/(2\pi)$.

see also LUNE (PLANE), SPHERE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 130, 1987.

Lunule

see LUNE (PLANE)

Lüroth's Theorem

If x and y are nonconstant rational functions of a parameter, the curve so defined has GENUS 0. Furthermore, x and y may be expressed rationally in terms of a parameter which is rational in them.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 246, 1959.

Lusin's Theorem

Let $f(x)$ be a finite and MEASURABLE FUNCTION in $(-\infty, \infty)$, and let ϵ be freely chosen. Then there is a function $g(x)$ such that

1. $g(x)$ is continuous in $(-\infty, \infty)$,
2. The MEASURE of $\{x : f(x) \neq g(x)\}$ is $< \epsilon$,
3. $M(|g|; R_1) \leq M(|f|; R_1)$,

where $M(f; S)$ denotes the upper bound of the aggregate of the values of $f(P)$ as P runs through all values of S.

References

Kestelman, H. §4.4 in *Modern Theories of Integration, 2nd rev. ed.* New York: Dover, pp. 30 and 109–112, 1960.

LUX Method

A method for constructing MAGIC SQUARES of SINGLY EVEN order $n \geq 6$.

see also MAGIC SQUARE

Lyapunov Characteristic Exponent

The Lyapunov characteristic exponent [LCE] gives the rate of exponential divergence from perturbed initial conditions. To examine the behavior of an orbit around a point $\mathbf{X}^*(t)$, perturb the system and write

$$\mathbf{X}(t) = \mathbf{X}^*(t) + U(t), \tag{1}$$

where $U(t)$ is the average deviation from the unperturbed trajectory at time t. In a CHAOTIC region, the LCE σ is independent of $\mathbf{X}^*(0)$. It is given by the OSEDELEC THEOREM, which states that

$$\sigma_i = \lim_{t\to\infty} \ln |\mathbf{U}(t)|. \tag{2}$$

For an n-dimensional mapping, the Lyapunov characteristic exponents are given by

$$\sigma_i = \lim_{N\to\infty} \ln |\lambda_i(N)| \tag{3}$$

for $i = 1, \ldots, n$, where λ_i is the LYAPUNOV CHARACTERISTIC NUMBER.

One Lyapunov characteristic exponent is always 0, since there is never any divergence for a perturbed trajectory in the direction of the unperturbed trajectory. The larger the LCE, the greater the rate of exponential divergence and the wider the corresponding SEPARATRIX of the CHAOTIC region. For the STANDARD MAP, an analytic estimate of the width of the CHAOTIC zone by Chirikov (1979) finds

$$\delta I = Be^{-AK^{-1/2}}. \tag{4}$$

Since the Lyapunov characteristic exponent increases with increasing K, some relationship likely exists connecting the two. Let a trajectory (expressed as a MAP) have initial conditions (x_0, y_0) and a nearby trajectory have initial conditions $(x', y') = (x_0 + dx, y_0 + dy)$. The distance between trajectories at iteration k is then

$$dk = \|(x' - x_0, y' - y_0)\|, \tag{5}$$

and the mean exponential rate of divergence of the trajectories is defined by

$$\sigma_1 = \lim_{k\to\infty} \frac{1}{k} \ln\left(\frac{d_k}{d_0}\right). \tag{6}$$

For an n-dimensional phase space (MAP), there are n Lyapunov characteristic exponents $\sigma_1 \geq \sigma_2 \geq \ldots > \sigma_n$. However, because the largest exponent σ_1 will dominate, this limit is practically useful only for finding the largest exponent. Numerically, since d_k increases exponentially with k, after a few steps the perturbed trajectory is no longer nearby. It is therefore necessary to renormalize frequently every t steps. Defining

$$r_{k\tau} \equiv \frac{d_{k\tau}}{d_0}, \tag{7}$$

one can then compute

$$\sigma_1 = \lim_{n\to\infty} \frac{1}{n\tau} \sum_{k=1}^{n} \ln r_{k\tau}. \tag{8}$$

Numerical computation of the second (smaller) Lyapunov exponent may be carried by considering the evolution of a 2-D surface. It will behave as

$$e^{(\sigma_1+\sigma_2)t}, \tag{9}$$

so σ_2 can be extracted if σ_1 is known. The process may be repeated to find smaller exponents.

For HAMILTONIAN SYSTEMS, the LCEs exist in additive inverse pairs, so if σ is an LCE, then so is $-\sigma$. One LCE is always 0. For a 1-D oscillator (with a 2-D phase space), the two LCEs therefore must be $\sigma_1 = \sigma_2 = 0$, so the motion is QUASIPERIODIC and cannot be CHAOTIC. For higher order HAMILTONIAN SYSTEMS, there are always at least two 0 LCEs, but other LCEs may enter in plus-and-minus pairs l and $-l$. If they, too, are both zero, the motion is integrable and not CHAOTIC. If they are NONZERO, the POSITIVE LCE l results in an exponential separation of trajectories, which corresponds to a CHAOTIC region. Notice that it is not possible to have all LCEs NEGATIVE, which explains why convergence of orbits is never observed in HAMILTONIAN SYSTEMS.

Now consider a dissipative system. For an arbitrary n-D phase space, there must always be one LCE equal to 0, since a perturbation along the path results in no divergence. The LCEs satisfy $\sum_i \sigma_i < 0$. Therefore, for a 2-D phase space of a dissipative system, $\sigma_1 = 0, \sigma_2 < 0$. For a 3-D phase space, there are three possibilities:

1. (Integrable): $\sigma_1 = 0, \sigma_2 = 0, \sigma_3 < 0$,

2. (Integrable): $\sigma_1 = 0, \sigma_2, \sigma_3 < 0$,
3. (CHAOTIC): $\sigma_1 = 0, \sigma_2 > 0, \sigma_3 < -\sigma_2 < 0$.

see also CHAOS, HAMILTONIAN SYSTEM, LYAPUNOV CHARACTERISTIC NUMBER, OSEDELEC THEOREM

References
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Lyapunov Characteristic Number

Given a LYAPUNOV CHARACTERISTIC EXPONENT σ_i, the corresponding Lyapunov characteristic number λ_i is defined as

$$\lambda_i \equiv e^{\sigma_i}. \qquad (1)$$

For an n-dimensional linear MAP,

$$\mathbf{X}_{n+1} = \mathsf{M}\mathbf{X}_n. \qquad (2)$$

The Lyapunov characteristic numbers $\lambda_1, \ldots, \lambda_n$ are the EIGENVALUES of the MAP MATRIX. For an arbitrary MAP

$$x_{n+1} = f_1(x_n, y_n) \qquad (3)$$

$$y_{n+1} = f_2(x_n, y_n), \qquad (4)$$

the Lyapunov numbers are the EIGENVALUES of the limit

$$\lim_{n\to\infty} [J(x_n, y_n) J(x_{n-1}, y_{n-1}) \cdots J(x_1, y_1)]^{1/n}, \qquad (5)$$

where $J(x, y)$ is the JACOBIAN

$$J(x,y) \equiv \begin{vmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{vmatrix}. \qquad (6)$$

If $\lambda_i = 0$ for all i, the system is not CHAOTIC. If $\lambda \neq 0$ and the MAP is AREA-PRESERVING (HAMILTONIAN), the product of EIGENVALUES is 1.

see also ADIABATIC INVARIANT, CHAOS, LYAPUNOV CHARACTERISTIC EXPONENT

Lyapunov Condition

If the third MOMENT exists for a DISTRIBUTION of x_i and the LEBESGUE INTEGRAL is given by

$${r_n}^3 = \sum_{i=1}^{n} \int_{-\infty}^{\infty} |x|^3 \, dF_i(x),$$

then if

$$\lim_{n\to\infty} \frac{r_n}{s_n} = 0,$$

the CENTRAL LIMIT THEOREM holds.

see also CENTRAL LIMIT THEOREM

Lyapunov Dimension

For a 2-D MAP with $\sigma_2 > \sigma_1$,

$$d_{\mathrm{Lya}} = 1 - \frac{\sigma_1}{\sigma_2},$$

where σ_n are the LYAPUNOV CHARACTERISTIC EXPONENTS.

see also CAPACITY DIMENSION, KAPLAN-YORKE CONJECTURE

References
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Nayfeh, A. H. and Balachandran, B. *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods.* New York: Wiley, p. 549, 1995.

Lyapunov's First Theorem

A NECESSARY and SUFFICIENT condition for all the EIGENVALUES of a REAL $n \times n$ matrix A to have NEGATIVE REAL PARTS is that the equation

$$\mathsf{A}^{\mathrm{T}}\mathsf{V} + \mathsf{V}\mathsf{A} = -\mathsf{I}$$

has as a solution where V is an $n \times n$ matrix and $(\mathbf{x}, \mathsf{V}\mathbf{x})$ is a positive definite quadratic form.

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1122, 1979.

Lyapunov Function

A function which is continuous, nonnegative, and has continuous PARTIAL DERIVATIVES. The existence of a Lyapunov function guarantees the NONLINEAR STABILITY of a FIXED POINT.

References
Jordan, D. W. and Smith, P. *Nonlinear Ordinary Differential Equations.* Oxford, England: Clarendon Press, p. 283, 1977.

Lyapunov's Second Theorem

If all the EIGENVALUES of a REAL MATRIX A have REAL PARTS, then to an arbitrary negative definite quadratic form $(\mathbf{x}, \mathsf{W}\mathbf{x})$ with $\mathbf{x} = \mathbf{x}(t)$ there corresponds a positive definite quadratic form $(\mathbf{x}, \mathsf{V}\mathbf{x})$ such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x},$$

then $(\mathbf{x}, \mathsf{W}\mathbf{x})$ and $(\mathbf{x}, \mathsf{W}\mathbf{x})$ satisfy

$$\frac{d}{dt}(\mathbf{x}, \mathsf{V}\mathbf{x}) = (\mathbf{x}, \mathsf{W}\mathbf{x}).$$

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1122, 1979.

Lyndon Word

A Lyndon word is an aperiodic notation for representing a NECKLACE.

see also DE BRUIJN SEQUENCE, NECKLACE

References

Ruskey, F. "Information on Necklaces, Lyndon Words, de Bruijn Sequences." http://sue.csc.uvic.ca/~cos/inf/neck/NecklaceInfo.html.

Sloane, N. J. A. Sequence A001037/M0116 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Lyons Group

The SPORADIC GROUP Ly.

see also SPORADIC GROUP

References

Wilson, R. A. "ATLAS of Finite Group Representation." http://for.mat.bham.ac.uk/atlas/Ly.html.

M

M-Estimate

A ROBUST ESTIMATION based on maximum likelihood argument.

see also L-ESTIMATE, R-ESTIMATE

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Robust Estimation." §15.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 694–700, 1992.

Mac Lane's Theorem

A theorem which treats constructions of FIELDS of CHARACTERISTIC p.

see also CHARACTERISTIC (FIELD), FIELD

Machin's Formula

$$\tfrac{1}{4}\pi = 4\tan^{-1}(\tfrac{1}{5}) - \tan^{-1}(\tfrac{1}{239}).$$

There are a whole class of MACHIN-LIKE FORMULAS with various numbers of terms (although only four such formulas with only two terms). The properties of these formulas are intimately connected with COTANGENT identities.

see also 196-ALGORITHM, GREGORY NUMBER, MACHIN-LIKE FORMULAS, PI

Machin-Like Formulas

Machin-like formulas have the form

$$m\cot^{-1}u + n\cot^{-1}v = \tfrac{1}{4}k\pi, \tag{1}$$

where u, v, and k are POSITIVE INTEGERS and m and n are NONNEGATIVE INTEGERS. Some such FORMULAS can be found by converting the INVERSE TANGENT decompositions for which $c_n \neq 0$ in the table of Todd (1949) to INVERSE COTANGENTS. However, this gives only Machin-like formulas in which the smallest term is ± 1.

Maclaurin-like formulas can be derived by writing

$$\cot^{-1}z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right) \tag{2}$$

and looking for a_k and u_k such that

$$\sum_k a_k \cot^{-1}u_k = \tfrac{1}{4}\pi, \tag{3}$$

so

$$\prod_k \left(\frac{u_k+i}{u_k-i}\right)^{a_k} = e^{2\pi i/4} = i. \tag{4}$$

Machin-like formulas exist IFF (4) has a solution in INTEGERS. This is equivalent to finding INTEGER values such that

$$(1-i)^k(u+i)^m(v+i)^n \tag{5}$$

is REAL (Borwein and Borwein 1987, p. 345). An equivalent formulation is to find all integral solutions to one of

$$1+x^2 = 2y^n \tag{6}$$

$$1+x^2 = y^n \tag{7}$$

for $n = 3, 5, \ldots$.

There are only four such FORMULAS,

$$\tfrac{1}{4}\pi = 4\tan^{-1}(\tfrac{1}{5}) - \tan^{-1}(\tfrac{1}{239}) \tag{8}$$

$$\tfrac{1}{4}\pi = \tan^{-1}(\tfrac{1}{2}) + \tan^{-1}(\tfrac{1}{3}) \tag{9}$$

$$\tfrac{1}{4}\pi = 2\tan^{-1}(\tfrac{1}{2}) - \tan^{-1}(\tfrac{1}{7}) \tag{10}$$

$$\tfrac{1}{4}\pi = 2\tan^{-1}(\tfrac{1}{3}) + \tan^{-1}(\tfrac{1}{7}), \tag{11}$$

known as MACHIN'S FORMULA, EULER'S MACHIN-LIKE FORMULA, HERMANN'S FORMULA, and HUTTON'S FORMULA. These follow from the identities

$$\left(\frac{5+i}{5-i}\right)^4 \left(\frac{239+i}{239-i}\right)^{-1} = i \tag{12}$$

$$\left(\frac{2+i}{2-i}\right)\left(\frac{3+i}{3-i}\right) = i \tag{13}$$

$$\left(\frac{2+i}{2-i}\right)^2 \left(\frac{7+i}{7-i}\right)^{-1} = i \tag{14}$$

$$\left(\frac{3+i}{3-i}\right)^2 \left(\frac{7+i}{7-i}\right) = i. \tag{15}$$

Machin-like formulas with two terms can also be generated which do not have integral arc cotangent arguments such as Euler's

$$\tfrac{1}{4}\pi = 5\tan^{-1}(\tfrac{1}{7}) + 2\tan^{-1}(\tfrac{3}{79}) \tag{16}$$

(Wetherfield 1996), and which involve inverse SQUARE ROOTS, such as

$$\frac{\pi}{2} = 2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{8}}\right). \tag{17}$$

Three-term Machin-like formulas include GAUSS'S MACHIN-LIKE FORMULA

$$\tfrac{1}{4}\pi = 12\cot^{-1}18 + 8\cot^{-1}57 - 5\cot^{-1}239, \tag{18}$$

STRASSNITZKY'S FORMULA

$$\tfrac{1}{4}\pi = \cot^{-1}2 + \cot^{-1}5 + \cot^{-1}8, \tag{19}$$

and the following,

$$\tfrac{1}{4}\pi = 6\cot^{-1}8 + 2\cot^{-1}57 + \cot^{-1}239 \qquad (20)$$

$$\tfrac{1}{4}\pi = 4\cot^{-1}5 - 1\cot^{-1}70 + \cot^{-1}99 \qquad (21)$$

$$\tfrac{1}{4}\pi = 1\cot^{-1}2 + 1\cot^{-1}5 + \cot^{-1}8 \qquad (22)$$

$$\tfrac{1}{4}\pi = 8\cot^{-1}10 - 1\cot^{-1}239 - 4\cot^{-1}515 \qquad (23)$$

$$\tfrac{1}{4}\pi = 5\cot^{-1}7 + 4\cot^{-1}53 + 2\cot^{-1}4443. \qquad (24)$$

The first is due to Størmer, the second due to Rutherford, and the third due to Dase.

Using trigonometric identities such as

$$\cot^{-1}x = 2\cot^{-1}(2x) - \cot^{-1}(4x^3 + 3x), \qquad (25)$$

it is possible to generate an infinite sequence of Machin-like formulas. Systematic searches therefore most often concentrate on formulas with particularly "nice" properties (such as "efficiency").

The efficiency of a FORMULA is the time it takes to calculate π with the POWER series for arctangent

$$\pi = a_1\cot(b_1) + a_2\cot(b_2) + \ldots, \qquad (26)$$

and can be roughly characterized using Lehmer's "measure" formula

$$e \equiv \sum \frac{1}{\log_{10} b_i}. \qquad (27)$$

The number of terms required to achieve a given precision is roughly proportional to e, so lower e-values correspond to better sums. The best currently known efficiency is 1.51244, which is achieved by the 6-term series

$$\tfrac{1}{4}\pi = 183\cot^{-1}239 + 32\cot^{-1}1023 - 68\cot^{-1}5832$$
$$+12\cot^{-1}110443 - 12\cot^{-1}4841182$$
$$-100\cot^{-1}6826318 \qquad (28)$$

discovered by C.-L. Hwang (1997). Hwang (1997) also discovered the remarkable identities

$$\tfrac{1}{4}\pi = P\cot^{-1}2 - M\cot^{-1}3 + L\cot^{-1}5 + K\cot^{-1}7$$
$$+(N + K + L - 2M + 3P - 5)\cot^{-1}8$$
$$+(2N + M - P + 2 - L)\cot^{-1}18$$
$$-(2P - 3 - M + L + K - N)\cot^{-1}57 - N\cot^{-1}239, \qquad (29)$$

where K, L, M, N, and P are POSITIVE INTEGERS, and

$$\tfrac{1}{4}\pi = (N+2)\cot^{-1}2 - N\cot^{-1}3 - (N+1)\cot^{-1}N. \qquad (30)$$

The following table gives the number $N(n)$ of Machin-like formulas of n terms in the compilation by Wetherfield and Hwang. Except for previously known identities (which are included), the criteria for inclusion are the following:

1. first term $<$ 8 digits: measure $<$ 1.8.
2. first term $=$ 8 digits: measure $<$ 1.9.
3. first term $=$ 9 digits: measure $<$ 2.0.
4. first term $=$10 digits: measure $<$ 2.0.

n	$N(n)$	min e
1	1	0
2	4	1.85113
3	106	1.78661
4	39	1.58604
5	90	1.63485
6	120	1.51244
7	113	1.54408
8	18	1.65089
9	4	1.72801
10	78	1.63086
11	34	1.6305
12	188	1.67458
13	37	1.71934
14	5	1.75161
15	24	1.77957
16	51	1.81522
17	5	1.90938
18	570	1.87698
19	1	1.94899
20	11	1.95716
21	1	1.98938
Total	1500	1.51244

see also EULER'S MACHIN-LIKE FORMULA, GAUSS'S MACHIN-LIKE FORMULA, GREGORY NUMBER, HERMANN'S FORMULA, HUTTON'S FORMULA, INVERSE COTANGENT, MACHIN'S FORMULA, PI, STØRMER NUMBER, STRASSNITZKY'S FORMULA

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$x_2, \ldots, c_1, c_2, \ldots, k$ de l'Equation...." *Archiv for Mathematik og Naturvidenskab* **B 19**, 75–85, 1896.
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Wetherfield, M. "The Enhancement of Machin's Formula by Todd's Process." *Math. Gaz.* **80**, 333–344, 1996.
Wetherfield, M. "Machin Revisited." *Math. Gaz.*, 121–123, March 1997.
Williams, R. "Arctangent Formulas for Pi." http://www.cacr.caltech.edu/~roy/pi.formulas.html. [Contains errors].

Machine

A method for producing infinite LOOP SPACES and spectra.

see also GADGET, LOOP SPACE, MAY-THOMASON UNIQUENESS THEOREM, TURING MACHINE

Mackey's Theorem

Let E and F be paired spaces with S a family of absolutely convex bounded sets of F such that the sets of S generate F and, if $B_1, B_2 \in S$, there exists a $B_3 \in S$ such that $B_3 \supset B_1$ and $B_3 \supset B_2$. Then the dual space of E_S is equal to the union of the weak completions of λB, where $\lambda > 0$ and $B \in S$.

see also GROTHENDIECK'S THEOREM

References

Iyanaga, S. and Kawada, Y. (Eds.). "Mackey's Theorem." §407M in *Encyclopedic Dictionary of Mathematics*. Cambridge, MA: MIT Press, p. 1274, 1980.

Maclaurin-Bezout Theorem

The Maclaurin-Bèzout theorem says that two curves of degree n intersect in n^2 points, so two CUBICS intersect in nine points. This means that $n(n+3)/2$ points do not always uniquely determine a single curve of order n.

see also CRAMÉR-EULER PARADOX

Maclaurin-Cauchy Theorem

If $f(x)$ is POSITIVE and decreases to 0, then an EULER CONSTANT

$$\gamma_f \equiv \lim_{n\to\infty}\left[\sum_{k=1}^{n} f(k) - \int_a^n f(x)\,dx\right]$$

can be defined. If $f(x) = 1/x$, then

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{dx}{x}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right),$$

where γ is the EULER-MASCHERONI CONSTANT.

Maclaurin Integral Test

see INTEGRAL TEST

Maclaurin Polynomial

see MACLAURIN SERIES

Maclaurin Series

A series expansion of a function about 0,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots, \quad (1)$$

named after the Scottish mathematician Maclaurin. Maclaurin series for common functions include

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \qquad -1 < x < 1 \quad (2)$$

$$\operatorname{cn}(x,k^2) = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}(1+4k^2)x^4 + \ldots \quad (3)$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 - \ldots \qquad -\infty < x < \infty \quad (4)$$

$$\cos^{-1} x = \tfrac{1}{2}\pi - x - \tfrac{1}{6}x^3 - \tfrac{3}{40}x^5 - \tfrac{5}{112}x^7 - \ldots \qquad -1 < x < 1 \quad (5)$$

$$\cosh x = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \tfrac{1}{720}x^6 + \tfrac{1}{40{,}320}x^8 + \ldots \quad (6)$$

$$\cosh^{-1}(1+x) = \sqrt{2x}\,(1 - \tfrac{1}{2}x + \tfrac{3}{160}x^2 - \tfrac{5}{896}x^3 + \ldots) \quad (7)$$

$$\cot x = x^{-1} - \tfrac{1}{3}x - \tfrac{1}{45}x^3 - \tfrac{2}{945}x^5 - \tfrac{1}{4725}x^7 - \ldots \quad (8)$$

$$\cot^{-1} x = \tfrac{1}{2}\pi - x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{9}x^9 + \ldots \quad (9)$$

$$= x^{-1} - \tfrac{1}{3}x^{-3} + \tfrac{1}{5}x^{-5} - \tfrac{1}{7}x^{-7} + \tfrac{1}{9}x^{-9} + \ldots \quad (10)$$

$$\coth x = x^{-1} + \tfrac{1}{3}x - \tfrac{1}{45}x^4 + \tfrac{2}{945}x^5 - \tfrac{1}{4725}x^7 + \ldots \quad (11)$$

$$\coth^{-1}(1+x) = \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln x + \tfrac{1}{4}x - \tfrac{1}{16}x^2 + \ldots \quad (12)$$

$$\csc x = x^{-1} + \tfrac{1}{6}x + \tfrac{7}{360}x^3 + \tfrac{31}{15120}x^5 + \ldots \quad (13)$$

$$\operatorname{csch} x = x^{-1} - \tfrac{1}{6}x + \tfrac{7}{360}x^3 + \tfrac{31}{15120}x^5 + \ldots \quad (14)$$

$$\operatorname{csch}^{-1} x = \ln 2 - \ln x + \tfrac{1}{4}x^2 - \tfrac{3}{32}x^4 + \tfrac{5}{96}x^6 - \ldots \quad (15)$$

$$\operatorname{dn}(x,k^2)x = 1 - \tfrac{1}{2!}k^2x^2 + \tfrac{1}{4!}k^2(4+k^2)x^4 + \ldots \quad (16)$$

$$\operatorname{erf} x = \frac{1}{\sqrt{\pi}}\left(2x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{21}x^7 + \ldots\right) \quad (17)$$

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \ldots \qquad -\infty < x < \infty \quad (18)$$

$${}_2F_1(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha\beta}{1!\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \ldots \quad (19)$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \qquad -1 < x < 1 \quad (20)$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \tfrac{2}{7}x^7 + \ldots \qquad -1 < x < 1 \quad (21)$$

$$\sec x = 1 + \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 + \tfrac{61}{720}x^6 + \tfrac{277}{8064}x^8 + \ldots \quad (22)$$

$$\operatorname{sech} x = 1 - \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 - \tfrac{61}{720}x^6 + \tfrac{277}{8064}x^8 + \ldots \quad (23)$$

$$\operatorname{sech}^{-1} x = \ln 2 - \ln x - \tfrac{1}{4}x^2 - \tfrac{3}{32}x^4 - \ldots \quad (24)$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \ldots$$

$$-\infty < x < \infty \quad (25)$$

$$\sin^{-1} x = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \tfrac{5}{112}x^7 + \tfrac{35}{1152}x^9 + \ldots \quad (26)$$

$$\sinh x = x + \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + \tfrac{1}{5040}x^7 + \tfrac{1}{362,880}x^9 + \ldots \quad (27)$$

$$\sinh^{-1} x = x - \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 - \tfrac{5}{112}x^7 + \tfrac{35}{1152}x^9 - \ldots \quad (28)$$

$$\operatorname{sn}(x, k^2) = \tfrac{1}{3!}(1+k^2)x^3 + \tfrac{1}{5!}(1+14k^2+k^4)x^5 + \ldots \quad (29)$$

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \tfrac{62}{2835}x^9 + \ldots \quad (30)$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ldots$$

$$-1 < x < 1 \quad (31)$$

$$\tan^{-1}(1+x) = \tfrac{1}{4}\pi + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{12}x^3 + \tfrac{1}{40}x^5 + \ldots \quad (32)$$

$$\tanh x = x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - \tfrac{17}{315}x^7 + \tfrac{62}{2835}x^9 + \ldots \quad (33)$$

$$\tanh^{-1} x = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \ldots. \quad (34)$$

The explicit forms for some of these are

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (35)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (36)$$

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(2^{2n-1}-1)B_{2n}}{(2n)!} x^{2n-1} \quad (37)$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad (38)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \quad (39)$$

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{2}{(2n-1)} x^{2n-1} \quad (40)$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \quad (41)$$

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1} \quad (42)$$

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1} \quad (43)$$

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} x^{2n-1} \quad (44)$$

$$\tanh^{-1} x = \sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n-1}, \quad (45)$$

where B_n are BERNOULLI NUMBERS and E_n are EULER NUMBERS.

see also ALCUIN'S SEQUENCE, LAGRANGE EXPANSION, LEGENDRE SERIES, TAYLOR SERIES

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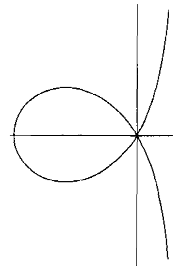
Maclaurin Trisectrix



A curve first studied by Colin Maclaurin in 1742. It was studied to provide a solution to one of the GEOMETRIC PROBLEMS OF ANTIQUITY, in particular TRISECTION of an ANGLE, whence the name trisectrix. The Maclaurin trisectrix is an ANALLAGMATIC CURVE, and the origin is a CRUNODE.

The Maclaurin trisectrix has CARTESIAN equation

$$y^2 = \frac{x^2(x+3a)}{a-x}, \quad (1)$$

or the parametric equations

$$x = a\frac{t^2-3}{t^2+1} \quad (2)$$

$$y = a\frac{t(t^2-3)}{t^2+1}. \quad (3)$$

The ASYMPTOTE has equation $x = a$, and the center of the loop is as $(2a, 0)$. Draw a line L with ANGLE 3α through the loop center. Then the angle made by the line through the center and point of intersection of L with the trisectrix is α. The trisectrix is sometimes defined instead as

$$x(x^2+y^2) = a(y^2-3x^2) \quad (4)$$

$$y^2 = \frac{x^2(3a+x)}{a-x} \quad (5)$$

$$r = \frac{2a\sin(3\theta)}{\sin(2\theta)}. \quad (6)$$

Another form of the equation is the POLAR EQUATION

$$r = a\sec(\tfrac{1}{3}\theta), \quad (7)$$

where the origin is inside the loop and the crossing point is on the NEGATIVE x-AXIS.

The tangents to the curve at the origin make angles of $\pm 60°$ with the x-AXIS. The AREA of the loop is

$$A_{\text{loop}} = 3\sqrt{3}\,a^2, \quad (8)$$

and the NEGATIVE x-intercept is $(-3a, 0)$ (MacTutor Archive).

The Maclaurin trisectrix is the PEDAL CURVE of the PARABOLA where the PEDAL POINT is taken as the reflection of the FOCUS in the DIRECTRIX.

see also CATALAN'S TRISECTRIX, STROPHOID

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Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 103–106, 1972.

Lee, X. "Trisectrix." http://www.best.com/~xah/SpecialPlaneCurves_dir/Trisectrix_dir/trisectrix.html.

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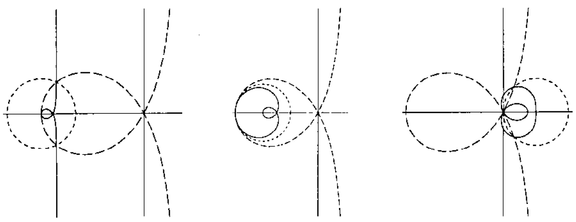
Maclaurin Trisectrix Inverse Curve



The INVERSE CURVE of the MACLAURIN TRISECTRIX with INVERSION CENTER at the NEGATIVE x-intercept is a TSCHIRNHAUSEN CUBIC.

MacMahon's Prime Number of Measurement

see PRIME NUMBER OF MEASUREMENT

MacRobert's E-Function

$$\begin{aligned}E(p;\alpha_r:\rho_s:x)&\\ \equiv{}&\frac{\Gamma(\alpha_{q+1})}{\Gamma(\rho_1-\alpha_1)\Gamma(\rho_2-\alpha_2)\cdots\Gamma(\rho_q-\alpha_q)}\\ &\times\prod_{\mu=1}^{q}\int_0^\infty {\lambda_\mu}^{\rho_\mu-\alpha_\mu-1}(1+\lambda_\mu)^{-\rho_\mu}\,d\lambda_\mu\\ &\times\prod_{\nu=2}^{p-q-1}\int_0^\infty e^{-\lambda_{q+\nu}}{\lambda_{q+\nu}}^{\alpha_{q+\nu}-1}\,d\lambda_{q+\nu}\times\\ \int_0^\infty e^{-\lambda_p}{\lambda_p}^{\alpha_p-1}&\left[1+\frac{\lambda_{q+2}\lambda_{q+3}\cdots\lambda_p}{(1+\lambda_1)\cdots(1+\lambda_q)x}\right]^{-\alpha_q+1}d\lambda_p,\end{aligned}$$

where $\Gamma(z)$ is the GAMMA FUNCTION and other details are discussed by Gradshteyn and Ryzhik (1980).

see also FOX'S H-FUNCTION, MEIJER'S G-FUNCTION

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 896–903 and 1071–1072, 1979.

Madelung Constants

The quantities obtained from cubic, hexagonal, etc., LATTICE SUMS, evaluated at $s=1$, are called Madelung constants. For cubic LATTICE SUMS, they are expressible in closed form for EVEN indices,

$$b_2(2)=-4\beta(1)\eta(1)=-4\tfrac{\pi}{4}\ln 2=-\pi\ln 2 \qquad (1)$$
$$b_4(2)=-8\eta(1)\eta(0)=-8\ln 2\cdot\tfrac{1}{2}=-4\ln 2. \qquad (2)$$

$b_3(1)$ is given by BENSON'S FORMULA,

$$\begin{aligned}-b_3(1)&=\sum_{i,\,j,\,k=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k+1}}{\sqrt{i^2+j^2+k^2}}\\ &=12\pi\sum_{m,\,n=1,3,\ldots}^{\infty}\operatorname{sech}^2\left(\tfrac{1}{2}\pi\sqrt{m^2+n^2}\right), \qquad (3)\end{aligned}$$

where the prime indicates that summation over (0, 0, 0) is excluded. $b_3(1)$ is sometimes called "the" Madelung constant, corresponds to the Madelung constant for a 3-D NaCl crystal, and is numerically equal to $-1.74756\ldots$.

For hexagonal LATTICE SUM, $h_2(2)$ is expressible in closed form as

$$h_2(2)=\pi\ln 3\sqrt{3}. \qquad (4)$$

see also BENSON'S FORMULA, LATTICE SUM

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Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, 1987.

Buhler, J. and Wagon, S. "Secrets of the Madelung Constant." *Mathematica in Education and Research* **5**, 49–55, Spring 1996.

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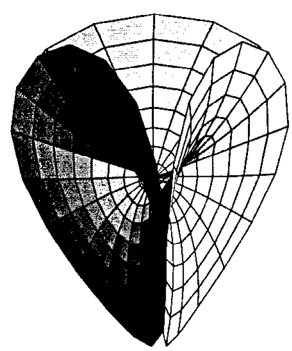
Maeder's Owl Minimal Surface



A MINIMAL SURFACE which resembles a CROSS-CAP. It is given by the polar equations

$$x=1 \qquad (1)$$
$$y=\sqrt{z} \qquad (2)$$
$$z=z, \qquad (3)$$

or the parametric equations

$$x = r\cos\theta - \tfrac{1}{2}r^2\cos(2\theta) \tag{4}$$
$$y = -r\sin\theta - \tfrac{1}{2}r^2\sin(2\theta), \tag{5}$$
$$z = \tfrac{4}{3}r^{3/2}\cos(\tfrac{3}{2}\theta). \tag{6}$$

see also CROSS-CAP, MINIMAL SURFACE

References
Maeder, R. *Programming in Mathematica, 3rd ed.* Reading, MA: Addison-Wesley, pp. 29–30, 1997.

Maehly's Procedure

A method for finding ROOTS which defines

$$P_j(x) = \frac{P(x)}{(x-x_1)\cdots(x-x_j)}, \tag{1}$$

so the derivative is

$$P_j'(x) = \frac{P'(x)}{(x-x_1)\cdots(x-x_j)} - \frac{P(x)}{(x-x_1)\cdots(x-x_j)}\sum_{i=1}^{j}(x-x_i)^{-1}. \tag{2}$$

One step of NEWTON'S METHOD can then be written as

$$x_{k+1} = x_k - \frac{P(x_k)}{P'(x_k) - P(x_k)\sum_{i=1}^{j}(x_k-x_i)^{-1}}. \tag{3}$$

Mainardi-Codazzi Equations

$$\frac{\partial e}{\partial v} - \frac{\partial f}{\partial u} = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11} \tag{1}$$
$$\frac{\partial f}{\partial v} - \frac{\partial g}{\partial u} = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}, \tag{2}$$

where e, f, and g are coefficients of the second FUNDAMENTAL FORM and Γ^k_{ij} are CHRISTOFFEL SYMBOLS OF THE SECOND KIND. Therefore,

$$\frac{\partial e}{\partial v} = \tfrac{1}{2}E_v\left(\frac{e}{E} + \frac{g}{G}\right) \tag{3}$$
$$\frac{\partial g}{\partial u} = \tfrac{1}{2}G_u\left(\frac{e}{E} + \frac{g}{G}\right) \tag{4}$$

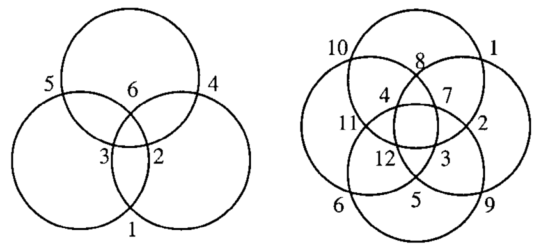
$$\frac{\partial(\ln f)}{\partial u} = \Gamma^1_{11} - \Gamma^2_{12} \tag{5}$$
$$\frac{\partial(\ln f)}{\partial v} = \Gamma^2_{22} - \Gamma^1_{12} \tag{6}$$

$$\frac{\partial}{\partial u}\left(\frac{\ln f}{\sqrt{EG-F^2}}\right) = -2\Gamma^2_{12} \tag{7}$$
$$\frac{\partial}{\partial v}\left(\frac{\ln f}{\sqrt{EG-F^2}}\right) = -2\Gamma^1_{12}, \tag{8}$$

where E, F, and G are coefficients of the first FUNDAMENTAL FORM.

References
Gray, A. "The Mainardi-Codazzi Equations." §20.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 401–402, 1993.
Green, A. E. and Zerna, W. *Theoretical Elasticity, 2nd ed.* New York: Dover, p. 37, 1992.

Magic Circles



A set of n magic circles is a numbering of the intersection of the n CIRCLES such that the sum over all intersections is the same constant for all circles. The above sets of three and four magic circles have magic constants 14 and 39 (Madachy 1979).

see also MAGIC GRAPH, MAGIC SQUARE

References
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, p. 86, 1979.

Magic Constant

The number

$$M_2(n) = \frac{1}{n}\sum_{k=1}^{n^2} k = \tfrac{1}{2}n(n^2+1)$$

to which the n numbers in any horizontal, vertical, or *main* diagonal line must sum in a MAGIC SQUARE. The first few values are 1, 5 (no such magic square), 15, 34, 65, 111, 175, 260, ... (Sloane's A006003). The magic constant for an nth order magic square starting with an INTEGER A and with entries in an increasing ARITHMETIC SERIES with difference D between terms is

$$M_2(n; A, D) = \tfrac{1}{2}n[2a + D(n^2-1)]$$

(Hunter and Madachy 1975, Madachy 1979). In a PANMAGIC SQUARE, in addition to the main diagonals, the broken diagonals also sum to $M_2(n)$.

For a MAGIC CUBE, the magic constant is

$$M_3(n) = \frac{1}{n^2}\sum_{k=1}^{n^3} k = \tfrac{1}{2}n(n^3+1) = \tfrac{1}{2}n(1+n)(n^2-n+1).$$

The first few values are 1, 9, 42, 130, 315, 651, 1204, ... (Sloane's A027441).

There is a corresponding multiplicative magic constant for MULTIPLICATION MAGIC SQUARES.

see also MAGIC CUBE, MAGIC GEOMETRIC CONSTANTS, MAGIC HEXAGON, MAGIC SQUARE, MULTIPLICATION MAGIC SQUARE, PANMAGIC SQUARE

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Hunter, J. A. H. and Madachy, J. S. "Mystic Arrays." Ch. 3 in *Mathematical Diversions.* New York: Dover, pp. 23–34, 1975.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, p. 86, 1979.

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Magic Cube

An $n \times n \times n$ 3-D version of the MAGIC SQUARE in which the n^2 rows, n^2 columns, n^2 pillars (or "files"), and four space diagonals each sum to a single number $M_3(n)$ known as the MAGIC CONSTANT. If the cross-section diagonals also sum to $M_3(n)$, the magic cube is called a PERFECT MAGIC CUBE; if they do not, the cube is called a SEMIPERFECT MAGIC CUBE, or sometimes an ANDREWS CUBE (Gardner 1988). A pandiagonal cube is a perfect or semiperfect magic cube which is magic not only along the main space diagonals, but also on the broken space diagonals.

A magic cube using the numbers 1, 2, ..., n^3, if it exists, has MAGIC CONSTANT

$$M_3(n) = \frac{1}{n^2}\sum_{k=1}^{n^3} k = \tfrac{1}{2}n(n^3+1) = \tfrac{1}{2}n(n+1)(n^2-n+1).$$

For $n = 1, 2, \ldots$, the magic constants are 1, 9, 42, 130, 315, 651, ... (Sloane's A027441).

4	12	26
11	25	6
27	5	10

20	7	15
9	14	19
13	21	8

18	23	1
22	3	17
2	16	24

60	37	12	21
13	20	61	36
56	41	8	25
1	32	49	48

7	26	55	42
50	47	2	31
11	22	59	38
62	35	14	19

57	40	9	24
16	17	64	33
53	44	5	28
4	29	52	45

6	27	54	43
51	46	3	30
10	23	58	39
63	34	15	18

The above semiperfect magic cubes of orders three (Hunter and Madachy 1975, p. 31; Ball and Coxeter 1987, p. 218) and four (Ball and Coxeter 1987, p. 220) have magic constants 42 and 130, respectively. There is a trivial semiperfect magic cube of order one, but no semiperfect cubes of orders two or three exist. Semiperfect cubes of ODD order with $n \geq 5$ and DOUBLY EVEN order can be constructed by extending the methods used for MAGIC SQUARES.

19	497	255	285	432	78	324	162
303	205	451	33	148	370	128	414
336	174	420	66	243	273	31	509
116	402	160	382	463	45	291	193
486	8	266	236	89	443	181	343
218	316	54	472	357	135	393	107
185	347	85	439	262	232	490	12
389	103	361	139	58	476	214	312

134	360	106	396	313	219	469	55
442	92	342	184	5	487	233	267
473	59	309	215	102	392	138	364
229	263	9	491	346	188	438	88
371	145	415	125	208	302	36	450
79	429	163	321	500	18	288	254
48	462	196	290	403	113	383	157
276	242	512	30	175	333	67	417

306	212	478	64	141	367	97	387
14	496	226	260	433	83	349	191
109	399	129	355	466	52	318	224
337	179	445	95	238	272	2	484
199	293	43	457	380	154	408	118
507	25	279	245	72	422	172	330
412	122	376	150	39	453	203	297
168	326	76	426	283	249	503	21

423	69	331	169	28	506	248	278
155	377	119	405	296	198	460	42
252	282	24	502	327	165	427	73
456	38	300	202	123	409	151	373
82	436	190	352	493	15	257	227
366	144	386	100	209	307	61	479
269	239	481	3	178	340	94	448
49	467	221	319	398	112	354	132

381	159	401	115	194	292	46	464
65	419	173	335	510	32	274	244
34	452	206	304	413	127	369	147
286	256	498	20	161	323	77	431
140	362	104	390	311	213	475	57
440	86	348	186	11	489	231	261
471	53	315	217	108	394	136	358
235	265	7	485	344	182	444	90

492	10	264	230	87	437	187	345
216	310	60	474	363	137	391	101
183	341	91	441	268	234	488	6
395	105	359	133	56	470	220	314
29	511	241	275	418	68	334	176
289	195	461	47	158	384	114	404
322	164	430	80	253	287	17	499
126	416	146	372	449	35	301	207

96	446	180	338	483	1	271	237
356	130	400	110	223	317	51	465
259	225	495	13	192	350	84	434
63	477	211	305	388	98	368	142
425	75	325	167	22	504	250	284
149	375	121	411	298	204	454	40
246	280	26	508	329	171	421	71
458	44	294	200	117	407	153	379

201	299	37	455	374	152	410	124
501	23	281	251	74	428	166	328
406	120	378	156	41	459	197	295
170	332	70	424	277	247	505	27
320	222	468	50	131	353	111	397
4	482	240	270	447	93	339	177
99	385	143	365	480	62	308	210
351	189	435	81	228	258	16	494

There are no perfect magic cubes of order four (Beeler *et al.* 1972, Item 50; Gardner 1988). No perfect magic cubes of order five are known, although it is known that such a cube must have a central value of 63 (Beeler *et al.* 1972, Item 51; Gardner 1988). No order-six perfect magic cubes are known, but Langman (1962) constructed a perfect magic cube of order seven. An order-eight perfect magic cube was published anonymously in 1875 (Barnard 1888, Benson and Jacoby 1981, Gardner 1988). The construction of such a cube is discussed in Ball and Coxeter (1987). Rosser and Walker rediscovered the order-eight cube in the late 1930s (but did not publish it), and Myers independently discovered the cube illustrated above in 1970 (Gardner 1988). Order 9 and 11 magic cubes have also been discovered, but none of order 10 (Gardner 1988).

Semiperfect pandiagonal cubes exist for all orders $8n$ and all ODD $n > 8$ (Ball and Coxeter 1987). A perfect pandiagonal magic cube has been constructed by Planck (1950), cited in Gardner (1988).

Berlekamp *et al.* (1982, p. 783) give a magic TESSERACT.

see also MAGIC CONSTANT, MAGIC GRAPH, MAGIC HEXAGON, MAGIC SQUARE

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Magic Geometric Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let E be a compact connected subset of d-dimensional EUCLIDEAN SPACE. Gross (1964) and Stadje (1981) proved that there is a unique REAL NUMBER $a(E)$ such that for all $x_1, x_2, \ldots, x_n \in E$, there exists $y \in E$ with

$$\frac{1}{n}\sum_{j=1}^{n}\sqrt{\sum_{k=1}^{d}(x_{j,k}-y_k)^2}=a(E). \tag{1}$$

The magic constant $m(E)$ of E is defined by

$$m(E)=\frac{a(E)}{\operatorname{diam}(E)}, \tag{2}$$

where

$$\operatorname{diam}(E)\equiv\max_{u,v\in E}\sqrt{\sum_{k=1}^{d}(u_k-v_k)^2}. \tag{3}$$

These numbers are also called DISPERSION NUMBERS and RENDEZVOUS VALUES. For any E, Gross (1964) and Stadje (1981) proved that

$$\tfrac{1}{2}\leq m(E)<1. \tag{4}$$

If I is a subinterval of the LINE and D is a circular DISK in the PLANE, then

$$m(I)=m(D)=\tfrac{1}{2}. \tag{5}$$

If C is a CIRCLE, then

$$m(C)=\frac{2}{\pi}=0.6366\ldots. \tag{6}$$

An expression for the magic constant of an ELLIPSE in terms of its SEMIMAJOR and SEMIMINOR AXES lengths is not known. Nikolas and Yost (1988) showed that for a REULEAUX TRIANGLE T

$$0.6675276\leq m(T)\leq 0.6675284. \tag{7}$$

Denote the MAXIMUM value of $m(E)$ in n-D space by $M(n)$. Then

$$M(1)=\tfrac{1}{2} \tag{8}$$

$$M(2): m(T)\leq M(2)\leq\frac{2+\sqrt{3}}{3\sqrt{3}}<0.7182336 \tag{9}$$

$$M(d):\frac{d}{d+1}\leq M(d)\leq\frac{[\Gamma(\frac{1}{2}d)]^2 2^{d-2}\sqrt{2d}}{\Gamma(d-\frac{1}{2})\sqrt{(d+1)\pi}}<\sqrt{\frac{d}{d+1}}, \tag{10}$$

where $\Gamma(z)$ is the GAMMA FUNCTION (Nikolas and Yost 1988).

An unrelated quantity characteristic of a given MAGIC SQUARE is also known as a MAGIC CONSTANT.

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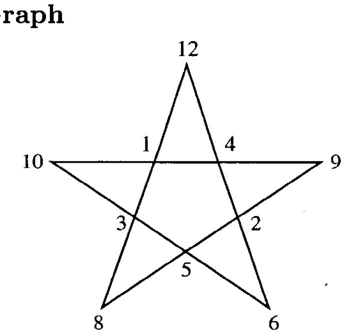
Magic Graph



A LABELLED GRAPH with e EDGES labeled with distinct elements $\{1, 2, \ldots, e\}$ so that the sum of the EDGE labels at each VERTEX is the same. Another type of magic graph, such as the PENTAGRAM shown above, has labelled VERTICES which give the same sum along every straight line segment (Madachy 1979).

see also ANTIMAGIC GRAPH, LABELLED GRAPH, MAGIC CIRCLES, MAGIC CONSTANT, MAGIC CUBE, MAGIC HEXAGON, MAGIC SQUARE

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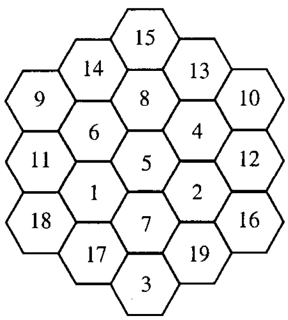
Magic Hexagon



An arrangement of close-packed HEXAGONS containing the numbers 1, 2, ..., $H_n = 3n(n-1)+1$, where H_n is the nth HEX NUMBER, such that the numbers along each straight line add up to the same sum. In the above magic hexagon, each line (those of lengths both 3 and 4) adds up to 38. This is the only magic hexagon of the counting numbers for any size hexagon. It was discovered by C. W. Adam, who worked on the problem from 1910 to 1957.

see also HEX NUMBER, HEXAGON, MAGIC GRAPH, MAGIC SQUARE, TALISMAN HEXAGON

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Magic Labelling

It is conjectured that every TREE with e edges whose nodes are all trivalent or monovalent can be given a "magic" labelling such that the INTEGERS 1, 2, ..., e can be assigned to the edges so that the SUM of the three meeting at a node is constant.

see also MAGIC CONSTANT, MAGIC CUBE, MAGIC GRAPH, MAGIC HEXAGON, MAGIC SQUARE

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Magic Number

see MAGIC CONSTANT

Magic Series

n numbers form a magic series of degree p if the sum of their kth POWERS is the MAGIC CONSTANT of degree k for all $k \in [1, p]$.

see also MAGIC CONSTANT, MAGIC SQUARE

References

Kraitchik, M. "Magic Series." §7.13.3 in *Mathematical Recreations.* New York: W. W. Norton, pp. 183–186, 1942.

Magic Square

8	1	6
3	5	7
4	9	2

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

A (normal) magic square consists of the distinct POSITIVE INTEGERS 1, 2, ..., n^2 such that the sum of the n numbers in any horizontal, vertical, or *main* diagonal line is always the same MAGIC CONSTANT

$$M_2(n) = \frac{1}{n}\sum_{k=1}^{n^2} k = \tfrac{1}{2}n(n^2+1).$$

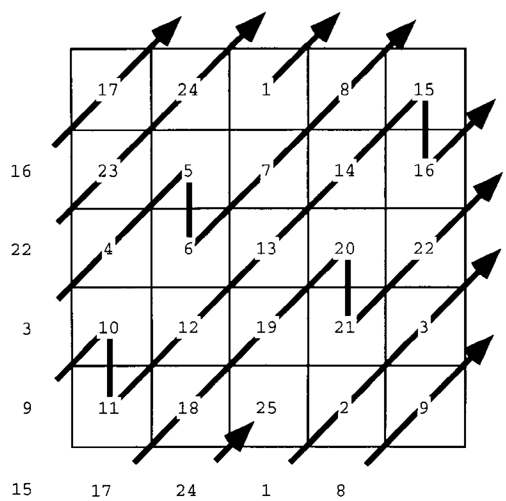
The unique normal square of order three was known to the ancient Chinese, who called it the LO SHU. A version of the order 4 magic square with the numbers 15 and 14 in adjacent middle columns in the bottom row is called DÜRER'S MAGIC SQUARE. Magic squares of order 3 through 8 are shown above.

The MAGIC CONSTANT for an nth order magic square starting with an INTEGER A and with entries in an increasing ARITHMETIC SERIES with difference D between terms is

$$M_2(n; A, D) = \tfrac{1}{2}n[2a + D(n^2-1)]$$

(Hunter and Madachy 1975). If every number in a magic square is subtracted from n^2+1, another magic square is obtained called the complementary magic square. Squares which are magic under multiplication instead of addition can be constructed and are known as MULTIPLICATION MAGIC SQUARES. In addition, squares which are magic under both addition *and* multiplication can be constructed and are known as ADDITION-MULTIPLICATION MAGIC SQUARES (Hunter and Madachy 1975).

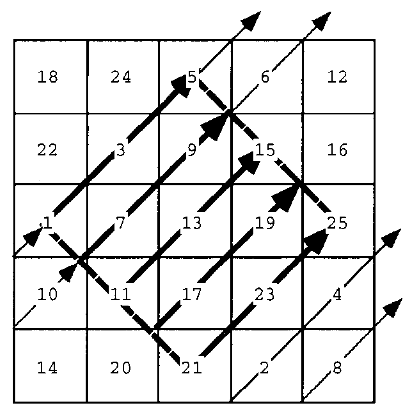
A square that fails to be magic only because one or both of the main diagonal sums do not equal the MAGIC CONSTANT is called a SEMIMAGIC SQUARE. If *all* diagonals (including those obtained by wrapping around) of a magic square sum to the MAGIC CONSTANT, the square is said to be a PANMAGIC SQUARE (also called a DIABOLICAL SQUARE or PANDIAGONAL SQUARE). If replacing each number n_i by its square $n_i{}^2$ produces another magic square, the square is said to be a BIMAGIC SQUARE (or DOUBLY MAGIC SQUARE). If a square is magic for n_i, $n_i{}^2$, and $n_i{}^3$, it is called a TREBLY MAGIC SQUARE. If all pairs of numbers symmetrically opposite the center sum to $n^2 + 1$, the square is said to be an ASSOCIATIVE MAGIC SQUARE.



Kraitchik (1942) gives general techniques of constructing EVEN and ODD squares of order n. For n ODD, a very straightforward technique known as the Siamese method can be used, as illustrated above (Kraitchik 1942, pp. 148–149). It begins by placing a 1 in any location (in the center square of the top row in the above example), then incrementally placing subsequent numbers in the square one unit above and to the right. The counting is wrapped around, so that falling off the top returns on the bottom and falling off the right returns on the left. When a square is encountered which is already filled, the next number is instead placed *below* the previous one and the method continues as before. The method, also called de la Loubere's method, is purported to have been first reported in the West when de la Loubere returned to France after serving as ambassador to Siam.

A generalization of this method uses an "ordinary vector" (x, y) which gives the offset for each noncolliding move and a "break vector" (u, v) which gives the offset to introduce upon a collision. The standard Siamese method therefore has ordinary vector (1, −1) and break vector (0, 1). In order for this to produce a magic square, each break move must end up on an unfilled cell. Special classes of magic squares can be constructed by considering the absolute sums $|u+v|$, $|(u-x)+(v-y)|$, $|u-v|$, and $|(u-x)-(v-y)| = |u+y-x-v|$. Call the set of these numbers the sumdiffs (sums and differences). If all sumdiffs are RELATIVELY PRIME to n and the square is a magic square, then the square is also a PANMAGIC SQUARE. This theory originated with de la Hire. The following table gives the sumdiffs for particular choices of ordinary and break vectors.

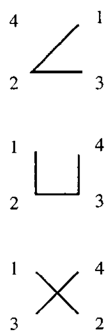
Ordinary Vector	Break Vector	Sumdiffs	Magic Squares	Panmagic Squares
(1, −1)	(0, 1)	(1, 3)	$2k+1$	none
(1, −1)	(0, 2)	(0, 2)	$6k \pm 1$	none
(2, 1)	(1, −2)	(1, 2, 3, 4)	$6k \pm 1$	none
(2, 1)	(1, −1)	(0, 1, 2, 3)	$6k \pm 1$	$6k \pm 1$
(2, 1)	(1, 0)	(0, 1, 2)	$2k+1$	none
(2, 1)	(1, 2)	(0, 1, 2, 3)	$6k \pm 1$	none



A second method for generating magic squares of ODD order has been discussed by J. H. Conway under the name of the "lozenge" method. As illustrated above, in this method, the ODD numbers are built up along diagonal lines in the shape of a DIAMOND in the central part of the square. The EVEN numbers which were missed are then added sequentially along the continuation of the diagonal obtained by wrapping around the square until the wrapped diagonal reaches its initial point. In the above square, the first diagonal therefore fills in 1, 3, 5, 2, 4, the second diagonal fills in 7, 9, 6, 8, 10, and so on.

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

An elegant method for constructing magic squares of DOUBLY EVEN order $n = 4m$ is to draw xs through each 4×4 subsquare and fill all squares in sequence. Then replace each entry a_{ij} on a crossed-off diagonal by $(n^2 + 1) - a_{ij}$ or, equivalently, reverse the order of the crossed-out entries. Thus in the above example for $n = 8$, the crossed-out numbers are originally 1, 4, ..., 61, 64, so entry 1 is replaced with 64, 4 with 61, etc.



68	65	96	93	4	1	32	29	60	57
L		L		L		L		L	
66	67	94	95	2	3	30	31	58	59
92	89	20	17	28	25	56	53	64	61
L		L		L		L		L	
90	91	18	19	26	27	54	55	62	63
16	13	24	21	49	52	80	77	88	85
L		L		U		L		L	
14	15	22	23	50	51	78	79	86	87
37	40	45	48	76	73	81	84	9	12
U		U		L		U		U	
38	39	46	47	74	75	82	83	10	11
41	44	69	72	97	100	5	8	33	36
X		X		X		X		X	
43	42	71	70	99	98	7	6	35	34

A very elegant method for constructing magic squares of SINGLY EVEN order $n = 4m + 2$ with $m \geq 1$ (there is no magic square of order 2) is due to J. H. Conway, who calls it the "LUX" method. Create an array consisting of $m + 1$ rows of Ls, 1 row of Us, and $m - 1$ rows of Xs, all of length $n/2 = 2m + 1$. Interchange the middle U with the L above it. Now generate the magic square of order $2m + 1$ using the Siamese method centered on the array of letters (starting in the center square of the top row), but fill each set of four squares surrounding a letter sequentially according to the order prescribed by the the letter. That order is illustrated on the left side of the above figure, and the completed square is illustrated to the right. The "shapes" of the letters L, U, and X naturally suggest the filling order, hence the name of the algorithm.

It is an unsolved problem to determine the number of magic squares of an arbitrary order, but the number of distinct magic squares (excluding those obtained by rotation and reflection) of order $n = 1, 2, \ldots$ are 1, 0, 1, 880, 275305224, ... (Sloane's A006052; Madachy 1979, p. 87). The 880 squares of order four were enumerated by Frenicle de Bessy in the seventeenth century, and are illustrated in Berlekamp *et al.* (1982, pp. 778–783). The number of 6×6 squares is not known.

67	1	43
13	37	61
31	73	7

3	61	19	37
43	31	5	41
7	11	73	29
67	17	23	13

The above magic squares consist only of PRIMES and were discovered by E. Dudeney (1970) and A. W. Johnson, Jr. (Dewdney 1988). Madachy (1979, pp. 93–96) and Rivera discuss other magic squares composed of PRIMES.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Benjamin Franklin constructed the above 8×8 PANMAGIC SQUARE having MAGIC CONSTANT 260. Any half-row or half-column in this square totals 130, and the four corners plus the middle total 260. In addition, bent diagonals (such as 52-3-5-54-10-57-63-16) also total 260 (Madachy 1979, p. 87).

1480028159	1480028153	1480028201
1480028213	1480028171	1480028129
1480028141	1480028189	1480028183

In addition to other special types of magic squares, a 3×3 square whose entries are consecutive PRIMES, illustrated above, has been discovered by H. Nelson (Rivera). Variations on magic squares can also be constructed using letters (either in defining the square or as entries in it), such as the ALPHAMAGIC SQUARE and TEMPLAR MAGIC SQUARE.

4	9	2
3	5	7
8	1	6

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

Various numerological properties have also been associated with magic squares. Pivari associates the squares illustrated above with Saturn, Jupiter, Mars, the Sun, Venus, Mercury, and the Moon, respectively. Attractive patterns are obtained by connecting consecutive numbers in each of the squares (with the exception of the Sun magic square).

see also ADDITION-MULTIPLICATION MAGIC SQUARE ALPHAMAGIC SQUARE, ANTIMAGIC SQUARE, ASSOCIATIVE MAGIC SQUARE, BIMAGIC SQUARE, BORDER SQUARE, DÜRER'S MAGIC SQUARE, EULER SQUARE, FRANKLIN MAGIC SQUARE, GNOMON MAGIC SQUARE, HETEROSQUARE, LATIN SQUARE, MAGIC CIRCLES, MAGIC CONSTANT, MAGIC CUBE, MAGIC HEXAGON, MAGIC LABELLING, MAGIC SERIES, MAGIC TOUR, MULTIMAGIC SQUARE, MULTIPLICATION MAGIC SQUARE, PANMAGIC SQUARE, SEMIMAGIC SQUARE, TALISMAN SQUARE, TEMPLAR MAGIC SQUARE, TRIMAGIC SQUARE

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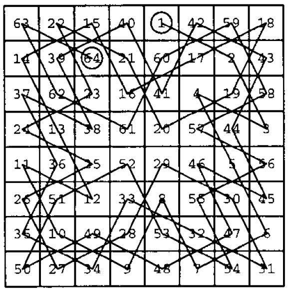
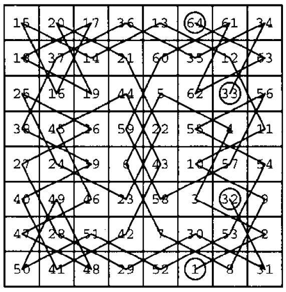
Magic Star

see MAGIC GRAPH

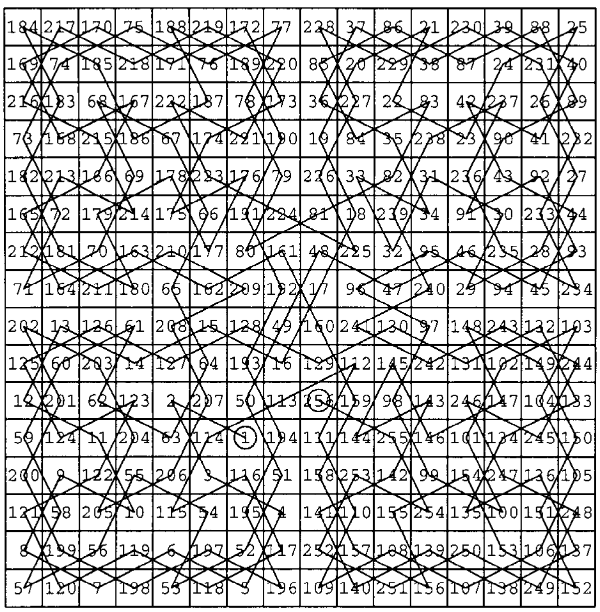
Magic Tour

Let a chess piece make a TOUR on an $n \times n$ CHESSBOARD whose squares are numbered from 1 to n^2 along the path of the chess piece. Then the TOUR is called a magic tour if the resulting arrangement of numbers is a MAGIC SQUARE. If the first and last squares traversed are connected by a move, the tour is said to be closed (or "re-entrant"); otherwise it is open. The MAGIC CONSTANT for the 8×8 CHESSBOARD is 260.

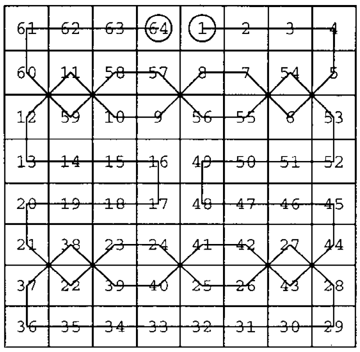




Magic KNIGHT'S TOURS are not possible on $n \times n$ boards for n ODD, and are believed to be impossible for $n = 8$. The "most magic" knight tour known on the 8×8 board is the SEMIMAGIC SQUARE illustrated in the above left figure (Ball and Coxeter 1987, p. 185) having main diagonal sums of 348 and 168. Combining two half-knights' tours one above the other as in the above right figure does, however, give a MAGIC SQUARE (Ball and Coxeter 1987, p. 185).



The above illustration shows a 16×16 closed magic KNIGHT'S TOUR (Madachy 1979).



A magic tour for king moves is illustrated above (Coxeter 1987, p. 186).

see also CHESSBOARD, KNIGHT'S TOUR, MAGIC SQUARE, SEMIMAGIC SQUARE, TOUR

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Mahler-Lech Theorem

Let K be a FIELD of CHARACTERISTIC 0 (e.g., the rationals $\mathbb{Q}$) and let $\{u_n\}$ be a SEQUENCE of elements of K which satisfies a difference equation of the form

$$u_n = c_0 u_n + c_1 u_{n+1} + \ldots + c_k u_{n+k},$$

where the COEFFICIENTS c_i are fixed elements of K. Then, for any $c \in K$, we have *either* $u_n = c$ for only finitely many values of n, *or* $u_n = c$ for the values of n in some ARITHMETIC PROGRESSION.

The proof involves embedding certain fields inside the p-ADIC NUMBERS $\mathbb{Q}_p$ for some PRIME p, and using properties of zeros of POWER series over $\mathbb{Q}_p$ (STRASSMAN'S THEOREM).

see also ARITHMETIC PROGRESSION, p-ADIC NUMBER, STRASSMAN'S THEOREM

Mahler's Measure

For a POLYNOMIAL P,

$$M(P) = \exp \int_0^{2\pi} \ln |P(e^{i\theta})| \frac{d\theta}{2\pi}.$$

It is related to JENSEN'S INEQUALITY.

see also JENSEN'S INEQUALITY

Major Axis

see SEMIMAJOR AXIS

Major Triangle Center

A TRIANGLE CENTER $\alpha : \beta : \gamma$ is called a major center if the TRIANGLE CENTER FUNCTION $\alpha = f(a, b, c, A, B, C)$ is a function of ANGLE A alone, and therefore β and γ of B and C alone, respectively.

see also REGULAR TRIANGLE CENTER, TRIANGLE CENTER

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Majorant

A function used to study ORDINARY DIFFERENTIAL EQUATIONS.

Makeham Curve

The function defined by

$$y = ks^x b^{q^x}$$

which is used in actuarial science for specifying a simplified mortality law. Using $s(x)$ as the probability that a newborn will achieve age x, the Makeham law (1860) uses

$$s(x) = \exp(-Ax - m(c^x - 1))$$

for $B > 0$, $A \geq -B$, $c > 1$, $x \geq 0$.

see also GOMPERTZ CURVE, LIFE EXPECTANCY, LOGISTIC GROWTH CURVE, POPULATION GROWTH

References

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Malfatti Circles

Three circles packed inside a RIGHT TRIANGLE which are tangent to each other and to two sides of the TRIANGLE.

see also MALFATTI'S RIGHT TRIANGLE PROBLEM

Malfatti Points

see AJIMA-MALFATTI POINTS

Malfatti's Right Triangle Problem

Find the maximum total AREA of three CIRCLES (of possibly different sizes) which can be packed inside a RIGHT TRIANGLE of any shape without overlapping. In 1803, Malfatti gave the solution as three CIRCLES (the MALFATTI CIRCLES) tangent to each other and to two sides of the TRIANGLE. In 1929, it was shown that the MALFATTI CIRCLES were not always the best solution. Then Goldberg (1967) showed that, even worse, they are *never* the best solution.

see also MALFATTI'S TANGENT TRIANGLE PROBLEM

References

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Goldberg, M. "On the Original Malfatti Problem." *Math. Mag.* **40**, 241–247, 1967.

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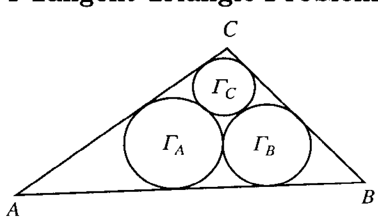
Malfatti's Tangent Triangle Problem



Draw within a given TRIANGLE three CIRCLES, each of which is TANGENT to the other two and to two sides of the TRIANGLE. Denote the three CIRCLES so constructed Γ_A, Γ_B, and Γ_C. Then Γ_A is tangent to AB and AC, Γ_B is tangent to BC and BA, and Γ_C is tangent to AC and BC.

see also AJIMA-MALFATTI POINTS, MALFATTI'S RIGHT TRIANGLE PROBLEM

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Malliavin Calculus

An infinite-dimensional DIFFERENTIAL CALCULUS on the WIENER SPACE. Also called STOCHASTIC CALCULUS OF VARIATIONS.

Mallow's Sequence

An INTEGER SEQUENCE given by the recurrence relation

$$a(n) = a(a(n-2)) + a(n - a(n-2))$$

with $a(1) = a(2) = 1$. The first few values are 1, 1, 2, 3, 3, 4, 5, 6, 6, 7, 7, 8, 9, 10, 10, 11, 12, 12, 13, 14, ... (Sloane's A005229).

see also HOFSTADTER-CONWAY \$10,000 SEQUENCE, HOFSTADTER'S Q-SEQUENCE

References

Mallows, C. "Conway's Challenging Sequence." *Amer. Math. Monthly* **98**, 5–20, 1991.

Sloane, N. J. A. Sequence A005229/M0441 in "An On-Line Version of the Encyclopedia of Integer Sequences."

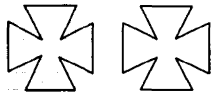
Malmstén's Differential Equation

$$y'' + \frac{r}{z} y' = \left(Az^m + \frac{s}{z^2} \right) y.$$

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 99–100, 1966.

Maltese Cross



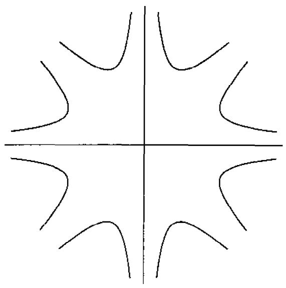
An irregular DODECAHEDRON CROSS shaped like a + sign but whose points flange out at the end: ✠. The conventional proportions as computed on a 5×5 grid as illustrated above.

see also CROSS, DISSECTION, DODECAHEDRON

References

Frederickson, G. "Maltese Crosses." Ch. 14 in *Dissections: Plane and Fancy.* New York: Cambridge University Press, pp. 157–162, 1997.

Maltese Cross Curve



The plane curve with Cartesian equation

$$xy(x^2 - y^2) = x^2 + y^2$$

and polar equation

$$r^2 = \frac{1}{\cos\theta \sin\theta(\cos^2\theta - \sin^2\theta)}.$$

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 71, 1989.

Malthusian Parameter

The parameter α in the exponential POPULATION GROWTH equation

$$N_1(t) = N_0 e^{\alpha t}.$$

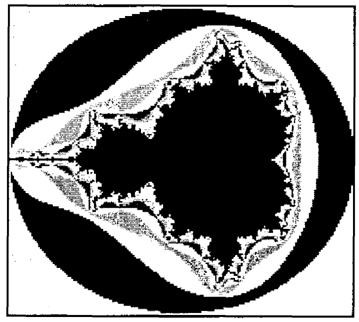
see also LIFE EXPECTANCY, POPULATION GROWTH

Mandelbar Set

A FRACTAL set analogous to the MANDELBROT SET or its generalization to a higher power with the variable z replaced by its COMPLEX CONJUGATE z^*.

see also MANDELBROT SET

Mandelbrot Set



The set obtained by the QUADRATIC RECURRENCE

$$z_{n+1} = z_n{}^2 + C, \tag{1}$$

where points C for which the orbit $z = 0$ does not tend to infinity are in the SET. It marks the set of points in the COMPLEX PLANE such that the corresponding JULIA SET is CONNECTED and not COMPUTABLE. The Mandelbrot set was originally called a μ MOLECULE by Mandelbrot.

J. Hubbard and A. Douady proved that the Mandelbrot set is CONNECTED. Shishikura (1994) proved that the boundary of the Mandelbrot set is a FRACTAL with HAUSDORFF DIMENSION 2. However, it is not yet known if the Mandelbrot set is pathwise-connected. If it is pathwise-connected, then Hubbard and Douady's proof implies that the Mandelbrot set is the image of a CIRCLE and can be constructed from a DISK by collapsing certain arcs in the interior (Douady 1986).

The AREA of the set is known to lie between 1.5031 and 1.5702; it is estimated as 1.50659....

Decomposing the COMPLEX coordinate $z = x + iy$ and $z_0 = a + ib$ gives

$$x' = x^2 - y^2 + a \tag{2}$$

$$y' = 2xy + b. \tag{3}$$

In practice, the limit is approximated by

$$\lim_{n\to\infty} |z_n| \approx \lim_{n\to n_{\max}} |z_n| < r_{\max}. \tag{4}$$

Beautiful computer-generated plots can be created by coloring nonmember points depending on how quickly they diverge to $r_{\max}$. A common choice is to define an INTEGER called the COUNT to be the largest n such that $|z_n| < r$, where r is usually taken as $r = 2$, and to color points of different COUNT different colors. The boundary between successive COUNTS defines a series of "LEMNISCATES," called EQUIPOTENTIAL CURVES by Peitgen and Saupe (1988), $|L_n(C)| = r$ which have distinctive shapes. The first few LEMNISCATES are

$$L_1(C) = C \tag{5}$$

$$L_2(C) = C(C+1) \tag{6}$$

$$L_3(C) = C + (C + C^2)^2 \tag{7}$$

$$L_4(C) = C + [C + (C + C^2)^2]^2. \tag{8}$$

When written in CARTESIAN COORDINATES, the first three of these are

$$r^2 = x^2 + y^2 \tag{9}$$

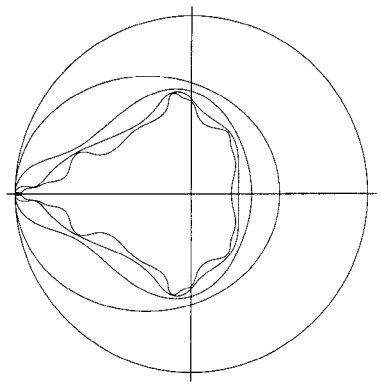
$$r^2 = (x^2 + y^2)[(x+1)^2 + y^2] \tag{10}$$

$$\begin{aligned} r^2 = {} & (x^2 + y^2)(1 + 2x + 5x^2 + 6x^3 + 6x^4 + 4x^5 + x^6 \\ & - 3y^2 - 2xy^2 + 8x^2y^2 + 8x^3y^2 \\ & + 3x^4y^2 + 2y^4 + 4xy^4 + 3x^2y^4 + y^6), \end{aligned} \tag{11}$$

which are a CIRCLE, an OVAL, and a PEAR CURVE. In fact, the second LEMNISCATE L_2 can be written in terms of a new coordinate system with $x' \equiv x - 1/2$ as

$$[(x' - \tfrac{1}{2})^2 + y^2][(x' + \tfrac{1}{2})^2 + y^2] = r^2, \tag{12}$$

which is just a CASSINI OVAL with $a = 1/2$ and $b^2 = r$. The LEMNISCATES grow increasingly convoluted with higher COUNT and approach the Mandelbrot set as the COUNT tends to infinity.



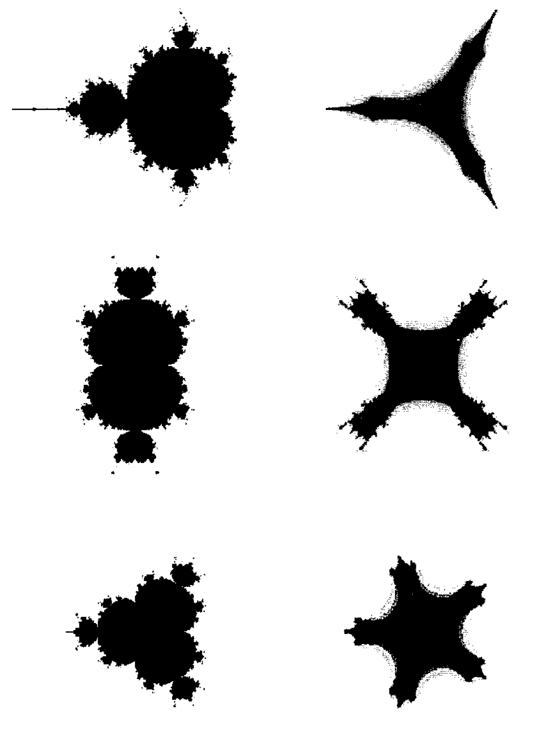
The kidney bean-shaped portion of the Mandelbrot set is bordered by a CARDIOID with equations

$$4x = 2\cos t - \cos(2t) \tag{13}$$

$$4y = 2\sin t - \sin(2t). \tag{14}$$

The adjoining portion is a CIRCLE with center at $(-1, 0)$ and RADIUS 1/4. One region of the Mandelbrot set containing spiral shapes is known as SEA HORSE VALLEY because the shape resembles the tail of a sea horse.

Generalizations of the Mandelbrot set can be constructed by replacing ${z_n}^2$ with ${z_n}^k$ or ${z_n^*}^k$, where k is a POSITIVE INTEGER and z^* denotes the COMPLEX CONJUGATE of z. The following figures show the FRACTALS obtained for $k = 2$, 3, and 4 (Dickau). The plots on the right have z replaced with z^* and are sometimes called "MANDELBAR SETS."



see also CACTUS FRACTAL, FRACTAL, JULIA SET, LEMNISCATE (MANDELBROT SET), MANDELBAR SET, QUADRATIC MAP, RANDELBROT SET, SEA HORSE VALLEY

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Branner, B. "The Mandelbrot Set." In *Chaos and Fractals: The Mathematics Behind the Computer Graphics, Proc. Sympos. Appl. Math., Vol. 39* (Ed. R. L. Devaney and L. Keen). Providence, RI: Amer. Math. Soc., 75–105, 1989.

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Fisher, Y. and Hill, J. "Bounding the Area of the Mandelbrot Set." Submitted.

Hill, J. R. "Fractals and the Grand Internet Parallel Processing Project." Ch. 15 in *Fractal Horizons: The Future Use of Fractals.* New York: St. Martin's Press, pp. 299–323, 1996.

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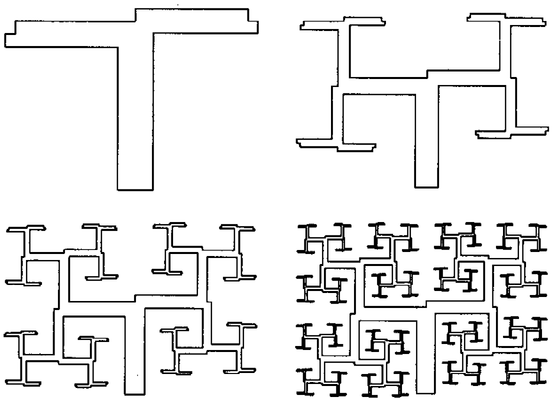
Peitgen, H.-O. and Saupe, D. (Eds.). *The Science of Fractal Images.* New York: Springer-Verlag, pp. 178–179, 1988.

Shishikura, M. "The Boundary of the Mandelbrot Set has Hausdorff Dimension Two." *Astérisque*, No. 222, **7**, 389–405, 1994.

Mandelbrot Tree



The FRACTAL illustrated above.

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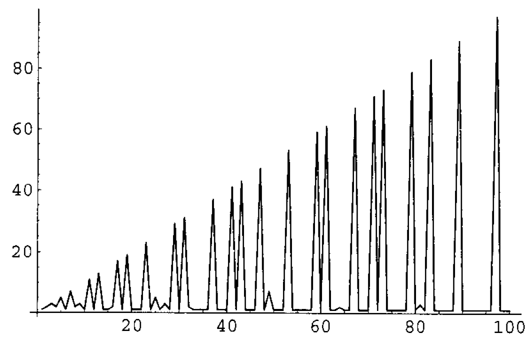
Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Mangoldt Function



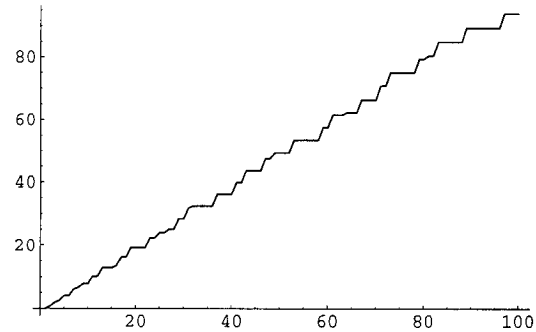
The function defined by

$$\Lambda(n) \equiv \begin{cases} \ln p & \text{if } n = p^k \text{ for } p \text{ a prime} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$\exp(\Lambda(n))$ is also given by $[1, 2, \ldots, n]/[1, 2, \ldots, n-1]$, where $[a, b, c, \ldots]$ denotes the LEAST COMMON MULTIPLE. The first few values of $\exp(\Lambda(n))$ for $n = 1, 2, \ldots$, plotted above, are 1, 2, 3, 2, 5, 1, 7, 2, ... (Sloane's A014963). The Mangoldt function is related to the RIEMANN ZETA FUNCTION $\zeta(z)$ by

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{2}$$

where $\Re[s] > 1$.



The SUMMATORY Mangoldt function, illustrated above, is defined by

$$\psi(x) \equiv \sum_{n \leq x} \Lambda(n), \tag{3}$$

where $\Lambda(n)$ is the MANGOLDT FUNCTION. This has the explicit formula

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \ln(2\pi) - \tfrac{1}{2}\ln(1 - x^2), \tag{4}$$

where the second SUM is over all complex zeros ρ of the RIEMANN ZETA FUNCTION $\zeta(s)$ and interpreted as

$$\lim_{t \to \infty} \sum_{|\Im(\rho)| < t} \frac{x^\rho}{\rho}. \tag{5}$$

Vardi (1991, p. 155) also gives the interesting formula

$$\ln([x]!) = \psi(x) + \psi(\tfrac{1}{2}x) + \psi(\tfrac{1}{3}x) + \ldots, \tag{6}$$

where $[x]$ is the NINT function and $n!$ is a FACTORIAL.

Vallée Poussin's version of the PRIME NUMBER THEOREM states that

$$\psi(x) = x + \mathcal{O}(xe^{-a\sqrt{\ln x}}) \tag{7}$$

for some a (Davenport 1980, Vardi 1991). The RIEMANN HYPOTHESIS is equivalent to

$$\psi(x) = x + \mathcal{O}(\sqrt{x}\,(\ln x)^2) \tag{8}$$

(Davenport 1980, p. 114; Vardi 1991).

see also BOMBIERI'S THEOREM, GREATEST PRIME FACTOR, LAMBDA FUNCTION, LEAST COMMON MULTIPLE, LEAST PRIME FACTOR, RIEMANN FUNCTION

References
Davenport, H. *Multiplicative Number Theory, 2nd ed.* New York: Springer-Verlag, p. 110, 1980.
Sloane, N. J. A. Sequence A014963 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 146–147, 152–153, and 249, 1991.

Manifold

Rigorously, an n-D (topological) manifold is a TOPOLOGICAL SPACE M such that any point in M has a NEIGHBORHOOD $U \subset M$ which is HOMEOMORPHIC to n-D EUCLIDEAN SPACE. The HOMEOMORPHISM is called a chart, since it lays that part of the manifold out flat, like charts of regions of the Earth. So a preferable statement is that any object which can be "charted" is a manifold.

The most important manifolds are DIFFERENTIABLE MANIFOLDS. These are manifolds where overlapping charts "relate smoothly" to each other, meaning that the inverse of one followed by the other is an infinitely differentiable map from EUCLIDEAN SPACE to itself.

Manifolds arise naturally in a variety of mathematical and physical applications as "global objects." For example, in order to precisely describe all the configurations of a robot arm or all the possible positions and momenta of a rocket, an object is needed to store all of these parameters. The objects that crop up are manifolds. From the geometric perspective, manifolds represent the profound idea having to do with global versus local properties.

Consider the ancient belief that the Earth was flat compared to the modern evidence that it is round. The discrepancy arises essentially from the fact that on the small scales that we see, the Earth does look flat. We cannot see it curve because we are too small (although the Greeks did notice that the last part of a ship to disappear over the horizon was the mast). We can detect curvature only indirectly from our vantage point on the Earth. The basic idea for this "problem" was codified by Poincaré. The problem is that on a small scale, the Earth is nearly flat. In general, any object which is nearly "flat" on small scales is a manifold, and so manifolds constitute a generalization of objects we could live on in which we would encounter the round/flat Earth problem.

see also COBORDANT MANIFOLD, COMPACT MANIFOLD, CONNECTED SUM DECOMPOSITION, DIFFERENTIABLE MANIFOLD, FLAG MANIFOLD, GRASSMANN MANIFOLD, HEEGAARD SPLITTING, ISOSPECTRAL MANIFOLDS, JACO-SHALEN-JOHANNSON TORUS DECOMPOSITION, KÄHLER MANIFOLD, POINCARÉ CONJECTURE, POISSON MANIFOLD, PRIME MANIFOLD, RIEMANNIAN MANIFOLD, SET, SMOOTH MANIFOLD, SPACE, STIEFEL MANIFOLD, STRATIFIED MANIFOLD, SUBMANIFOLD, SURGERY, SYMPLECTIC MANIFOLD, THURSTON'S GEOMETRIZATION CONJECTURE, TOPOLOGICAL MANIFOLD, TOPOLOGICAL SPACE, WHITEHEAD MANIFOLD, WIEDERSEHEN MANIFOLD

References

Conlon, L. *Differentiable Manifolds: A First Course.* Boston, MA: Birkhäuser, 1993.

Mantissa

For a REAL NUMBER x, the mantissa is defined as the POSITIVE fractional part $x - \lfloor x \rfloor = \texttt{frac(x)}$, where $\lfloor x \rfloor$ denotes the FLOOR FUNCTION.

see also CHARACTERISTIC (REAL NUMBER), FLOOR FUNCTION, SCIENTIFIC NOTATION

Map

A way of associating unique objects to every point in a given SET. So a map from $A \mapsto B$ is an object f such that for every $a \in A$, there is a unique object $f(a) \in B$. The terms FUNCTION and MAPPING are synonymous with map.

The following table gives several common types of complex maps.

Mapping	Formula	Domain
inversion	$f(z) = \frac{1}{z}$	
magnification	$f(z) = az$	$a \in \mathbb{R} \neq 0$
magnification+rotation	$f(z) = az$	$a \in \mathbb{C} \neq 0$
Möbius	$f(z) = \frac{az+b}{cz+d}$	$a, b, c, d \in \mathbb{C}$
rotation	$f(z) = e^{i\theta z}$	
translation	$f(z) = z + a$	$a \in \mathbb{C}$

see also $2x$ MOD 1 MAP, ARNOLD'S CAT MAP, BAKER'S MAP, BOUNDARY MAP, CONFORMAL MAP, FUNCTION, GAUSS MAP, GINGERBREADMAN MAP, HARMONIC MAP, HÉNON MAP, IDENTITY MAP, INCLUSION MAP, KAPLAN-YORKE MAP, LOGISTIC MAP, MANDELBROT SET, MAP PROJECTION, PULLBACK MAP, QUADRATIC MAP, TANGENT MAP, TENT MAP, TRANSFORMATION, ZASLAVSKII MAP

References

Arfken, G. "Mapping." §6.6 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 384–392, 1985.

Lee, X. "Transformation of the Plane." `http://www.best.com/~xah/MathGraphicsGallery_dir/Transform2DPlot_dir/transform2DPlot.html`.

Map Coloring

Given a map with GENUS $g > 0$, Heawood showed in 1890 that the maximum number N_u of colors necessary to color a map (the CHROMATIC NUMBER) on an unbounded surface is

$$N_u \equiv \left\lfloor \tfrac{1}{2}(7 + \sqrt{48g+1}\,) \right\rfloor = \left\lfloor \tfrac{1}{2}(7 + \sqrt{49 - 24\chi}\,) \right\rfloor,$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, g is the GENUS, and χ is the EULER CHARACTERISTIC. This is the HEAWOOD CONJECTURE. In 1968, for any orientable surface other than the SPHERE (or equivalently, the PLANE) and any nonorientable surface other than the KLEIN BOTTLE, N_u was shown to be not merely a maximum, but the actual number needed (Ringel and Youngs 1968).

When the FOUR-COLOR THEOREM was proven, the Heawood FORMULA was shown to hold also for all orientable and nonorientable surfaces with the exception of the

KLEIN BOTTLE. For this case (which has EULER CHARACTERISTIC 1, and therefore can be considered to have $g = 1/2$), the actual number of colors N needed is six—*one less than* $N_u = 7$ (Franklin 1934; Saaty 1986, p. 45).

surface	g	N_u	N
Klein bottle	1	7	6
Möbius strip	$\frac{1}{2}$	6	6
plane	0	4	4
projective plane	$\frac{1}{2}$	6	6
sphere	0	4	4
torus	1	7	7

see also CHROMATIC NUMBER, FOUR-COLOR THEOREM, HEAWOOD CONJECTURE, SIX-COLOR THEOREM, TORUS COLORING

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Ringel, G. and Youngs, J. W. T. "Solution of the Heawood Map-Coloring Problem." *Proc. Nat. Acad. Sci. USA* **60**, 438–445, 1968.

Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, 1986.

Map Folding

A general FORMULA giving the number of distinct ways of folding an $N = m \times n$ rectangular map is not known. A distinct folding is defined as a permutation of N numbered cells reading from the top down. Lunnon (1971) gives values up to $n = 28$.

n	$1\times n$	$2\times n$	$3\times n$	$4\times n$	$5\times n$
1	1	1			
2	2	8			
3	6	60	1368		
4	16	1980		300608	
5	59	19512			18698669
6	144	15552			

The limiting ratio of the number of $1 \times (n+1)$ strips to the number of $1 \times n$ strips is given by

$$\lim_{n\to\infty} \frac{[1\times(n+1)]}{[1\times n]} \in [3.3868, 3.9821].$$

see also STAMP FOLDING

References

Gardner, M. "The Combinatorics of Paper Folding." Ch. 7 in *Wheels, Life, and Other Mathematical Amusements.* New York: W. H. Freeman, 1983.

Koehler, J. E. "Folding a Strip of Stamps." *J. Combin. Th.* **5**, 135–152, 1968.

Lunnon, W. F. "A Map-Folding Problem." *Math. Comput.* **22**, 193–199, 1968.

Lunnon, W. F. "Multi-Dimensional Strip Folding." *Computer J.* **14**, 75–79, 1971.

Map Projection

A projection which maps a SPHERE (or SPHEROID) onto a PLANE. No projection can be simultaneously CONFORMAL and AREA-PRESERVING.

see also AIRY PROJECTION, ALBERS EQUAL-AREA CONIC PROJECTION, AXONOMETRY, AZIMUTHAL EQUIDISTANT PROJECTION, AZIMUTHAL PROJECTION, BEHRMANN CYLINDRICAL EQUAL-AREA PROJECTION, BONNE PROJECTION, CASSINI PROJECTION, CHROMATIC NUMBER, CONIC EQUIDISTANT PROJECTION, CONIC PROJECTION, CYLINDRICAL EQUAL-AREA PROJECTION, CYLINDRICAL EQUIDISTANT PROJECTION, CYLINDRICAL PROJECTION, ECKERT IV PROJECTION, ECKERT VI PROJECTION, FOUR-COLOR THEOREM, GNOMIC PROJECTION, GUTHRIE'S PROBLEM, HAMMER-AITOFF EQUAL-AREA PROJECTION, LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION, LAMBERT CONFORMAL CONIC PROJECTION, MAP COLORING, MERCATOR PROJECTION, MILLER CYLINDRICAL PROJECTION, MOLLWEIDE PROJECTION, ORTHOGRAPHIC PROJECTION, POLYCONIC PROJECTION, PSEUDOCYLINDRICAL PROJECTION, RECTANGULAR PROJECTION, SINUSOIDAL PROJECTION, SIX-COLOR THEOREM, STEREOGRAPHIC PROJECTION, VAN DER GRINTEN PROJECTION, VERTICAL PERSPECTIVE PROJECTION

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Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, 1987.

Mapes' Method

A method for computing the PRIME COUNTING FUNCTION. Define the function

$$T_k(x,a) = (-1)^{\beta_0+\beta_1+\ldots+\beta_{a-1}} \left\lfloor \frac{x}{{p_1}^{\beta_0}{p_2}^{\beta_1}\cdots{p_a}^{\beta_{a-1}}} \right\rfloor, \tag{1}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and the β_i are the binary digits (0 or 1) in

$$k = 2^{a-1}\beta_{a-1} + 2^{a-2}\beta_{a-2} + \ldots + 2^1\beta_1 + 2^0\beta_0. \tag{2}$$

The LEGENDRE SUM can then be written

$$\phi(x,a) = \sum_{k=0}^{2^a-1} T_k(x,a). \tag{3}$$

The first few values of $T_k(x,3)$ are

$$T_0(x,3) = \lfloor x \rfloor \tag{4}$$

$$T_1(x,3) = -\left\lfloor \frac{x}{p_1} \right\rfloor \tag{5}$$

$$T_2(x,3) = -\left\lfloor \frac{x}{p_2} \right\rfloor \tag{6}$$

$$T_3(x,3) = \left\lfloor \frac{x}{p_1 p_2} \right\rfloor \tag{7}$$

$$T_4(x,3) = -\left\lfloor \frac{x}{p_3} \right\rfloor \tag{8}$$

$$T_5(x,3) = \left\lfloor \frac{x}{p_1 p_3} \right\rfloor \tag{9}$$

$$T_6(x,3) = \left\lfloor \frac{x}{p_2 p_3} \right\rfloor \tag{10}$$

$$T_7(x,3) = -\left\lfloor \frac{x}{p_1 p_2 p_3} \right\rfloor. \tag{11}$$

Mapes' method takes time $\sim x^{0.7}$, which is slightly faster than the LEHMER-SCHUR METHOD.

see also LEHMER-SCHUR METHOD, PRIME COUNTING FUNCTION

References

Mapes, D. C. "Fast Method for Computing the Number of Primes Less than a Given Limit." *Math. Comput.* **17**, 179–185, 1963.

Riesel, H. "Mapes' Method." *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, p. 23, 1994.

Mapping (Function)

see MAP

Mapping Space

Let Y^X be the set of continuous mappings $f: X \to Y$. Then the TOPOLOGICAL SPACE for Y^X supplied with a compact-open topology is called a mapping space.

see also LOOP SPACE

References

Iyanaga, S. and Kawada, Y. (Eds.). "Mapping Spaces." §204B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 658, 1980.

Marginal Analysis

Let $R(x)$ be the revenue for a production x, $C(x)$ the cost, and $P(x)$ the profit. Then

$$P(x) = R(x) - C(x),$$

and the marginal profit for the x_0th unit is defined by

$$P'(x_0) = R'(x_0) - C'(x_0),$$

where $P'(x)$, $R'(x)$, and $C'(x)$ are the DERIVATIVES of $P(x)$, $R(x)$, and $C(x)$, respectively.

see also DERIVATIVE

Marginal Probability

Let S be partitioned into $r \times s$ disjoint sets E_i and F_j where the general subset is denoted $E_i \cap F_j$. Then the marginal probability of E_i is

$$P(E_i) = \sum_{j=1}^{s} P(E_i \cap F_j).$$

Markoff's Formulas

Formulas obtained from differentiating NEWTON'S FORWARD DIFFERENCE FORMULA,

$$f'(a_0 + ph) = \frac{1}{h}\Big[\Delta_0 + \tfrac{1}{2}(2p-1)\Delta_0^2 + \tfrac{1}{6}(3p^2 - 6p + 2)\Delta_0^3 + \ldots + \frac{d}{dp}\binom{p}{n}\Delta_0^n\Big] + R_n',$$

where

$$R_n' = h^n f^{(n+1)}(\xi)\frac{d}{dp}\binom{p}{n+1} + h^{n+1}\binom{p}{n+1}\frac{d}{dx}f^{(n+1)}(\xi),$$

$\binom{n}{k}$ is a BINOMIAL COEFFICIENT, and $a_0 < \xi < a_n$. Abramowitz and Stegun (1972) and Beyer (1987) give derivatives $h^n f_0^{(n)}$ in terms of Δ^k and derivatives in terms of δ^k and ∇^k.

see also FINITE DIFFERENCE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 883, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 449–450, 1987.

Markoff Number

see MARKOV NUMBER

Markov Algorithm

An ALGORITHM which constructs allowed mathematical statements from simple ingredients.

Markov Chain

A collection of random variables $\{X_t\}$, where the index t runs through 0, 1, ...

References

Kemeny, J. G. and Snell, J. L. *Finite Markov Chains.* New York: Springer-Verlag, 1976.

Stewart, W. J. *Introduction to the Numerical Solution of Markov Chains.* Princeton, NJ: Princeton University Press, 1995.

Markov's Inequality

If x takes only NONNEGATIVE values, then

$$P(x \geq a) \leq \frac{\langle x \rangle}{a}.$$

To prove the theorem, write

$$\langle x \rangle = \int_0^\infty x f(x)\,dx = \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx.$$

Since $P(x)$ is a probability density, it must be ≥ 0. We have stipulated that $x \geq 0$, so

$$\langle x \rangle = \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

…

$$= a \int_a^\infty f(x)\,dx = aP(x \geq a),$$

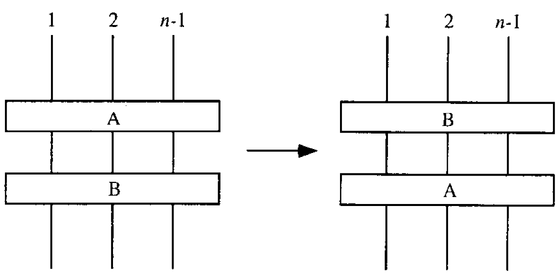
Q. E. D.

Markov Matrix

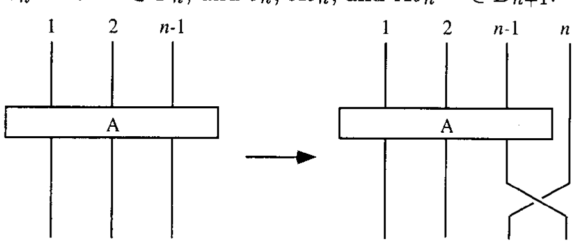
see STOCHASTIC MATRIX

Markov Moves

A type I move (CONJUGATION) takes $AB \to BA$ for A, $B \in B_n$ where B_n is a BRAID GROUP.



A type II move (STABILIZATION) takes $A \to Ab_n$ or $A \to Ab_n{}^{-1}$ for $A \in B_n$, and b_n, Ab_n, and $Ab_n{}^{-1} \in B_{n+1}$.



see also BRAID GROUP, CONJUGATION, REIDEMEISTER MOVES, STABILIZATION

Markov Number

The Markov numbers m occur in solutions to the DIOPHANTINE EQUATION

$$x^2 + y^2 + z^2 = 3xyz,$$

and are related to LAGRANGE NUMBERS L_n by

$$L_n = \sqrt{9 - \frac{4}{m^2}}.$$

The first few solutions are $(x, y, z) = (1,1,1)$, (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), ... The solutions can be arranged in an infinite tree with two smaller branches on each trunk. It is not known if two different regions can have the same label. Strangely, the regions adjacent to 1 have alternate FIBONACCI NUMBERS 1, 2, 5, 13, 34, …

…

N, then

$$M(n) = C(\ln N) + \mathcal{O}((\ln N)^{1+\epsilon}),$$

where $C \approx 0.180717105$ (Guy 1994, p. 166).

see also HURWITZ EQUATION, HURWITZ'S IRRATIONAL NUMBER THEOREM, LAGRANGE NUMBER (RATIONAL APPROXIMATION) LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, LIOUVILLE-ROTH CONSTANT, ROTH'S THEOREM, SEGRE'S THEOREM, THUE-SIEGEL-ROTH THEOREM

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 187–189, 1996.

Guy, R. K. "Markoff Numbers." §D12 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 166–168, 1994.

Markov Process

A random process whose future probabilities are determined by its most recent values.

see also DOOB'S THEOREM

Markov Spectrum

A SPECTRUM containing the REAL NUMBERS larger than FREIMAN'S CONSTANT.

see also FREIMAN'S CONSTANT, SPECTRUM SEQUENCE

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 188–189, 1996.

Markov's Theorem

Published by A. A. Markov in 1935, Markov's theorem states that equivalent BRAIDS expressing the same LINK are mutually related by successive applications of two types of MARKOV MOVES. Markov's theorem is difficult to apply in practice, so it is difficult to establish the equivalence or nonequivalence of LINKS having different BRAID representations.

see also BRAID, LINK, MARKOV MOVES

Marriage Theorem

If a group of men and women may date only if they have previously been introduced, then a complete set of dates is possible IFF every subset of men has collectively been introduced to at least as many women, and vice versa.

References

Chartrand, G. *Introductory Graph Theory.* New York: Dover, p. 121, 1985.

Married Couples Problem

Also called the MÉNAGE PROBLEM. In how many ways can n married couples be seated around a circular table in such a manner than there is always one man between two women and none of the men is next to his own wife? The solution (Ball and Coxeter 1987, p. 50) uses DISCORDANT PERMUTATIONS and can be given in terms of LAISANT'S RECURRENCE FORMULA

$$(n-1)A_{n+1} = (n^2-1)A_n + (n+1)A_{n-1} + 4(-1)^n,$$

with $A_1 = A_2 = 1$. A closed form expression due to Touchard (1934) is

$$A_n = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!(-1)^k,$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT (Vardi 1991).

The first few values of A_n are -1, 1, 0, 2, 13, 80, 579, ... (Sloane's A000179), which are sometimes called MÉNAGE NUMBERS. The desired solution is then $2n!A_n$ The numbers A_n can be considered a special case of a restricted ROOKS PROBLEM.

see also DISCORDANT PERMUTATION, LAISANT'S RECURRENCE FORMULA, ROOKS PROBLEM

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 50, 1987.

Dörrie, H. §8 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 27–33, 1965.

Halmos, P. R.; Vaughan, H. E. "The Marriage Problem." *Amer. J. Math.* **72**, 214–215, 1950.

Lucas, E. *Théorie des Nombres.* Paris, pp. 215 and 491–495, 1891.

MacMahon, P. A. *Combinatory Analysis, Vol. 1.* London: Cambridge University Press, pp. 253–256, 1915.

Newman, D. J. "A Problem in Graph Theory." *Amer. Math. Monthly* **65**, 611, 1958.

Sloane, N. J. A. Sequence A000179/M2062 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Touchard, J. "Sur un probléme de permutations." *C. R. Acad. Sci. Paris* **198**, 631–633, 1934.

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, p. 123, 1991.

Marshall-Edgeworth Index

The statistical INDEX

$$P_{ME} \equiv \frac{\sum p_n(q_0+q_n)}{\sum (v_0+v_n)},$$

where p_n is the price per unit in period n, q_n is the quantity produced in period n, and $v_n \equiv p_n q_n$ is the value of the n units.

see also INDEX

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 66–67, 1962.

Martingale

A sequence of random variates such that the CONDITIONAL PROBABILITY of x_{n+1} given $x_1, x_2, \ldots, x_n$ is x_n. The term was first used to describe a type of wagering in which the bet is doubled or halved after a loss or win, respectively.

see also GAMBLER'S RUIN, SAINT PETERSBURG PARADOX

Mascheroni Constant

see EULER-MASCHERONI CONSTANT

Mascheroni Construction

A geometric construction done with a movable COMPASS alone. All constructions possible with a COMPASS and STRAIGHTEDGE are possible with a *movable* COMPASS alone, as was proved by Mascheroni (1797). Mascheroni's results are now known to have been anticipated largely by Mohr (1672).

see also COMPASS, GEOMETRIC CONSTRUCTION, NEUSIS CONSTRUCTION, STRAIGHTEDGE

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 96–97, 1987.

Bogomolny, A. "Geometric Constructions with the Compass Alone." `http://www.cut-the-knot.com/do_you_know/compass.html`.

Courant, R. and Robbins, H. "Constructions with Other Tools. Mascheroni Constructions with Compass Alone." §3.5 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 146–158, 1996.

Dörrie, H. "Mascheroni's Compass Problem." §33 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 160–164, 1965.

Gardner, M. "Mascheroni Constructions." Ch. 17 in *Mathematical Circus: More Puzzles, Games, Paradoxes and Other Mathematical Entertainments from Scientific American.* New York: Knopf, pp. 216–231, 1979.

Mascheroni, L. *Geometry of Compass.* Pavia, Italy, 1797.

Mohr, G. *Euclides Danicus.* Amsterdam, Netherlands, 1672.

Maschke's Theorem

If a MATRIX GROUP is reducible, then it is completely reducible, i.e., if the MATRIX GROUP is equivalent to the MATRIX GROUP in which every MATRIX has the reduced form

$$\begin{bmatrix} D_i^{(1)} & X_i \\ 0 & D_i^{(2)} \end{bmatrix},$$

then it is equivalent to the MATRIX GROUP obtained by putting $X_i = 0$.

see also MATRIX GROUP

References

Lomont, J. S. *Applications of Finite Groups.* New York: Dover, p. 49, 1987.

Mason's abc Theorem

see MASON'S THEOREM

Mason's Theorem

Let there be three POLYNOMIALS $a(x)$, $b(x)$, and $c(x)$ with no common factors such that

$$a(x) + b(x) = c(x).$$

Then the number of distinct ROOTS of the three POLYNOMIALS is one or more greater than their largest degree. The theorem was first proved by Stothers (1981).

Mason's theorem may be viewed as a very special case of a Wronskian estimate (Chudnovsky and Chudnovsky 1984). The corresponding Wronskian identity in the proof by Lang (1993) is

$$c^3 * W(a,b,c) = W(W(a,c), W(b,c)),$$

so if a, b, and c are linearly dependent, then so are $W(a,c)$ and $W(b,c)$. More powerful Wronskian estimates with applications toward diophantine approximation of solutions of linear differential equations may be found in Chudnovsky and Chudnovsky (1984) and Osgood (1985).

The rational function case of FERMAT'S LAST THEOREM follows trivially from Mason's theorem (Lang 1993, p. 195).

see also ABC CONJECTURE

References

Chudnovsky, D. V. and Chudnovsky, G. V. "The Wronskian Formalism for Linear Differential Equations and Padé Approximations." *Adv. Math.* **53**, 28–54, 1984.

Lang, S. "Old and New Conjectured Diophantine Inequalities." *Bull. Amer. Math. Soc.* **23**, 37–75, 1990.

Lang, S. *Algebra, 3rd ed.* Reading, MA: Addison-Wesley, 1993.

Osgood, C. F. "Sometimes Effective Thue-Siegel-Roth-Schmidt-Nevanlinna Bounds, or Better." *J. Number Th.* **21**, 347–389, 1985.

Stothers, W. W. "Polynomial Identities and Hauptmodulen." *Quart. J. Math. Oxford Ser. II* **32**, 349–370, 1981.

Masser-Gramain Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

If $f(z)$ is an ENTIRE FUNCTION such that $f(n)$ is an INTEGER for each POSITIVE INTEGER n. Pólya (1915) showed that if

$$\limsup_{r\to\infty} \frac{\ln M_r}{r} < \ln 2 = 0.693\ldots, \tag{1}$$

where

$$M_r = \sup_{|z|\leq r} |f(x)| \tag{2}$$

is the SUPREMUM, then f is a POLYNOMIAL. Furthermore, $\ln 2$ is the best constant (i.e., counterexamples exist for every smaller value).

If $f(z)$ is an ENTIRE FUNCTION with $f(n)$ a GAUSSIAN INTEGER for each GAUSSIAN INTEGER n, then Gelfond (1929) proved that there exists a constant α such that

$$\limsup_{r\to\infty} \frac{\ln M_r}{r^2} < \alpha \tag{3}$$

implies that f is a POLYNOMIAL. Gramain (1981, 1982) showed that the best such constant is

$$\alpha = \frac{\pi}{2e} = 0.578\ldots. \tag{4}$$

Maser (1980) proved the weaker result that f must be a POLYNOMIAL if

$$\limsup_{r\to\infty} \frac{\ln M_r}{r^2} < \alpha_0 = \tfrac{1}{2}\exp\left(-\delta + \frac{4c}{\pi}\right), \tag{5}$$

where

$$c = \gamma\beta(1) + \beta'(1) = 0.6462454398948114\ldots, \tag{6}$$

γ is the EULER-MASCHERONI CONSTANT, $\beta(z)$ is the DIRICHLET BETA FUNCTION,

$$\delta \equiv \lim_{n\to\infty}\left(\sum_{k=2}^{\infty} \frac{1}{\pi {r_k}^2} - \ln n\right), \tag{7}$$

and r_k is the minimum NONNEGATIVE r for which there exists a COMPLEX NUMBER z for which the CLOSED DISK with center z and radius r contains at least k distinct GAUSSIAN INTEGER. Gosper gave

$$c = \pi\{-\ln[\Gamma(\tfrac{1}{4})] + \tfrac{3}{4}\pi + \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\gamma\}. \tag{8}$$

Gramain and Weber (1985, 1987) have obtained

$$1.811447299 < \delta < 1.897327117, \tag{9}$$

which implies

$$0.1707339 < \alpha_0 < 0.1860446. \tag{10}$$

Gramain (1981, 1982) conjectured that

$$\alpha_0 = \frac{1}{2e}, \tag{11}$$

which would imply

$$\delta = 1 + \frac{4c}{\pi} = 1.822825249\ldots. \tag{12}$$

References
Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/masser/masser.html.
Gramain, F. "Sur le théorème de Fukagawa-Gel'fond." *Invent. Math.* **63**, 495–506, 1981.
Gramain, F. "Sur le théorème de Fukagawa-Gel'fond-Gruman-Masser." *Séminaire Delange-Pisot-Poitou (Théorie des Nombres), 1980–1981.* Boston, MA: Birkhäuser, 1982.
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Masser, D. W. "Sur les fonctions entières à valeurs entières." *C. R. Acad. Sci. Paris Sér. A-B* **291**, A1–A4, 1980.

Match Problem

Given n matches, find the number of topologically distinct planar arrangements $T(n)$ which can be made. The first few values are 1, 1, 3, 5, 10, 19, 39, ... (Sloane's A003055).

see also CIGARETTES, MATCHSTICK GRAPH

References
Gardner, M. "The Problem of the Six Matches." In *The Unexpected Hanging and Other Mathematical Diversions.* Chicago, IL: Chicago University Press, pp. 79–81, 1991.
Sloane, N. J. A. Sequence A003055/M2464 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Matchstick Graph

A PLANAR GRAPH whose EDGES are all unit line segments. The minimal number of EDGES for matchstick graphs of various degrees are given in the table below. The minimal degree 1 matchstick graph is a single EDGE, and the minimal degree 2 graph is an EQUILATERAL TRIANGLE.

n	e	v
1	1	2
2	3	3
3	12	8
4		≤ 42

Mathematical Induction

see INDUCTION

Mathematics

Mathematics is a broad-ranging field of study in which the properties and interactions of idealized objects are examined. Whereas mathematics began merely as a calculational tool for computation and tabulation of quantities, it has blossomed into an extremely rich and diverse set of tools, terminologies, and approaches which range from the purely abstract to the utilitarian.

Bertrand Russell once whimsically defined mathematics as, "The subject in which we never know what we are talking about nor whether what we are saying is true" (Bergamini 1969).

The term "mathematics" is often shortened to "math" in informal American speech and, consistent with the British penchant for adding superfluous letters, "maths" in British English.

see also METAMATHEMATICS

References
Bergamini, D. *Mathematics.* New York: Time-Life Books, p. 9, 1969.

Mathematics Prizes

Several prizes are awarded periodically for outstanding mathematical achievement. There is no Nobel Prize in mathematics, and the most prestigious mathematical award is known as the FIELDS MEDAL. In rough order of importance, other awards are the $100,000 Wolf Prize of the Wolf Foundation of Israel, the Leroy P. Steele Prize of the American Mathematical Society, followed by the Bôcher Memorial Prize, Frank Nelson Cole Prizes in Algebra and Number Theory, and the Delbert Ray Fulkerson Prize, all presented by the American Mathematical Society.

see also FIELDS MEDAL

References
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Monastyrsky, M. *Modern Mathematics in the Light of the Fields Medals.* Wellesley, MA: A. K. Peters, 1997.
"Wolf Prize Recipients in Mathematics." http://www.aquanet.co.il/wolf/wolf5.html.

Mathieu Differential Equation

$$\frac{d^2V}{dv^2} + [b - 2q\cos(2v)]V = 0.$$

It arises in separation of variables of LAPLACE'S EQUATION in ELLIPTIC CYLINDRICAL COORDINATES. Whittaker and Watson (1990) use a slightly different form to define the MATHIEU FUNCTIONS.

The modified Mathieu differential equation

$$\frac{d^2U}{du^2} - [b - 2q\cosh(2u)]U = 0$$

arises in SEPARATION OF VARIABLES of the HELMHOLTZ DIFFERENTIAL EQUATION in ELLIPTIC CYLINDRICAL COORDINATES.

see also MATHIEU FUNCTION

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 722, 1972.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 556–557, 1953.
Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Mathieu Function

The form given by Whittaker and Watson (1990, p. 405) defines the Mathieu function based on the equation

$$\frac{d^2u}{dz^2} + [a + 16q\cos(2z)]u = 0. \qquad (1)$$

This equation is closely related to HILL'S DIFFERENTIAL EQUATION. For an EVEN Mathieu function,

$$G(\eta) = \lambda \int_{-\pi}^{\pi} e^{k\cos\eta\cos\theta} G(\theta)\,d\theta, \qquad (2)$$

where $k \equiv \sqrt{32q}$. For an ODD Mathieu function,

$$G(\eta) = \lambda \int_{-\pi}^{\pi} \sin(k\sin\eta\sin\theta)G(\theta)\,d\theta. \qquad (3)$$

Both EVEN and ODD functions satisfy

$$G(\eta) = \lambda \int_{-\pi}^{\pi} e^{ik\sin\eta\sin\theta} G(\theta)\,d\theta. \qquad (4)$$

Letting $\zeta \equiv \cos^2 z$ transforms the MATHIEU DIFFERENTIAL EQUATION to

$$4\zeta(1-\zeta)\frac{d^2u}{d\zeta^2} + 2(1-2\zeta)\frac{du}{d\zeta} + (a - 16q + 32q\zeta)u = 0. \qquad (5)$$

see also MATHIEU DIFFERENTIAL EQUATION

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Mathieu Functions." Ch. 20 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 721–746, 1972.
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Mathieu Groups

The first SIMPLE SPORADIC GROUPS discovered. M_{11}, M_{12}, M_{22}, M_{23}, M_{24} were discovered in 1861 and 1873 by Mathieu. Frobenius showed that all the Mathieu groups are SUBGROUPS of M_{24}.

The Mathieu groups are most simply defined as AUTOMORPHISM groups of STEINER SYSTEMS. M_{11} corresponds to $S(4,5,11)$ and M_{23} corresponds to $S(4,7,23)$. M_{11} and M_{23} are TRANSITIVE PERMUTATION GROUPS of 11 and 23 elements.

The ORDERS of the Mathieu groups are

$$\begin{aligned}|M_{11}| &= 2^4 \cdot 3^2 \cdot 5 \cdot 11\\ |M_{12}| &= 2^6 \cdot 3^3 \cdot 5 \cdot 11\\ |M_{22}| &= 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11\\ |M_{23}| &= 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23.\end{aligned}$$

see also SPORADIC GROUP

References
Conway, J. H. and Sloane, N. J. A. "The Golay Codes and the Mathieu Groups." Ch. 11 in *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, pp. 299–330, 1993.
Rotman, J. J. Ch. 9 in *An Introduction to the Theory of Groups, 4th ed.* New York: Springer-Verlag, 1995.
Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/M11.html`, `M12.html`, `M22.html`, `M23.html`, and `M24.html`.

Matrix

The TRANSFORMATION given by the system of equations

$$\begin{aligned}x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n\\ &\vdots\\ x_m' &= a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n\end{aligned}$$

is denoted by the MATRIX EQUATION

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

In concise notation, this could be written

$$\mathbf{x}' = \mathsf{A}\mathbf{x},$$

where $\mathbf{x}'$ and $\mathbf{x}$ are VECTORS and A is called an $n \times m$ matrix. A matrix is said to be SQUARE if $m = n$. Special types of SQUARE MATRICES include the IDENTITY MATRIX I, with $a_{ij} = \delta_{ij}$ (where δ_{ij} is the KRONECKER DELTA) and the DIAGONAL MATRIX $a_{ij} = c_i\delta_{ij}$ (where c_i are a set of constants).

For every linear transformation there exists one and only one corresponding matrix. Conversely, every matrix corresponds to a unique linear transformation. The matrix is an important concept in mathematics, and was first formulated by Sylvester and Cayley.

Two matrices may be added (MATRIX ADDITION) or multiplied (MATRIX MULTIPLICATION) together to yield a new matrix. Other common operations on a single matrix are diagonalization, inversion (MATRIX INVERSE), and transposition (MATRIX TRANSPOSE). The DETERMINANT $\det(\mathsf{A})$ or $|\mathsf{A}|$ of a matrix A is an very important quantity which appears in many diverse applications. Matrices provide a concise notation which is extremely useful in a wide range of problems involving linear equations (e.g., LEAST SQUARES FITTING).

see also ADJACENCY MATRIX, ADJUGATE MATRIX, ANTISYMMETRIC MATRIX, BLOCK MATRIX, CARTAN MATRIX, CIRCULANT MATRIX, CONDITION NUMBER, CRAMER'S RULE, DETERMINANT, DIAGONAL MATRIX, DIRAC MATRICES, EIGENVECTOR, ELEMENTARY MATRIX, EQUIVALENT MATRIX, FOURIER MATRIX, GRAM MATRIX, HILBERT MATRIX, HYPERMATRIX, IDENTITY MATRIX, INCIDENCE MATRIX, IRREDUCIBLE MATRIX, KAC MATRIX, LU DECOMPOSITION, MARKOV MATRIX, MATRIX ADDITION, MATRIX DECOMPOSITION THEOREM, MATRIX INVERSE, MATRIX MULTIPLICATION, MCCOY'S THEOREM, MINIMAL MATRIX, NORMAL MATRIX, PAULI MATRICES, PERMUTATION MATRIX, POSITIVE DEFINITE MATRIX, RANDOM MATRIX, RATIONAL CANONICAL FORM, REDUCIBLE MATRIX, ROTH'S REMOVAL RULE, SHEAR MATRIX, SKEW SYMMETRIC MATRIX, SMITH NORMAL FORM, SPARSE MATRIX, SPECIAL MATRIX, SQUARE MATRIX, STOCHASTIC MATRIX, SUBMATRIX, SYMMETRIC MATRIX, TOURNAMENT MATRIX

References

Arfken, G. "Matrices." §4.2 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 176–191, 1985.

Matrix Addition

Denote the sum of two MATRICES A and B (of the same dimensions) by $\mathsf{C} = \mathsf{A}+\mathsf{B}$. The sum is defined by adding entries with the same indices

$$c_{ij} \equiv a_{ij} + b_{ij}$$

over all i and j. For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}.$$

Matrix addition is therefore both COMMUTATIVE and ASSOCIATIVE.

see also MATRIX, MATRIX MULTIPLICATION

Matrix Decomposition Theorem

Let P be a MATRIX of EIGENVECTORS of a given MATRIX A and D a MATRIX of the corresponding EIGENVALUES. Then A can be written

$$\mathsf{A} = \mathsf{PDP}^{-1}, \tag{1}$$

where D is a DIAGONAL MATRIX and the columns of P are ORTHOGONAL VECTORS. If P is not a SQUARE MATRIX, then it cannot have a MATRIX INVERSE. However, if P is $m \times n$ (with $m > n$), then A can be written

$$\mathsf{A} = \mathsf{UDV}^{\mathrm{T}}, \tag{2}$$

where U and V are $n \times n$ SQUARE MATRICES with ORTHOGONAL columns,

$$\mathsf{U}^{\mathrm{T}}\mathsf{U} = \mathsf{V}^{\mathrm{T}} = \mathsf{I}. \tag{3}$$

Matrix Diagonalization

Diagonalizing a MATRIX is equivalent to finding the EIGENVECTORS and EIGENVALUES. The EIGENVALUES make up the entries of the diagonalized MATRIX, and the EIGENVECTORS make up the new set of axes corresponding to the DIAGONAL MATRIX.

see also DIAGONAL MATRIX, EIGENVALUE, EIGENVECTOR

References

Arfken, G. "Diagonalization of Matrices." §4.6 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 217–229, 1985.

Matrix Direct Product

see DIRECT PRODUCT (MATRIX)

Matrix Equality

Two MATRICES A and B are said to be equal IFF

$$a_{ij} \equiv b_{ij}$$

for all i, j. Therefore,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

while

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}.$$

Matrix Equation

Nonhomogeneous matrix equations of the form

$$\mathsf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

can be solved by taking the MATRIX INVERSE to obtain

$$\mathbf{x} = \mathsf{A}^{-1}\mathbf{b}. \tag{2}$$

This equation will have a nontrivial solution IFF the DETERMINANT $\det(\mathsf{A}) \neq 0$. In general, more numerically stable techniques of solving the equation include GAUSSIAN ELIMINATION, LU DECOMPOSITION, or the SQUARE ROOT METHOD.

For a homogeneous $n \times n$ MATRIX equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

to be solved for the x_is, consider the DETERMINANT

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{4}$$

Now multiply by x_1, which is equivalent to multiplying the first row (or any row) by x_1,

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{5}$$

The value of the DETERMINANT is unchanged if multiples of columns are added to other columns. So add x_2 times column 2, ..., and x_n times column n to the first column to obtain

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \tag{6}$$

But from the original MATRIX, each of the entries in the first columns is zero since

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = 0, \tag{7}$$

so

$$\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0. \tag{8}$$

Therefore, if there is an $x_1 \neq 0$ which is a solution, the DETERMINANT is zero. This is also true for x_2, ..., x_n, so the original homogeneous system has a nontrivial solution for all x_is only if the DETERMINANT is 0. This approach is the basis for CRAMER'S RULE.

Given a numerical solution to a matrix equation, the solution can be iteratively improved using the following technique. Assume that the numerically obtained solution to

$$\mathsf{A}\mathbf{x} = \mathbf{b} \tag{9}$$

is $\mathbf{x}_1 = \mathbf{x} + \delta\mathbf{x}_1$, where $\delta\mathbf{x}_1$ is an error term. The first solution therefore gives

$$\mathsf{A}(\mathbf{x} + \delta\mathbf{x}_1) = \mathbf{b} + \delta\mathbf{b} \tag{10}$$

$$\mathsf{A}\delta\mathbf{x}_1 = \delta\mathbf{b}, \tag{11}$$

where $\delta\mathbf{b}$ is found by solving (10)

$$\delta\mathbf{b} = \mathsf{A}\mathbf{x}_1 - \mathbf{b}. \tag{12}$$

Combining (11) and (12) then gives

$$\delta\mathbf{x}_1 = \mathsf{A}^{-1}\delta\mathbf{b} = \mathsf{A}^{-1}(\mathsf{A}\mathbf{x}_1 - \mathbf{b}) = \mathbf{x}_1 - \mathsf{A}^{-1}\mathbf{b}, \tag{13}$$

so the next iteration to obtain $\mathbf{x}$ accurately should be

$$\mathbf{x}_2 = \mathbf{x}_1 - \delta\mathbf{x}_1. \tag{14}$$

see also CRAMER'S RULE, GAUSSIAN ELIMINATION, LU DECOMPOSITION, MATRIX, MATRIX ADDITION, MATRIX INVERSE, MATRIX MULTIPLICATION, NORMAL EQUATION, SQUARE ROOT METHOD

Matrix Exponential

Given a SQUARE MATRIX A, the matrix exponential is defined by

$$\exp(\mathsf{A}) \equiv e^{\mathsf{A}} = \sum_{n=0}^{\infty} \frac{\mathsf{A}^n}{n!} = \mathsf{I} + \mathsf{A} + \frac{\mathsf{AA}}{2!} + \frac{\mathsf{AAA}}{3!} + \ldots,$$

where I is the IDENTITY MATRIX.

see also EXPONENTIAL FUNCTION, MATRIX

Matrix Group

A GROUP in which the elements are SQUARE MATRICES, the group multiplication law is MATRIX MULTIPLICATION, and the group inverse is simply the MATRIX INVERSE. Every matrix group is equivalent to a unitary matrix group (Lomont 1987, pp. 47–48).

see also MASCHKE'S THEOREM

References

Lomont, J. S. "Matrix Groups." §3.1 in *Applications of Finite Groups.* New York: Dover, pp. 46–52, 1987.

Matrix Inverse

A MATRIX A has an inverse IFF the DETERMINANT $|\mathsf{A}| \neq 0$. For a 2×2 MATRIX

$$\mathsf{A} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1}$$

the inverse is

$$\mathsf{A}^{-1} = \frac{1}{|\mathsf{A}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{2}$$

For a 3×3 MATRIX,

$$\mathsf{A}^{-1} = \frac{1}{|\mathsf{A}|} \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}. \tag{3}$$

A general $n \times n$ matrix can be inverted using methods such as the GAUSS-JORDAN ELIMINATION, GAUSSIAN ELIMINATION, or LU DECOMPOSITION.

The inverse of a PRODUCT AB of MATRICES A and B can be expressed in terms of A^{-1} and B^{-1}. Let

$$\mathsf{C} \equiv \mathsf{AB}. \tag{4}$$

Then

$$\mathsf{B} = \mathsf{A}^{-1}\mathsf{AB} = \mathsf{A}^{-1}\mathsf{C} \tag{5}$$

and

$$\mathsf{A} = \mathsf{ABB}^{-1} = \mathsf{CB}^{-1}. \tag{6}$$

Therefore,

$$\mathsf{C} = \mathsf{AB} = (\mathsf{CB}^{-1})(\mathsf{A}^{-1}\mathsf{C}) = \mathsf{CB}^{-1}\mathsf{A}^{-1}\mathsf{C}, \tag{7}$$

so

$$\mathsf{CB}^{-1}\mathsf{A}^{-1} = \mathsf{I}, \tag{8}$$

where I is the IDENTITY MATRIX, and

$$\mathsf{B}^{-1}\mathsf{A}^{-1} = \mathsf{C}^{-1} = (\mathsf{AB})^{-1}. \tag{9}$$

see also MATRIX, MATRIX ADDITION, MATRIX MULTIPLICATION, MOORE-PENROSE GENERALIZED MATRIX INVERSE, STRASSEN FORMULAS

References

Ben-Israel, A. and Greville, T. N. E. *Generalized Inverses: Theory and Applications.* New York: Wiley, 1977.

Nash, J. C. *Compact Numerical Methods for Computers: Linear Algebra and Function Minimisation, 2nd ed.* Bristol, England: Adam Hilger, pp. 24–26, 1990.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Is Matrix Inversion an N^3 Process?" §2.11 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 95–98, 1992.

Matrix Multiplication

The product C of two MATRICES A and B is defined by

$$c_{ik} = a_{ij} b_{jk}, \tag{1}$$

where j is summed over for all possible values of i and k. Therefore, in order for multiplication to be defined, the dimensions of the MATRICES must satisfy

$$(n \times m)(m \times p) = (n \times p), \tag{2}$$

where $(a \times b)$ denotes a MATRIX with a rows and b columns. Writing out the product explicitly,

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}, \tag{3}$$

where

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1m}b_{m1} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + \ldots + a_{1m}b_{m2} \\ c_{1p} &= a_{11}b_{1p} + a_{12}b_{2p} + \ldots + a_{1m}b_{mp} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2m}b_{m1} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + \ldots + a_{2m}b_{m2} \\ c_{2p} &= a_{21}b_{1p} + a_{22}b_{2p} + \ldots + a_{2m}b_{mp} \\ c_{n1} &= a_{n1}b_{11} + a_{n2}b_{21} + \ldots + a_{nm}b_{m1} \\ c_{n2} &= a_{n1}b_{12} + a_{n2}b_{22} + \ldots + a_{nm}b_{m2} \\ c_{np} &= a_{n1}b_{1p} + a_{n2}b_{2p} + \ldots + a_{nm}b_{mp}. \end{aligned}$$

MATRIX MULTIPLICATION is ASSOCIATIVE, as can be seen by taking

$$[(ab)c]_{ij} = (ab)_{ik}c_{kj} = (a_{il}b_{lk})c_{kj}. \tag{4}$$

Now, since a_{il}, b_{lk}, and c_{kj} are SCALARS, use the ASSOCIATIVITY of SCALAR MULTIPLICATION to write

$$(a_{il}b_{lk})c_{kj} = a_{il}(b_{lk}c_{kj}) = a_{il}(bc)_{lj} = [a(bc)]_{ij}. \tag{5}$$

Since this is true for all i and j, it must be true that

$$[(ab)c]_{ij} = [a(bc)]_{ij}. \tag{6}$$

That is, MATRIX multiplication is ASSOCIATIVE. However, MATRIX MULTIPLICATION is *not* COMMUTATIVE unless A and B are DIAGONAL (and have the same dimensions).

The product of two BLOCK MATRICES is given by multiplying each block

$$\begin{bmatrix} o & o & & & & \\ o & o & & & & \\ & & o & & & \\ & & & o & o & o \\ & & & o & o & o \\ & & & o & o & o \end{bmatrix} \begin{bmatrix} x & x & & & & \\ x & x & & & & \\ & & x & & & \\ & & & x & x & x \\ & & & x & x & x \\ & & & x & x & x \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} o & o \\ o & o \end{bmatrix} \begin{bmatrix} x & x \\ x & x \end{bmatrix} & & \\ & [o][x] & \\ & & \begin{bmatrix} o & o & o \\ o & o & o \\ o & o & o \end{bmatrix} \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \end{bmatrix}. \quad (7)$$

see also MATRIX, MATRIX ADDITION, MATRIX INVERSE, STRASSEN FORMULAS

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 178–179, 1985.

Matrix Norm

Given a SQUARE MATRIX A with COMPLEX (or REAL) entries, a MATRIX NORM $||\mathsf{A}||$ is a NONNEGATIVE number associated with A having the properties

1. $||\mathsf{A}|| > 0$ when $\mathsf{A} \neq 0$ and $||\mathsf{A}|| = 0$ IFF $\mathsf{A} = 0$,
2. $||k\mathsf{A}|| = |k|\,||\mathsf{A}||$ for any SCALAR k,
3. $||\mathsf{A} + \mathsf{B}|| \leq ||\mathsf{A}|| + ||\mathsf{B}||$,
4. $||\mathsf{AB}|| \leq ||\mathsf{A}||\,||\mathsf{B}||$.

For an $n \times n$ MATRIX A and an $n \times n$ UNITARY MATRIX U,

$$||\mathsf{AU}|| = ||\mathsf{UA}|| = ||\mathsf{A}||.$$

Let $\lambda_1, \ldots, \lambda_n$ be the EIGENVALUES of A, then

$$\frac{1}{||\mathsf{A}^{-1}||} \leq |\lambda| \leq ||\mathsf{A}||.$$

The MAXIMUM ABSOLUTE COLUMN SUM NORM $||\mathsf{A}||_1$, SPECTRAL NORM $||\mathsf{A}||_2$, and MAXIMUM ABSOLUTE ROW SUM NORM $||\mathsf{A}||_\infty$ satisfy

$$(||\mathsf{A}||_2)^2 \leq ||\mathsf{A}||_1\,||\mathsf{A}||_\infty.$$

For a SQUARE MATRIX, the SPECTRAL NORM, which is the SQUARE ROOT of the maximum EIGENVALUE of $\mathsf{A}^\dagger\mathsf{A}$ (where $\mathsf{A}^\dagger$ is the ADJOINT MATRIX), is often referred to as "the" matrix norm.

see also COMPATIBLE, HILBERT-SCHMIDT NORM, MAXIMUM ABSOLUTE COLUMN SUM NORM, MAXIMUM ABSOLUTE ROW SUM NORM, NATURAL NORM, NORM, POLYNOMIAL NORM, SPECTRAL NORM, SPECTRAL RADIUS, VECTOR NORM

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1114–1125, 1979.

Matrix Polynomial Identity

see CAYLEY-HAMILTON THEOREM

Matrix Product

The result of a MATRIX MULTIPLICATION.

see also PRODUCT

Matrix Transpose

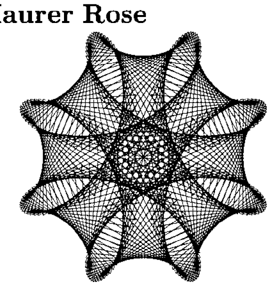
see TRANSPOSE

Matroid

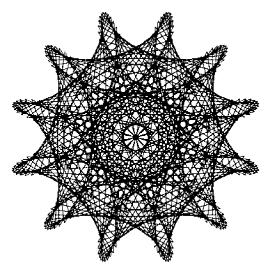
Roughly speaking, a finite set together with a generalization of a concept from linear algebra that satisfies a natural set of properties for that concept. For example, the finite set could be the rows of a MATRIX, and the generalizing concept could be linear dependence and independence of any subset of rows of the MATRIX. The number of matroids with n points are 1, 1, 2, 4, 9, 26, 101, 950, ... (Sloane's A002773).

References
Sloane, N. J. A. Sequences A002773/M1197 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Sloane, N. J. A. and Plouffe, S. Extended entry in *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.
Whitely, W. "Matroids and Rigid Structures." In Matroid Applications, *Encyclopedia of Mathematics and Its Applications* (Ed. N. White), Vol. 40. New York: Cambridge University Press, pp. 1–53, 1992.

Maurer Rose



$n = 4, d = 120$



$n = 6, d = 72$

A Maurer rose is a plot of a "walk" along an n- (or $2n$-) leafed ROSE in steps of a fixed number d degrees, including all cosets.

see also STARR ROSE

References
Maurer, P. "A Rose is a Rose..." *Amer. Math. Monthly* **94**, 631–645, 1987.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 96–102, 1991.

Max Sequence

A sequence defined from a FINITE sequence $a_0, a_1, \ldots, a_n$ by defining $a_{n+1} = \max_i(a_i + a_{n-i})$.

see also MEX SEQUENCE

References
Guy, R. K. "Max and Mex Sequences." §E27 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 227–228, 1994.

Maximal Ideal

A maximal ideal of a RING R is an IDEAL I, not equal to R, such that there are no IDEALS "in between" I and R. In other words, if J is an IDEAL which contains I as a SUBSET, then either $J = I$ or $J = R$. For example, $n\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ IFF n is PRIME, where $\mathbb{Z}$ is the RING of INTEGERS.

see also IDEAL, PRIME IDEAL, REGULAR LOCAL RING, RING

Maximal Sum-Free Set

A maximal sum-free set is a set $\{a_1, a_2, \ldots, a_n\}$ of distinct NATURAL NUMBERS such that a maximum l of them satisfy $a_{i_j} + a_{i_k} \neq a_m$, for $1 \leq j < k \leq l$, $1 \leq m \leq n$.

see also MAXIMAL ZERO-SUM-FREE SET

References

Guy, R. K. "Maximal Sum-Free Sets." §C14 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 128–129, 1994.

Maximal Zero-Sum-Free Set

A set having the largest number k of distinct residue classes modulo m so that no SUBSET has zero sum.

see also MAXIMAL SUM-FREE SET

References

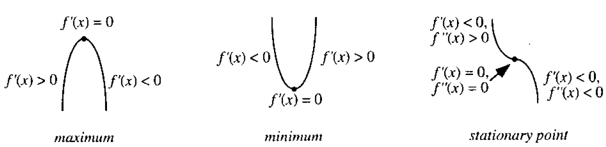
Guy, R. K. "Maximal Zero-Sum-Free Sets." §C15 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 129–131, 1994.

Maximally Linear Independent

A set of VECTORS is maximally linearly independent if including any other VECTOR in the VECTOR SPACE would make it LINEARLY DEPENDENT (i.e., if any other VECTOR in the SPACE can be expressed as a linear combination of elements of a maximal set—the BASIS).

Maximum

The largest value of a set, function, etc. The maximum value of a set of elements $A = \{a_i\}_{i=1}^N$ is denoted $\max A$ or $\max_i a_i$, and is equal to the last element of a sorted (i.e., ordered) version of A. For example, given the set $\{3, 5, 4, 1\}$, the sorted version is $\{1, 3, 4, 5\}$, so the maximum is 5. The maximum and MINIMUM are the simplest ORDER STATISTICS.



A continuous FUNCTION may assume a maximum at a single point or may have maxima at a number of points. A GLOBAL MAXIMUM of a FUNCTION is the largest value in the entire RANGE of the FUNCTION, and a LOCAL MAXIMUM is the largest value in some local neighborhood.

For a function $f(x)$ which is CONTINUOUS at a point x_0, a NECESSARY but not SUFFICIENT condition for $f(x)$ to have a RELATIVE MAXIMUM at $x = x_0$ is that x_0 be a CRITICAL POINT (i.e., $f(x)$ is either not DIFFERENTIABLE at x_0 or x_0 is a STATIONARY POINT, in which case $f'(x_0) = 0$).

The FIRST DERIVATIVE TEST can be applied to CONTINUOUS FUNCTIONS to distinguish maxima from MINIMA. For twice differentiable functions of one variable, $f(x)$, or of two variables, $f(x, y)$, the SECOND DERIVATIVE TEST can sometimes also identify the nature of an EXTREMUM. For a function $f(x)$, the EXTREMUM TEST succeeds under more general conditions than the SECOND DERIVATIVE TEST.

see also CRITICAL POINT, EXTREMUM, EXTREMUM TEST, FIRST DERIVATIVE TEST, GLOBAL MAXIMUM, INFLECTION POINT, LOCAL MAXIMUM, MIDRANGE, MINIMUM, ORDER STATISTIC, SADDLE POINT (FUNCTION), SECOND DERIVATIVE TEST, STATIONARY POINT

References

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Tikhomirov, V. M. *Stories About Maxima and Minima.* Providence, RI: Amer. Math. Soc., 1991.

Maximum Absolute Column Sum Norm

The NATURAL NORM induced by the L_1-NORM is called the maximum absolute column sum norm and is defined by

$$||\mathsf{A}||_1 = \max_j \sum_{i=1}^n |a_{ij}|$$

for a MATRIX A.

see also L_1-NORM, MAXIMUM ABSOLUTE ROW SUM NORM

Maximum Absolute Row Sum Norm

The NATURAL NORM induced by the L_∞-NORM is called the maximum absolute row sum norm and is defined by

$$||\mathsf{A}||_\infty = \max_i \sum_{j=1}^n |a_{ij}|$$

for a MATRIX A.

see also L_∞-NORM, MAXIMUM ABSOLUTE COLUMN SUM NORM

Maximum Clique Problem

see PARTY PROBLEM

Maximum Entropy Method

A DECONVOLUTION ALGORITHM (sometimes abbreviated MEM) which functions by minimizing a smoothness function ("ENTROPY") in an image. Maximum entropy is also called the ALL-POLES MODEL or AUTOREGRESSIVE MODEL. For images with more than a million pixels, maximum entropy is faster than the CLEAN ALGORITHM.

MEM is commonly employed in astronomical synthesis imaging. In this application, the resolution depends on the signal to NOISE ratio, which must be specified. Therefore, resolution is image dependent and varies across the map. MEM is also biased, since the ensemble average of the estimated noise is NONZERO. However, this bias is much smaller than the NOISE for pixels with a SNR $\gg 1$. It can yield super-resolution, which can usually be trusted to an order of magnitude in SOLID ANGLE.

Several definitions of "ENTROPY" normalized to the flux in the image are

$$H_1 \equiv \sum_k \ln\left(\frac{I_k}{M_k}\right) \tag{1}$$

$$H_2 \equiv -\sum_k I_k \ln\left(\frac{I_k}{M_k e}\right), \tag{2}$$

where M_k is a "default image" and I_k is the smoothed image. Some unnormalized entropy measures (Cornwell 1982, p. 3) are given by

$$H_1 \equiv -\sum f_i \ln(f_i) \tag{3}$$

$$H_2 \equiv \sum \ln(f_i) \tag{4}$$

$$H_3 \equiv -\sum \frac{1}{\ln(f_i)} \tag{5}$$

$$H_4 \equiv -\sum \frac{1}{[\ln(f_i)]^2} \tag{6}$$

$$H_5 \equiv \sum \sqrt{\ln(f_i)}\,. \tag{7}$$

see also CLEAN ALGORITHM, DECONVOLUTION, LUCY

References

Cornwell, T. J. "Can CLEAN be Improved?" VLA Scientific Memorandum No. 141, March 1982.

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Christiansen, W. N. and Högbom, J. A. *Radiotelescopes, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 217–218, 1985.

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Thompson, A. R.; Moran, J. M.; and Swenson, G. W. Jr. §3.2 in *Interferometry and Synthesis in Radio Astronomy.* New York: Wiley, pp. 349–352, 1986.

Maximum Likelihood

The procedure of finding the value of one or more parameters for a given statistic which makes the *known* LIKELIHOOD distribution a MAXIMUM. The maximum likelihood estimate for a parameter μ is denoted $\hat{\mu}$.

For a BERNOULLI DISTRIBUTION,

$$\frac{d}{d\theta}\left[\binom{N}{Np}\theta^{Np}(1-\theta)^{Nq}\right] = Np(1-\theta)-\theta Nq = 0, \tag{1}$$

so maximum likelihood occurs for $\theta = p$. If p is not known ahead of time, the likelihood function is

$$\begin{aligned} f(x_1,\ldots,x_n|p) &= P(X_1 = x_1,\ldots,X_n = x_n|p) \\ &= p^{x_1}(1-p)^{1-x_1}\cdots p^{x_n}(1-p)^{1-x_1 n} \\ &= p^{\Sigma x_i}(1-p)^{\Sigma(1-x_i)} = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}, \end{aligned} \tag{2}$$

where $x = 0$ or 1, and $i = 1, \ldots, n$.

$$\ln f = \sum x_i \ln p + \left(n - \sum x_i\right)\ln(1-p) \tag{3}$$

$$\frac{d(\ln f)}{dp} = \frac{\sum x_i}{p} - \frac{n-\sum x_i}{1-p} = 0 \tag{4}$$

$$\sum x_i - p\sum x_i = np - p\sum x_i \tag{5}$$

$$\hat{p} = \frac{\sum x_i}{n}. \tag{6}$$

For a GAUSSIAN DISTRIBUTION,

$$\begin{aligned} f(x_1,\ldots,x_n|\mu,\sigma) &= \prod \frac{1}{\sigma\sqrt{2\pi}}e^{-(x_i-\mu)^2/2\sigma^2} \\ &= \frac{(2\pi)^{-n/2}}{\sigma^n}\exp\left[-\frac{\sum(x_i-\mu)^2}{2\sigma^2}\right] \end{aligned} \tag{7}$$

$$\ln f = -\tfrac{1}{2}n\ln(2\pi) - n\ln\sigma - \frac{\sum(x_i-\mu)^2}{2\sigma^2} \tag{8}$$

$$\frac{\partial(\ln f)}{\partial\mu} = \frac{\sum(x_i-\mu)}{\sigma^2} = 0 \tag{9}$$

gives

$$\hat{\mu} = \frac{\sum x_i}{n}. \tag{10}$$

$$\frac{\partial(\ln f)}{\partial\sigma} = -\frac{n}{\sigma} + \frac{\sum(x_i-\mu)^2}{\sigma^3} \tag{11}$$

gives

$$\hat{\sigma} = \sqrt{\frac{\sum (x_i - \hat{\mu})^2}{n}}. \tag{12}$$

Note that in this case, the maximum likelihood STANDARD DEVIATION is the sample STANDARD DEVIATION, which is a BIASED ESTIMATOR for the population STANDARD DEVIATION.

For a weighted GAUSSIAN DISTRIBUTION,

$$\begin{aligned} f(x_1, \ldots, x_n | \mu, \sigma) &= \prod \frac{1}{\sigma_i \sqrt{2\pi}} e^{-(x_i - \mu)^2 / 2{\sigma_i}^2} \\ &= \frac{(2\pi)^{-n/2}}{\sigma^n} \exp\left[-\frac{\sum (x_i - \mu)^2}{2\sigma^2}\right] \end{aligned} \tag{13}$$

$$\ln f = -\tfrac{1}{2} n \ln(2\pi) - n \sum \ln \sigma_i - \sum \frac{(x_i - \mu)^2}{2{\sigma_i}^2} \tag{14}$$

$$\frac{\partial(\ln f)}{\partial \mu} = \sum \frac{(x_i - \mu)}{{\sigma_i}^2} = \sum \frac{x_i}{{\sigma_i}^2} - \mu \sum \frac{1}{{\sigma_i}^2} = 0 \tag{15}$$

gives

$$\hat{\mu} = \frac{\sum \frac{x_i}{{\sigma_i}^2}}{\sum \frac{1}{{\sigma_i}^2}}. \tag{16}$$

The VARIANCE of the MEAN is then

$${\sigma_\mu}^2 = \sum {\sigma_i}^2 \left(\frac{\partial \mu}{\partial x_i}\right)^2. \tag{17}$$

But

$$\frac{\partial \mu}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\sum (x_i / {\sigma_i}^2)}{\sum (1/{\sigma_i}^2)} = \frac{1/{\sigma_i}^2}{\sum (1/{\sigma_i}^2)}, \tag{18}$$

so

$$\begin{aligned} {\sigma_\mu}^2 &= \sum {\sigma_i}^2 \left(\frac{1/{\sigma_i}^2}{\sum (1/{\sigma_i}^2)}\right)^2 \\ &= \sum \frac{1/{\sigma_i}^2}{\left[\sum (1/{\sigma_i}^2)\right]^2} = \frac{1}{\sum (1/{\sigma_i}^2)}. \end{aligned} \tag{19}$$

For a POISSON DISTRIBUTION,

$$f(x_1, \ldots, x_n | \lambda) = \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \cdots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda} \lambda^{\sum x_i}}{x_1! \cdots x_n!} \tag{20}$$

$$\ln f = -n\lambda + (\ln \lambda) \sum x_i - \ln\left(\prod x_i!\right) \tag{21}$$

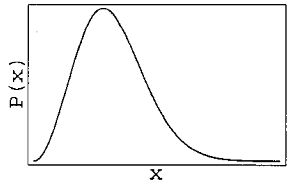
$$\frac{d(\ln f)}{\lambda} = -n + \frac{\sum x_i}{\lambda} = 0 \tag{22}$$

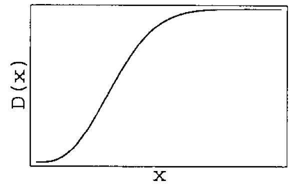
$$\hat{\lambda} = \frac{\sum x_i}{n}. \tag{23}$$

see also BAYESIAN ANALYSIS

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Least Squares as a Maximum Likelihood Estimator." §15.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 651–655, 1992.

Maxwell Distribution





The distribution of speeds of molecules in thermal equilibrium as given by statistical mechanics. The probability and cumulative distributions are

$$P(x) = \sqrt{\frac{2}{\pi}} a^{3/2} x^2 e^{-ax^2/2} \tag{1}$$

$$D(x) = \frac{2\gamma(\frac{3}{2}, \frac{1}{2}ax^2)}{\sqrt{\pi}}, \tag{2}$$

where $\gamma(a, x)$ is an incomplete GAMMA FUNCTION and $x \in [0, \infty)$. The moments are

$$\mu = 2\sqrt{\frac{2}{\pi a}} \tag{3}$$

$$\mu_2 = \frac{3}{a} \tag{4}$$

$$\mu_3 = 8\sqrt{\frac{2}{a^3 \pi}} \tag{5}$$

$$\mu_4 = \tfrac{15}{2}, \tag{6}$$

and the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = 2\sqrt{\frac{2}{\pi a}} \tag{7}$$

$$\sigma^2 = \frac{3\pi - 8}{\pi a} \tag{8}$$

$$\gamma_1 = \frac{8}{3}\sqrt{\frac{2}{3\pi}} \tag{9}$$

$$\gamma_2 = -\tfrac{4}{3}. \tag{10}$$

see also EXPONENTIAL DISTRIBUTION, GAUSSIAN DISTRIBUTION, RAYLEIGH DISTRIBUTION

References
Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, p. 119, 1992.
von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 252, 1993.

May's Theorem

Simple majority vote is the only procedure which is ANONYMOUS, DUAL, and MONOTONIC.

References
May, K. "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision." *Econometrica* **20**, 680–684, 1952.

May-Thomason Uniqueness Theorem

For every infinite LOOP SPACE MACHINE E, there is a natural equivalence of spectra between EX and Segal's spectrum $\mathbf{B}X$.

References

May, J. P. and Thomason, R. W. "The Uniqueness of Infinite Loop Space Machines." *Topology* **17**, 205–224, 1978.

Weibel, C. A. "The Mathematical Enterprises of Robert Thomason." *Bull. Amer. Math. Soc.* **34**, 1–13, 1996.

Maze

A maze is a drawing of impenetrable line segments (or curves) with "paths" between them. The goal of the maze is to start at one given point and find a path which reaches a second given point.

References

Gardner, M. "Mazes." Ch. 10 in *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 112–118, 1961.

Jablan, S. "Roman Mazes." `http://members.tripod.com/~modularity/mazes.htm`.

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Pappas, T. "Mazes." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 192–194, 1989.

Phillips, A. "The Topology of Roman Mazes." *Leonardo* **25**, 321–329, 1992.

Shepard, W. *Mazes and Labyrinths: A Book of Puzzles.* New York: Dover, 1961.

Mazur's Theorem

The generalization of the SCHÖNFLIES THEOREM to n-D. A smoothly embedded n-HYPERSPHERE in an $(n+1)$-HYPERSPHERE separates the $(n+1)$-HYPERSPHERE into two components, each HOMEOMORPHIC to $(n+1)$-BALLS. It can be proved using MORSE THEORY.

see also BALL, HYPERSPHERE

McCay Circle

If the VERTEX A_1 of a TRIANGLE describes the NEUBERG CIRCLE n_1, its MEDIAN POINT describes a circle whose radius is 1/3 that of the NEUBERG CIRCLE. Such a CIRCLE is known as a McKay circle, and the three McCay circles are CONCURRENT at the MEDIAN POINT M. Three homologous collinear points lie on the McCay circles.

see also CIRCLE, CONCURRENT, MEDIAN POINT, NEUBERG CIRCLES

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 290 and 306, 1929.

McCoy's Theorem

If two SQUARE $n \times n$ MATRICES A and B are simultaneously upper triangularizable by similarity transforms, then there is an ordering $a_1, \ldots, a_n$ of the EIGENVALUES of A and $b_1, \ldots, b_n$ of the EIGENVALUES of B so that, given any POLYNOMIAL $p(x,y)$ in noncommuting variables, the EIGENVALUES of $p(A,B)$ are the numbers $p(a_i,b_i)$ with $i = 1, \ldots, n$. McCoy's theorem states the converse: If every POLYNOMIAL exhibits the correct EIGENVALUES in a consistent ordering, then A and B are simultaneously triangularizable.

References

Luchins, E. H. and McLoughlin, M. A. "In Memoriam: Olga Taussky-Todd." *Not. Amer. Math. Soc.* **43**, 838–847, 1996.

McLaughlin Group

The SPORADIC GROUP *McL*.

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/McL.html`.

McMohan's Theorem

Consider a GAUSSIAN BIVARIATE DISTRIBUTION. Let $f(x_1, x_2)$ be an arbitrary FUNCTION. Then

$$\frac{\partial^2 \langle f \rangle}{\partial \rho^n} = \left\langle \frac{\partial^{2n} f}{\partial {x_1}^n \partial {x_2}^n} \right\rangle.$$

see also GAUSSIAN BIVARIATE DISTRIBUTION

McNugget Number

A number which can be obtained from an order of McDonald's® Chicken McNuggets™ (prior to consuming any), which originally came in boxes of 6, 9, and 20. All integers are McNugget numbers except 1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 37, and 43. Since the Happy Meal™-sized nugget box (4 to a box) can now be purchased separately, the modern McNugget numbers are a linear combination of 4, 6, 9, and 20. These new-fangled numbers are much less interesting than before, with only 1, 2, 3, 5, 7, and 11 remaining as non-McNugget numbers.

The GREEDY ALGORITHM can be used to find a McNugget expansion of a given INTEGER.

see also COMPLETE SEQUENCE, GREEDY ALGORITHM

References

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 19–20 and 233–234, 1991.

Wilson, D. `rec.puzzles` newsgroup posting, March 20, 1990.

Mean

A mean is HOMOGENEOUS and has the property that a mean μ of a set of numbers x_i satisfies

$$\min(x_1, \ldots, x_n) \leq \mu \leq \max(x_1, \ldots, x_n).$$

There are several statistical quantities called means, e.g., ARITHMETIC-GEOMETRIC MEAN, GEOMETRIC MEAN, HARMONIC MEAN, QUADRATIC MEAN, ROOT-MEAN-SQUARE. However, the quantity referred to as "the" mean is the ARITHMETIC MEAN, also called the AVERAGE.

see also ARITHMETIC-GEOMETRIC MEAN, AVERAGE, GENERALIZED MEAN, GEOMETRIC MEAN, HARMONIC MEAN, QUADRATIC MEAN, ROOT-MEAN-SQUARE

Mean Cluster Count Per Site

see s-CLUSTER

Mean Cluster Density

see s-CLUSTER

Mean Curvature

Let κ_1 and κ_2 be the PRINCIPAL CURVATURES, then their MEAN

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2) \tag{1}$$

is called the mean curvature. Let R_1 and R_2 be the radii corresponding to the PRINCIPAL CURVATURES, then the multiplicative inverse of the mean curvature H is given by the multiplicative inverse of the HARMONIC MEAN,

$$H \equiv \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{R_1 + R_2}{2R_1R_2}. \tag{2}$$

In terms of the GAUSSIAN CURVATURE K,

$$H = \tfrac{1}{2}(R_1 + R_2)K. \tag{3}$$

The mean curvature of a REGULAR SURFACE in $\mathbb{R}^3$ at a point $\mathbf{p}$ is formally defined as

$$H(\mathbf{p}) = \tfrac{1}{2}\operatorname{tr}(S(\mathrm{p})), \tag{4}$$

where S is the SHAPE OPERATOR and $\operatorname{tr}(S)$ denotes the TRACE. For a MONGE PATCH with $z = h(x,y)$,

$$H = \frac{(1 + {h_v}^2)h_{uu} - 2h_u h_v h_{uv} + (1 + {h_u}^2)h_{vv}}{(1 + {h_u}^2 + {h_v}^2)^{3/2}} \tag{5}$$

(Gray 1993, p. 307).

If $\mathbf{x} : U \to \mathbb{R}^3$ is a REGULAR PATCH, then the mean curvature is given by

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}, \tag{6}$$

where E, F, and G are coefficients of the first FUNDAMENTAL FORM and e, f, and g are coefficients of the second FUNDAMENTAL FORM (Gray 1993, p. 282). It can also be written

$$H = \frac{\det(\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v)|\mathbf{x}_v|^2 - 2\det(\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v)(\mathbf{x}_u \cdot \mathbf{x}_v)}{2[|\mathbf{x}_u|^2|\mathbf{x}_v|^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2]^{3/2}} + \frac{\det(\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v)|\mathbf{x}_u|^2}{2[|\mathbf{x}_u|^2|\mathbf{x}_v|^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2]^{3/2}} \tag{7}$$

Gray (1993, p. 285).

The GAUSSIAN and mean curvature satisfy

$$H^2 \geq K, \tag{8}$$

with equality only at UMBILIC POINTS, since

$$H^2 - K^2 = \tfrac{1}{4}(\kappa_1 - \kappa_2)^2. \tag{9}$$

If $\mathbf{p}$ is a point on a REGULAR SURFACE $M \subset \mathbb{R}^3$ and $\mathbf{v_p}$ and $\mathbf{w_p}$ are tangent vectors to M at $\mathbf{p}$, then the mean curvature of M at $\mathbf{p}$ is related to the SHAPE OPERATOR S by

$$S(\mathbf{v_p}) \times \mathbf{w_p} + \mathbf{v_p} \times S(\mathbf{w_p}) = 2H(\mathbf{p})\mathbf{v_p} \times \mathbf{w_p}. \tag{10}$$

Let $\mathbf{Z}$ be a nonvanishing VECTOR FIELD on M which is everywhere PERPENDICULAR to M, and let V and W be VECTOR FIELDS tangent to M such that $V \times W = \mathbf{Z}$, then

$$H = -\frac{\mathbf{Z} \cdot (D_V\mathbf{Z} \times W + V \times D_W\mathbf{Z})}{2|\mathbf{Z}|^3} \tag{11}$$

(Gray 1993, pp. 291–292).

Wente (1985, 1986, 1987) found a nonspherical finite surface with constant mean curvature, consisting of a self-intersecting three-lobed toroidal surface. A family of such surfaces exists.

see also GAUSSIAN CURVATURE, PRINCIPAL CURVATURES, SHAPE OPERATOR

References

Gray, A. "The Gaussian and Mean Curvatures." §14.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 279–285, 1993.

Isenberg, C. *The Science of Soap Films and Soap Bubbles.* New York: Dover, p. 108, 1992.

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Wente, H. C. "A Counterexample in 3-Space to a Conjecture of H. Hopf." In *Workshop Bonn 1984, Proceedings of the 25th Mathematical Workshop Held at the Max-Planck Institut für Mathematik, Bonn, June 15–22, 1984* (Ed. F. Hirzebruch, J. Schwermer, and S. Suter). New York: Springer-Verlag, pp. 421–429, 1985.

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Mean Deviation

The MEAN of the ABSOLUTE DEVIATIONS,

$$\mathrm{MD} \equiv \frac{1}{N}\sum_{i=1}^{N}|x_i - \bar{x}|,$$

where $\bar{x}$ is the MEAN of the distribution.

see also ABSOLUTE DEVIATION

Mean Distribution

For an infinite population with MEAN μ, STANDARD DEVIATION σ^2, SKEWNESS γ_1, and KURTOSIS γ_2, the corresponding quantities for the *distribution of means* are

$$\mu_{\bar{x}} = \mu \tag{1}$$

$$\sigma_{\bar{x}}{}^2 = \frac{\sigma^2}{N} \tag{2}$$

$$\gamma_{1,\bar{x}} = \frac{\gamma_1}{\sqrt{N}} \tag{3}$$

$$\gamma_{2,\bar{x}} = \frac{\gamma_2}{N}. \tag{4}$$

For a population of M (Kenney and Keeping 1962, p. 181),

$$\mu_{\bar{x}}^{(M)} = \mu \tag{5}$$

$$\sigma^{2(M)} = \frac{\sigma^2}{N}\frac{M-N}{M-1}. \tag{6}$$

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, 1962.

Mean Run Count Per Site

see s-RUN

Mean Run Density

see s-RUN

Mean Square Error

see ROOT-MEAN-SQUARE

Mean-Value Theorem

Let $f(x)$ be DIFFERENTIABLE on the OPEN INTERVAL (a, b) and CONTINUOUS on the CLOSED INTERVAL $[a, b]$. Then there is at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

see also EXTENDED MEAN-VALUE THEOREM, GAUSS'S MEAN-VALUE THEOREM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1097–1098, 1993.

Measurable Function

A function $f : X \to Y$ for which the pre-image of every measurable set in Y is measurable in X. For a BOREL MEASURE, all continuous functions are measurable.

Measurable Set

If F is a SIGMA ALGEBRA and A is a SUBSET of X, then A is called measurable if A is a member of F. X need not have, a priori, a topological structure. Even if it does, there may be no connection between the open sets in the topology and the given SIGMA ALGEBRA.

see also MEASURABLE SPACE, SIGMA ALGEBRA

Measurable Space

A SET considered together with the SIGMA ALGEBRA on the SET.

see also MEASURABLE SET, MEASURE SPACE, SIGMA ALGEBRA

Measure

The terms "measure," "measurable," etc., have very precise technical definitions (usually involving SIGMA ALGEBRAS) which makes them a little difficult to understand. However, the technical nature of the definitions is extremely important, since it gives a firm footing to concepts which are the basis for much of ANALYSIS (including some of the slippery underpinnings of CALCULUS).

For example, every definition of an INTEGRAL is based on a particular measure: the RIEMANN INTEGRAL is based on JORDAN MEASURE, and the LEBESGUE INTEGRAL is based on LEBESGUE MEASURE. The study of measures and their application to INTEGRATION is known as MEASURE THEORY.

A measure is formally defined as a MAP $m : F \to \mathbb{R}$ (the reals) such that $m(\varnothing) = 0$ and, if A_n is a COUNTABLE SEQUENCE in F and the A_n are pairwise DISJOINT, then

$$m\left(\bigcup_n A_n\right) = \sum_n m(A_n).$$

If, in addition, $m(X) = 1$, then m is said to be a PROBABILITY MEASURE.

A measure m may also be defined on SETS other than those in the SIGMA ALGEBRA F. By adding to F all sets to which m assigns measure zero, we again obtain a SIGMA ALGEBRA and call this the "completion" of F with respect to m. Thus, the completion of a SIGMA ALGEBRA is the smallest SIGMA ALGEBRA containing F and all sets of measure zero.

see also ALMOST EVERYWHERE, BOREL MEASURE, ERGODIC MEASURE, EULER MEASURE, GAUSS MEASURE, HAAR MEASURE, HAUSDORFF MEASURE, HELSON-SZEGŐ MEASURE, INTEGRAL, JORDAN MEASURE, LEBESGUE MEASURE, LIOUVILLE MEASURE, MAHLER'S

MEASURE, MEASURABLE SPACE, MEASURE ALGEBRA, MEASURE SPACE, MINKOWSKI MEASURE, NATURAL MEASURE, PROBABILITY MEASURE, WIENER MEASURE

Measure Algebra

A Boolean SIGMA ALGEBRA which possesses a MEASURE.

Measure Polytope

see HYPERCUBE

Measure-Preserving Transformation

see ENDOMORPHISM

Measure Space

A measure space is a MEASURABLE SPACE possessing a NONNEGATIVE MEASURE. Examples of measure spaces include n-D EUCLIDEAN SPACE with LEBESGUE MEASURE and the unit interval with LEBESGUE MEASURE (i.e., probability).

see also LEBESGUE MEASURE, MEASURABLE SPACE

Measure Theory

The mathematical theory of how to perform INTEGRATION in arbitrary MEASURE SPACES.

see also CANTOR SET, FRACTAL, INTEGRAL, MEASURABLE FUNCTION, MEASURABLE SET, MEASURABLE SPACE, MEASURE, MEASURE SPACE

References

Doob, J. L. *Measure Theory.* New York: Springer-Verlag, 1994.

Evans, L. C. and Gariepy, R. F. *Measure Theory and Finite Properties of Functions.* Boca Raton, FL: CRC Press, 1992.

Gordon, R. A. *The Integrals of Lebesgue, Denjoy, Perron, and Henstock.* Providence, RI: Amer. Math. Soc., 1994.

Halmos, P. R. *Measure Theory.* New York: Springer-Verlag, 1974.

Henstock, R. *The General Theory of Integration.* Oxford, England: Clarendon Press, 1991.

Kestelman, H. *Modern Theories of Integration, 2nd rev. ed.* New York: Dover, 1960.

Rao, M. M. *Measure Theory And Integration.* New York: Wiley, 1987.

Strook, D. W. *A Concise Introduction to the Theory of Integration, 2nd ed.* Boston, MA: Birkhäuser, 1994.

Mechanical Quadrature

see GAUSSIAN QUADRATURE

Mecon

Buckminster Fuller's term for the TRUNCATED OCTAHEDRON.

see also DYMAXION

Medial Axis

The boundaries of the cells of a VORONOI DIAGRAM.

Medial Deltoidal Hexecontahedron

The DUAL of the RHOMBIDODECADODECAHEDRON.

Medial Disdyakis Triacontahedron

The DUAL of the TRUNCATED DODECADODECAHEDRON.

Medial Hexagonal Hexecontahedron

The DUAL of the SNUB ICOSIDODECADODECAHEDRON.

Medial Icosacronic Hexecontahedron

The DUAL of the ICOSIDODECADODECAHEDRON.

Medial Inverted Pentagonal Hexecontahedron

The DUAL of the INVERTED SNUB DODECADODECAHEDRON.

Medial Pentagonal Hexecontahedron

The DUAL of the SNUB DODECADODECAHEDRON.

Medial Rhombic Triacontahedron

A ZONOHEDRON which is the DUAL of the DODECADODECAHEDRON. It is also called the SMALL STELLATED TRIACONTAHEDRON.

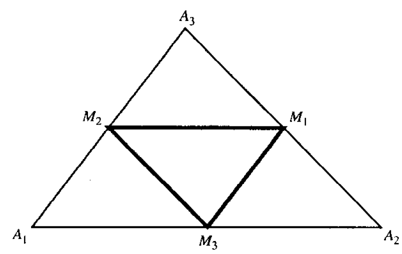
References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 125, 1989.

Medial Triambic Icosahedron

The DUAL of the DITRIGONAL DODECADODECAHEDRON.

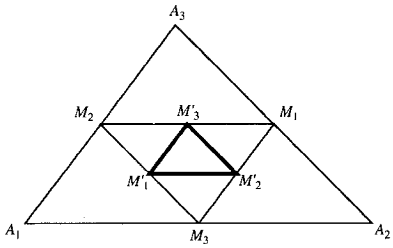
Medial Triangle



The TRIANGLE $\Delta M_1M_2M_3$ formed by joining the MIDPOINTS of the sides of a TRIANGLE $\Delta A_1A_2A_3$. The medial triangle is sometimes also called the AUXILIARY TRIANGLE (Dixon 1991). The medial triangle has TRILINEAR COORDINATES

$$\begin{aligned} A' &= 0 : b^{-1} : c^{-1} \\ B' &= a^{-1} : 0 : c^{-1} \\ C' &= a^{-1} : b^{-1} : 0. \end{aligned}$$

The medial triangle $\Delta M_1'M_2'M_3'$ of the medial triangle $\Delta M_1M_2M_3$ of a TRIANGLE $\Delta A_1A_2A_3$ is similar to $\Delta A_1A_2A_3$.



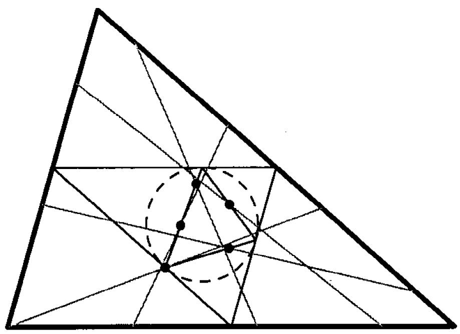
see also ANTICOMPLEMENTARY TRIANGLE

References

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 18–20, 1967.

Dixon, R. *Mathographics.* New York: Dover, p. 56, 1991.

Medial Triangle Locus Theorem



Given an original triangle (thick line), find the MEDIAL TRIANGLE (outer thin line) and its INCIRCLE. Take the PEDAL TRIANGLE (inner thin line) of the MEDIAL TRIANGLE with the INCENTER as the PEDAL POINT. Now pick any point on the original triangle, and connect it to the point located a half-PERIMETER away (gray lines). Then the locus of the MIDPOINTS of these lines (the •s in the above diagram) is the PEDAL TRIANGLE.

References

Honsberger, R. *More Mathematical Morsels.* Washington, DC: Math. Assoc. Amer., pp. 261–267, 1991.

Tsintsifas, G. "Problem 674." *Crux Math.,* p. 256, 1982.

Median Point

see CENTROID (GEOMETRIC)

Median (Statistics)

The middle value of a distribution or average of the two middle items, denoted $\mu_{1/2}$ or $\tilde{x}$. For small samples, the MEAN is more efficient than the median and approximately $\pi/2$ less. It is less sensitive to outliers than the MEAN (Kenney and Keeping 1962, p. 211).

For large N samples with population median $\tilde{x}_0$,

$$\mu_{\tilde{x}} = \tilde{x}_0$$

$$\sigma_{\tilde{x}}{}^2 = \frac{1}{8Nf^2(\tilde{x}_0)}.$$

The median is an L-ESTIMATE (Press *et al.* 1992).

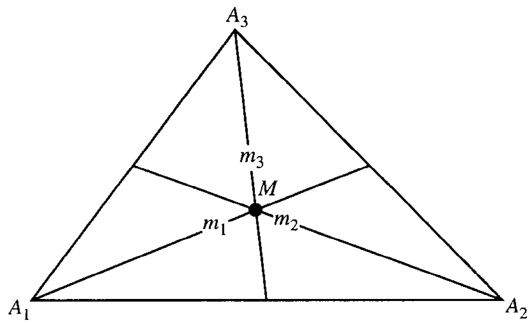
see also MEAN, MIDRANGE, MODE

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, 1962.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 694, 1992.

Median (Triangle)



The CEVIAN from a TRIANGLE'S VERTEX to the MIDPOINT of the opposite side is called a median of the TRIANGLE. The three medians of any TRIANGLE are CONCURRENT, meeting in the TRIANGLE'S CENTROID (which has TRILINEAR COORDINATES $1/a : 1/b : 1/c$). In addition, the medians of a TRIANGLE divide one another in the ratio 2:1. A median also bisects the AREA of a TRIANGLE.

Let m_i denote the length of the median of the ith side a_i. Then

$$m_1{}^2 = \tfrac{1}{4}(2a_2{}^2 + 2a_3{}^2 - a_1{}^2) \qquad (1)$$

$$m_1{}^2 + m_2{}^2 + m_3{}^2 = \tfrac{3}{4}(a_1{}^2 + a_2{}^2 + a_3{}^2) \qquad (2)$$

(Johnson 1929, p. 68). The AREA of a TRIANGLE can be expressed in terms of the medians by

$$A = \tfrac{4}{3}\sqrt{s_m(s_m - m_1)(s_m - m_2)(s_m - m_3)}, \qquad (3)$$

where

$$s_m \equiv \tfrac{1}{2}(m_1 + m_2 + m_3). \qquad (4)$$

see also BIMEDIAN, EXMEDIAN, EXMEDIAN POINT, HERONIAN TRIANGLE, MEDIAL TRIANGLE

References

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 7–8, 1967.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 68 and 173–175, 1929.

Median Triangle

A TRIANGLE whose sides are equal and PARALLEL to the MEDIANS of a given TRIANGLE. The median triangle of the median triangle is similar to the given TRIANGLE in the ratio 3/4.

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 282–283, 1929.

Mediant

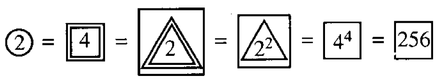
Given a FAREY SEQUENCE with consecutive terms h/k and h'/k', then the mediant is defined as the reduced form of the fraction $(h+h')/(k+k')$.

see also FAREY SEQUENCE

References
Conway, J. H. and Guy, R. K. "Farey Fractions and Ford Circles." *The Book of Numbers.* New York: Springer-Verlag, pp. 152–154, 1996.

Mega

Defined in terms of CIRCLE NOTATION by Steinhaus (1983, pp. 28–29) as



where STEINHAUS-MOSER NOTATION has also been used.

see also MEGISTRON, MOSER, STEINHAUS-MOSER NOTATION

References
Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, 1983.

Megistron

A very LARGE NUMBER defined in terms of CIRCLE NOTATION by Steinhaus (1983) as 10.

see also MEGA, MOSER

References
Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, pp. 28–29, 1983.

Mehler's Bessel Function Formula

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \sin(x\cosh t)\,dt,$$

where $J_0(x)$ is a zeroth order BESSEL FUNCTION OF THE FIRST KIND.

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1472, 1980.

Mehler-Dirichlet Integral

$$P_n(\cos\alpha) = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{\cos[(n+\frac{1}{2})\phi]}{\sqrt{\cos\phi-\cos\alpha}}\,d\phi,$$

where $P_n(x)$ is a LEGENDRE POLYNOMIAL.

Mehler's Hermite Polynomial Formula

$$\sum_{n=0}^\infty \frac{H_n(x)H_n(y)}{n!}(\tfrac{1}{2}w)^n = (1+4w^2)^{-1/2}\exp\left[\frac{2xyw-(x^2+y^2)w^2}{1-w^2}\right],$$

where $H_n(x)$ is a HERMITE POLYNOMIAL.

References
Almqvist, G. and Zeilberger, D. "The Method of Differentiating Under the Integral Sign." *J. Symb. Comput.* **10**, 571–591, 1990.
Foata, D. "A Combinatorial Proof of the Mehler Formula." *J. Comb. Th. Ser. A* **24**, 250–259, 1978.
Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, pp. 194–195, 1996.
Rainville, E. D. *Special Functions.* New York: Chelsea, p. 198, 1971.
Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 380, 1975.

Mehler Quadrature

see JACOBI-GAUSS QUADRATURE

Meijer's *G*-Function

$$G_{p,q}^{m,n}\left(x\middle|{}_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right) \equiv \frac{1}{2\pi i} \times \int_{\gamma_L} \frac{\prod_{j=1}^m \Gamma(b_j-z)\prod_{j=1}^n(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+z)\prod_{j=n+1}^q \Gamma(q_j-z)} x^z\,dz,$$

where $\Gamma(z)$ is the GAMMA FUNCTION. The CONTOUR γ_L and other details are discussed by Gradshteyn and Ryzhik (1980, pp. 896–903 and 1068–1071). Prudnikov *et al.* (1990) contains an extensive nearly 200-page listing of formulas for the Meijer G-function.

see also FOX'S H-FUNCTION, G-FUNCTION, MACROBERT'S E-FUNCTION, RAMANUJAN g- AND G-FUNCTIONS

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, 1979.
Luke, Y. L. *The Special Functions and Their Approximations, 2 vols.* New York: Academic Press, 1969.
Mathai, A. M. *A Handbook of Generalized Special Functions for Statistical and Physical Sciences.* New York: Oxford University Press, 1993.
Prudnikov, A. P.; Marichev, O. I.; and Brychkov, Yu. A.; *Integrals and Series, Vol. 3: More Special Functions.* Newark, NJ: Gordon and Breach, 1990.

Meissel's Formula

A modification of LEGENDRE'S FORMULA for the PRIME COUNTING FUNCTION $\pi(x)$. It starts with

$$\lfloor x \rfloor = 1 + \sum_{1\le i\le a} \left\lfloor \frac{x}{p_i} \right\rfloor - \sum_{1\le i<j\le a} \left\lfloor \frac{x}{p_i p_j} \right\rfloor + \sum_{1\le i<j<k\le a} \left\lfloor \frac{x}{p_i p_j p_k} \right\rfloor - \ldots + \pi(x) - a + P_2(x,a) + P_3(x,a) + \ldots, \quad (1)$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, $P_2(x,a)$ is the number of INTEGERS $p_i p_j \le x$ with $a+1 \le j \le j$, and $P_3(x,a)$ is the number of INTEGERS $p_i p_j p_k \le x$ with $a+1 \le i \le j \le k$. Identities satisfied by the Ps include

$$P_2(x,a) = \sum \left[\pi\left(\frac{x}{p_i}\right) - (i-1)\right] \quad (2)$$

for $p_a < p_i \le \sqrt{x}$ and

$$P_3(x,a) = \sum_{i>a} P_2\left(\frac{x}{p_i}, a\right) = \sum_{i=a+1}^{c} \sum_{j=i}^{\pi(\sqrt{x/p_i})} \left[\pi\left(\frac{x}{p_i p_j}\right) - (j-1)\right]. \quad (3)$$

Meissel's formula is

$$\pi(x) = \lfloor x \rfloor - \sum_{i=1}^{c} \left\lfloor \frac{x}{p_i} \right\rfloor + \sum_{1\le i\le j\le c} \left\lfloor \frac{x}{p_i p_j} \right\rfloor - \ldots + \tfrac{1}{2}(b+c-2)(b-c+1) - \sum_{c\le i\le b} \pi\left(\frac{x}{p_i}\right), \quad (4)$$

where

$$b \equiv \pi(x^{1/2}) \quad (5)$$

$$c \equiv \pi(x^{1/3}). \quad (6)$$

Taking the derivation one step further yields LEHMER'S FORMULA.

see also LEGENDRE'S FORMULA, LEHMER'S FORMULA, PRIME COUNTING FUNCTION

References

Riesel, H. "Meissel's Formula." *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, p. 12, 1994.

Mellin Transform

$$\phi(z) = \int_0^\infty t^{z-1} f(t)\, dt$$

$$f(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} t^{-z} \phi(z)\, dz.$$

see also STRASSEN FORMULAS

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 795, 1985.

Bracewell, R. *The Fourier Transform and Its Applications.* New York: McGraw-Hill, pp. 254–257, 1965.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 469–471, 1953.

Melnikov-Arnold Integral

$$A_m(\lambda) \equiv \int_{-\infty}^{\infty} \cos\left[\tfrac{1}{2} m\phi(t) - \lambda t\right] dt,$$

where the function

$$\phi(t) \equiv 4\tan^{-1}(e^t) - \pi$$

describes the motion along the pendulum SEPARATRIX. Chirikov (1979) has shown that this integral has the approximate value

$$A_m(\lambda) \approx \begin{cases} \frac{4\pi(2\lambda)^{m-1}}{\Gamma(m)} e^{-\pi\lambda/2} & \text{for } \lambda > 0 \\ -\frac{4e^{-\pi|\lambda|/2}}{(2|l|)^{m+1}} \Gamma(m+1)\sin(\pi m) & \text{for } \lambda < 0. \end{cases}$$

References

Chirikov, B. V. "A Universal Instability of Many-Dimensional Oscillator Systems." *Phys. Rep.* **52**, 264–379, 1979.

Melodic Series

If $a_1, a_2, a_3, \ldots$ is an ARTISTIC SERIES, then $1/a_1, 1/a_2, 1/a_3, \ldots$ is a MELODIC SERIES. The RECURRENCE RELATION obeyed by melodic series is

$$b_{i+3} = \frac{b_i {b_{i+2}}^2}{{b_{i+1}}^2} + \frac{{b_{i+2}}^2}{b_{i+1}} - b_{i+2}.$$

see also ARTISTIC SERIES

References

Duffin, R. J. "On Seeing Progressions of Constant Cross Ratio." *Amer. Math. Monthly* **100**, 38–47, 1993.

MEM

see MAXIMUM ENTROPY METHOD

Memoryless

A variable x is memoryless with respect to t if, for all s with $t \neq 0$,

$$P(x > s + t | x > t) = P(x > s). \tag{1}$$

Equivalently,

$$\frac{P(x > s + t, x > t)}{P(x > t)} = P(x > s) \tag{2}$$

$$P(x > s + t) = P(x > s) P(x > t). \tag{3}$$

The EXPONENTIAL DISTRIBUTION, which satisfies

$$P(x > t) = e^{-\lambda t} \tag{4}$$

$$P(x > s + t) = e^{-\lambda(s+t)}, \tag{5}$$

and therefore

$$\begin{aligned} P(x > s + t) = P(x > s) P(x > t) &= e^{-\lambda s} e^{-\lambda t} \\ &= e^{-\lambda(s+t)}, \end{aligned} \tag{6}$$

is the only memoryless random distribution.

see also EXPONENTIAL DISTRIBUTION

Ménage Number

see MARRIED COUPLES PROBLEM

Ménage Problem

see MARRIED COUPLES PROBLEM

Menasco's Theorem

For a BRAID with M strands, R components, P positive crossings, and N negative crossings,

$$\begin{cases} P - N \leq U_+ + M - R & \text{if } P \geq N \\ P - N \leq U_- + M - R & \text{if } P \leq N, \end{cases}$$

where $U_\pm$ are the smallest number of positive and negative crossings which must be changed to crossings of the opposite sign. These inequalities imply BENNEQUIN'S CONJECTURE. Menasco's theorem can be extended to arbitrary knot diagrams.

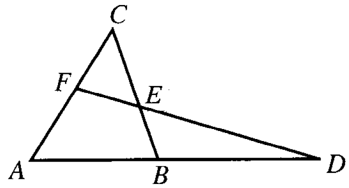
see also BENNEQUIN'S CONJECTURE, BRAID, UNKNOTTING NUMBER

References

Cipra, B. "From Knot to Unknot." *What's Happening in the Mathematical Sciences, Vol. 2.* Providence, RI: Amer. Math. Soc., pp. 8–13, 1994.

Menasco, W. W. "The Bennequin-Milnor Unknotting Conjectures." *C. R. Acad. Sci. Paris Sér. I Math.* **318**, 831–836, 1994.

Menelaus' Theorem



For TRIANGLES in the PLANE,

$$AD \cdot BE \cdot CF = BD \cdot CE \cdot AF. \tag{1}$$

For SPHERICAL TRIANGLES,

$$\sin AD \cdot \sin BE \cdot \sin CF = \sin BD \cdot \sin CE \cdot \sin AF. \tag{2}$$

This can be generalized to n-gons $P = [V_1, \ldots, V_n]$, where a transversal cuts the side $V_i V_{i+1}$ in W_i for $i = 1$, ..., n, by

$$\prod_{i=1}^{n} \left[\frac{V_i W_i}{W_i V_{i+1}} \right] = (-1)^n. \tag{3}$$

Here, $AB || CD$ and

$$\left[\frac{AB}{CD} \right] \tag{4}$$

is the ratio of the lengths $[A, B]$ and $[C, D]$ with a PLUS or MINUS SIGN depending if these segments have the same or opposite directions (Grünbaum and Shepard 1995). The case $n = 3$ is PASCH'S AXIOM.

see also CEVA'S THEOREM, HOEHN'S THEOREM, PASCH'S AXIOM

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 122, 1987.

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 66–67, 1967.

Grünbaum, B. and Shepard, G. C. "Ceva, Menelaus, and the Area Principle." *Math. Mag.* **68**, 254–268, 1995.

Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., p. xxi, 1995.

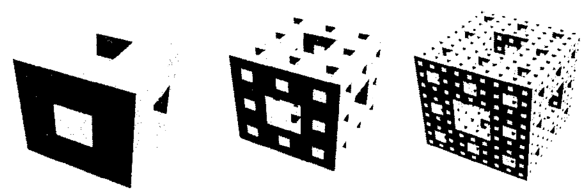
Menger's n-Arc Theorem

Let G be a graph with A and B two disjoint n-tuples of VERTICES. Then either G contains n pairwise disjoint AB-paths, each connecting a point of A and a point of B, or there exists a set of fewer than n VERTICES that separate A and B.

References

Menger, K. *Kurventheorie.* Leipzig, Germany: Teubner, 1932.

Menger Sponge



A FRACTAL which is the 3-D analog of the SIERPIŃSKI CARPET. Let N_n be the number of filled boxes, L_n the length of a side of a hole, and V_n the fractional VOLUME after the nth iteration.

$$N_n = 20^n \tag{1}$$
$$L_n = (\tfrac{1}{3})^n = 3^{-n} \tag{2}$$
$$V_n = {L_n}^3 N_n = (\tfrac{20}{27})^n. \tag{3}$$

The CAPACITY DIMENSION is therefore

$$\begin{aligned} d_{\text{cap}} &= -\lim_{n\to\infty} \frac{\ln N_n}{\ln L_n} = -\lim_{n\to\infty} \frac{\ln(20^n)}{\ln(3^{-n})} \\ &= \frac{\ln 20}{\ln 3} = \frac{\ln(2^2 \cdot 5)}{\ln 3} = \frac{2\ln 2 + \ln 5}{\ln 3} \\ &= 2.726833028\ldots. \end{aligned} \tag{4}$$

J. Mosely is leading an effort to construct a large Menger sponge out of old business cards.

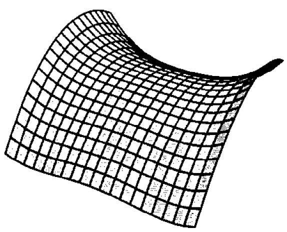
see also SIERPIŃSKI CARPET, TETRIX

References

Dickau, R. M. "Menger (Sierpinski) Sponge." http://forum.swarthmore.edu/advanced/robertd/sponge.html.

Mosely, J. "Menger's Sponge (Depth 3)." http://world.std.com/~j9/sponge/.

Menn's Surface



A surface given by the parametric equations

$$x(u,v) = u$$
$$y(u,v) = v$$
$$z(u,v) = au^4 + u^2v - v^2.$$

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 631, 1993.

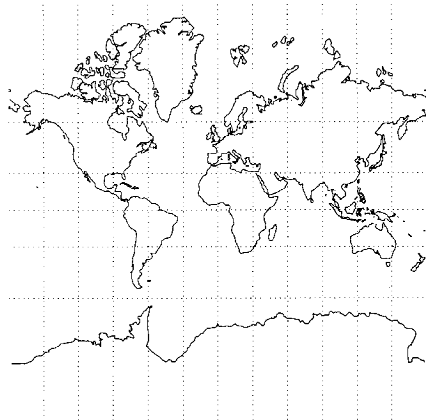
Mensuration Formula

A mensuration formula is simply a formula for computing the length-related properties of an object (such as AREA, CIRCUMRADIUS, etc., of a POLYGON) based on other known lengths, areas, etc. Beyer (1987) gives a collection of such formulas for various plane and solid geometric figures.

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 121–133, 1987.

Mercator Projection



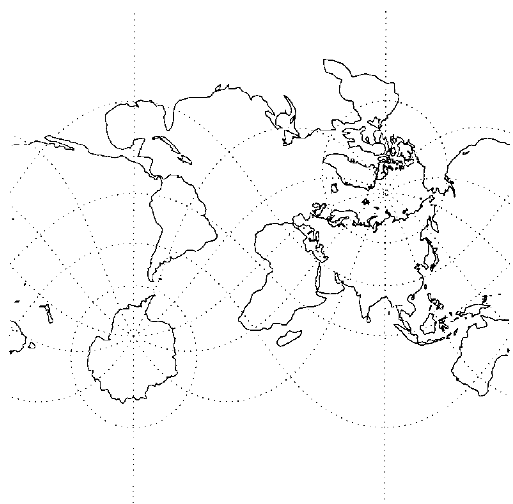
The following equations place the x-AXIS of the projection on the equator and the y-AXIS at LONGITUDE λ_0, where λ is the LONGITUDE and ϕ is the LATITUDE.

$$x = \lambda - \lambda_0 \tag{1}$$
$$y = \ln[\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi)] \tag{2}$$
$$= \tfrac{1}{2}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right) \tag{3}$$
$$= \sinh^{-1}(\tan\phi) \tag{4}$$
$$= \tanh^{-1}(\sin\phi) \tag{5}$$
$$= \ln(\tan\phi + \sec\phi). \tag{6}$$

The inverse FORMULAS are

$$\phi = \tfrac{1}{2}\pi - 2\tan^{-1}(e^{-y}) = \tan^{-1}(\sinh y) \tag{7}$$
$$\lambda = x + \lambda_0. \tag{8}$$

LOXODROMES are straight lines and GREAT CIRCLES are curved.



An oblique form of the Mercator projection is illustrated above. It has equations

$$x = \frac{\tan^{-1}[\tan\phi\cos\phi_p + \sin\phi_p\sin(\lambda-\lambda_0)]}{\cos(\lambda-\lambda_0)} \tag{9}$$

$$y = \tfrac{1}{2}\ln\left(\frac{1+A}{1-A}\right) = \tanh^{-1} A, \tag{10}$$

where

$$\lambda_p = \tan^{-1}\left(\frac{\cos\phi_1\sin\phi_2\cos\lambda_1 - \sin\phi_1\cos\phi_2\cos\lambda_2}{\sin\phi_1\cos\phi_2\sin\lambda_2 - \cos\phi_1\sin\phi_2\sin\lambda_1}\right) \tag{11}$$

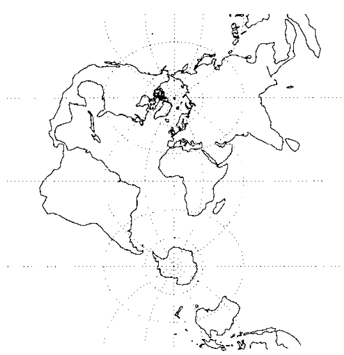
$$\phi_p = \tan^{-1}\left(-\frac{\cos(\lambda_p-\lambda_1)}{\tan\phi_1}\right) \tag{12}$$

$$A = \sin\phi_p\sin\phi - \cos\phi_p\cos\phi\sin(\lambda-\lambda_0). \tag{13}$$

The inverse FORMULAS are

$$\phi = \sin^{-1}\left(\sin\phi_p\tanh y + \frac{\cos\phi_p\sin x}{\cosh y}\right) \tag{14}$$

$$\lambda = \lambda_0 + \tan^{-1}\left(\frac{\sin\phi_p\sin x - \cos\phi_p\sinh y}{\cos x}\right). \tag{15}$$



There is also a transverse form of the Mercator projection, illustrated above. It is given by the equations

$$x = \tfrac{1}{2}\ln\left(\frac{1+B}{1-B}\right) = \tanh^{-1} B \tag{16}$$

$$y = \tan^{-1}\left[\frac{\tan\phi}{\cos(\lambda-\lambda_0)}\right] - \phi_0 \tag{17}$$

$$\phi = \sin^{-1}\left(\frac{\sin D}{\cosh x}\right) \tag{18}$$

$$\lambda = \lambda_0 + \tan^{-1}\left(\frac{\sinh x}{\cos D}\right), \tag{19}$$

where

$$B \equiv \cos\phi\sin(\lambda-\lambda_0) \tag{20}$$

$$D \equiv y + \phi_0. \tag{21}$$

Finally, the "universal transverse Mercator projection" is a MAP PROJECTION which maps the SPHERE into 60 zones of 6° each, with each zone mapped by a transverse Mercator projection with central MERIDIAN in the center of the zone. The zones extend from 80° S to 84° N (Dana).

see also SPHERICAL SPIRAL

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Mercator Series

The TAYLOR SERIES for the NATURAL LOGARITHM

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

which was found by Newton, but independently discovered and first published by Mercator in 1668.

see also LOGARITHMIC NUMBER, NATURAL LOGARITHM

Mercer's Theorem

see RIEMANN-LEBESGUE LEMMA

Mergelyan-Wesler Theorem

Let $P = \{D_1, D_2, \ldots\}$ be an infinite set of disjoint open DISKS D_n of radius r_n such that the union is almost the unit DISK. Then

$$\sum_{n=1}^{\infty} r_n = \infty. \tag{1}$$

Define

$$M_x(P) \equiv \sum_{n=1}^{\infty} {r_n}^x. \tag{2}$$

Then there is a number $e(P)$ such that $M_x(P)$ diverges for $x < e(P)$ and converges for $x > e(P)$. The above theorem gives

$$1 < e(P) < 2. \tag{3}$$

There exists a constant which improves the inequality, and the best value known is

$$S = 1.306951\ldots. \tag{4}$$

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Meridian

A line of constant LONGITUDE on a SPHEROID (or SPHERE). More generally, a meridian of a SURFACE OF REVOLUTION is the intersection of the surface with a PLANE containing the axis of revolution.

see also LATITUDE, LONGITUDE, PARALLEL (SURFACE OF REVOLUTION), SURFACE OF REVOLUTION

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Meromorphic

A meromorphic FUNCTION is complex analytic in all but a discrete subset of its domain, and at those singularities it must go to infinity like a POLYNOMIAL (i.e., have no ESSENTIAL SINGULARITIES). An equivalent definition of a meromorphic function is a complex analytic MAP to the RIEMANN SPHERE.

see also ESSENTIAL SINGULARITY, RIEMANN SPHERE

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Mersenne Number

A number of the form

$$M_n \equiv 2^n - 1 \tag{1}$$

for n an INTEGER is known as a Mersenne number. The Mersenne numbers are therefore 2-REPDIGITS, and also the numbers obtained by setting $x = 1$ in a FERMAT POLYNOMIAL. The first few are 1, 3, 7, 15, 31, 63, 127, 255, ... (Sloane's A000225).

The number of digits D in the Mersenne number M_n is

$$D = \lfloor \log(2^n - 1) + 1 \rfloor, \tag{2}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, which, for large n, gives

$$D \approx \lfloor n \log 2 + 1 \rfloor \approx \lfloor 0.301029n + 1 \rfloor = \lfloor 0.301029n \rfloor + 1. \tag{3}$$

In order for the Mersenne number M_n to be PRIME, n must be PRIME. This is true since for COMPOSITE n with factors r and s, $n = rs$. Therefore, $2^n - 1$ can be written as $2^{rs} - 1$, which is a BINOMIAL NUMBER and can be factored. Since the most interest in Mersenne numbers arises from attempts to factor them, many authors prefer to define a Mersenne number as a number of the above form

$$M_p = 2^p - 1, \tag{4}$$

but with p restricted to PRIME values.

The search for MERSENNE PRIMES is one of the most computationally intensive and actively pursued areas of advanced and distributed computing.

see also CUNNINGHAM NUMBER, EBERHART'S CONJECTURE, FERMAT NUMBER, LUCAS-LEHMER TEST, MERSENNE PRIME, PERFECT NUMBER, REPUNIT, RIESEL NUMBER, SIERPIŃSKI NUMBER OF THE SECOND KIND, SOPHIE GERMAIN PRIME, SUPERPERFECT NUMBER, WIEFERICH PRIME

References

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Mersenne Prime

A MERSENNE NUMBER which is PRIME is called a Mersenne prime. In order for the Mersenne number M_n defined by

$$M_n \equiv 2^n - 1$$

for n an INTEGER to be PRIME, n must be PRIME. This is true since for COMPOSITE n with factors r and s, $n = rs$. Therefore, $2^n - 1$ can be written as $2^{rs} - 1$, which is a BINOMIAL NUMBER and can be factored. Every MERSENNE PRIME gives rise to a PERFECT NUMBER.

If $n \equiv 3 \pmod 4$ is a PRIME, then $2n + 1$ DIVIDES M_n IFF $2n+1$ is PRIME. It is also true that PRIME divisors of $2^p - 1$ must have the form $2kp + 1$ where k is a POSITIVE INTEGER and simultaneously of either the form $8n+1$ or $8n - 1$ (Uspensky and Heaslet). A PRIME factor p of a Mersenne number $M_q = 2^q - 1$ is a WIEFERICH PRIME IFF $p^2 | 2^q - 1$, Therefore, MERSENNE PRIMES are *not* WIEFERICH PRIMES. All known Mersenne numbers M_p with p PRIME are SQUAREFREE. However, Guy (1994) believes that there are M_p which are not SQUAREFREE.

TRIAL DIVISION is often used to establish the COMPOSITENESS of a potential Mersenne prime. This test immediately shows M_p to be COMPOSITE for $p = 11$, 23, 83, 131, 179, 191, 239, and 251 (with small factors 23, 47, 167, 263, 359, 383, 479, and 503, respectively). A much more powerful primality test for M_p is the LUCAS-LEHMER TEST.

It has been conjectured that there exist an infinite number of Mersenne primes, although finding them is computationally very challenging. The table below gives the index p of known Mersenne primes (Sloane's A000043) M_p, together with the number of digits, discovery years, and discoverer. A similar table has been compiled by C. Caldwell. Note that the region after the 35th known Mersenne prime has not been completely searched, so identification of "the" 36th Mersenne prime is tentative. L. Welsh maintains an extensive bibliography and history of Mersenne numbers. G. Woltman has organized a distributed search program via the Internet in which hundreds of volunteers use their personal computers to perform pieces of the search.

#	p	Digits	Year	Published Reference
1	2	1	Anc.	
2	3	1	Anc.	
3	5	2	Anc.	
4	7	3	Anc.	
5	13	4	1461	Reguis 1536, Cataldi 1603
6	17	6	1588	Cataldi 1603
7	19	6	1588	Cataldi 1603
8	31	10	1750	Euler 1772
9	61	19	1883	Pervouchine 1883, Seelhoff 1886
10	89	27	1911	Powers 1911
11	107	33	1913	Powers 1914
12	127	39	1876	Lucas 1876
13	521	157	1952	Lehmer 1952–3
14	607	183	1952	Lehmer 1952–3
15	1279	386	1952	Lehmer 1952–3
16	2203	664	1952	Lehmer 1952–3
17	2281	687	1952	Lehmer 1952–3
18	3217	969	1957	Riesel 1957
19	4253	1281	1961	Hurwitz 1961
20	4423	1332	1961	Hurwitz 1961
21	9689	2917	1963	Gillies 1964
22	9941	2993	1963	Gillies 1964
23	11213	3376	1963	Gillies 1964
24	19937	6002	1971	Tuckerman 1971
25	21701	6533	1978	Noll and Nickel 1980
26	23209	6987	1979	Noll 1980
27	44497	13395	1979	Nelson and Slowinski 1979
28	86243	25962	1982	Slowinski 1982
29	110503	33265	1988	Colquitt and Welsh 1991
30	132049	39751	1983	Slowinski 1988
31	216091	65050	1985	Slowinski 1989
32	756839	227832	1992	Gage and Slowinski 1992
33	859433	258716	1994	Gage and Slowinski 1994
34	1257787	378632	1996	Slowinski and Gage
35	1398269	420921	1996	Armengaud, Woltman, *et al.*
36?	2976221	895832	1997	Spence
37?	3021377	909526	1998	Clarkson, Woltman, *et al.*

see also CUNNINGHAM NUMBER, FERMAT-LUCAS NUMBER, FERMAT NUMBER, FERMAT NUMBER (LUCAS), FERMAT POLYNOMIAL, LUCAS-LEHMER TEST, MERSENNE NUMBER, PERFECT NUMBER, REPUNIT, SUPERPERFECT NUMBER

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Mertens Conjecture

Given MERTENS FUNCTION defined by

$$M(n) \equiv \sum_{k=1}^{n} \mu(k), \tag{1}$$

where $\mu(n)$ is the MÖBIUS FUNCTION, Mertens (1897) conjecture states that

$$|M(x)| < x^{1/2} \tag{2}$$

for $x > 1$. The conjecture has important implications, since the truth of any equality of the form

$$|M(x)| \leq cx^{1/2} \tag{3}$$

for any fixed c (the form of Mertens conjecture with $c = 1$) would imply the RIEMANN HYPOTHESIS. In 1885, Stieltjes claimed that he had a proof that $M(x)x^{-1/2}$ always stayed between two fixed bounds. However, it seems likely that Stieltjes was mistaken.

Mertens conjecture was proved false by Odlyzko and te Riele (1985). Their proof is indirect and does not produce a specific counterexample, but it does show that

$$\limsup_{x\to\infty} M(x)x^{-1/2} > 1.06 \tag{4}$$

$$\liminf_{x\to\infty} M(x)x^{-1/2} < -1.009. \tag{5}$$

Odlyzko and te Riele (1985) believe that there are no counterexamples to Mertens conjecture for $x \leq 10^{20}$, or even 10^{30}. Pintz (1987) subsequently showed that at least one counterexample to the conjecture occurs for $x \leq 10^{65}$, using a weighted integral average of $M(x)/x$ and a discrete sum involving nontrivial zeros of the RIEMANN ZETA FUNCTION.

It is still not known if

$$\limsup_{x\to\infty} |M(x)|x^{-1/2} = \infty, \tag{6}$$

although it seems very probable (Odlyzko and te Riele 1985).

see also MERTENS FUNCTION, MÖBIUS FUNCTION, RIEMANN HYPOTHESIS

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Mertens Constant

A constant related to the TWIN PRIMES CONSTANT which appears in the FORMULA for the sum of inverse PRIMES

$$\sum_{p \text{ prime}}^{x} \frac{1}{p} = \ln \ln x + B_1 + o(1) \tag{1}$$

which is given by

$$B_1 = \gamma + \sum_{p \text{ prime}} \left[\ln(1 - p^{-1}) + \frac{1}{p}\right] \approx 0.261497. \tag{2}$$

Flajolet and Vardi (1996) show that

$$e^{B_1} = e^{\gamma} \prod_{m=2}^{\infty} \zeta(m)^{\mu(m)/m}, \tag{3}$$

where γ is the EULER-MASCHERONI CONSTANT, $\zeta(n)$ is the RIEMANN ZETA FUNCTION, and $\mu(n)$ is the MÖBIUS FUNCTION. The constant B_1 also occurs in the SUMMATORY FUNCTION of the number of DISTINCT PRIME FACTORS,

$$\sum_{k=2}^{n} \omega(k) = n \ln \ln n + B_1 n + o(n) \tag{4}$$

(Hardy and Wright 1979, p. 355).

The related constant

$$B_2 = \gamma + \sum_{p \text{ prime}} \left[\ln(1 - p^{-1}) + \frac{1}{p-1}\right] \approx 1.034653 \tag{5}$$

appears in the SUMMATORY FUNCTION of the DIVISOR FUNCTION $\sigma_0(n) = \Omega(n)$,

$$\sum_{k=2}^{n} \Omega(k) = n \ln \ln n + B_2 + o(n) \tag{6}$$

(Hardy and Wright 1979, p. 355).

see also BRUN'S CONSTANT, PRIME NUMBER, TWIN PRIMES CONSTANT

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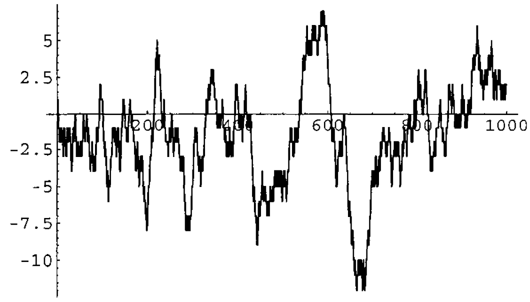
Mertens Function



The summary function

$$M(n) \equiv \sum_{k=1}^{n} \mu(k) = \frac{6}{\pi^2} n + \mathcal{O}(\sqrt{n}),$$

where $\mu(n)$ is the MÖBIUS FUNCTION. The first few values are 1, 0, -1, -1, -2, -1, -2, -2, -2, -1, -2, -2, ... (Sloane's A002321). The first few values of n at which $M(n) = 0$ are 2, 39, 40, 58, 65, 93, 101, 145, 149, 150, ... (Sloane's A028442).

Mertens function obeys

$$\sum_{n=1}^{x} M\left(\frac{x}{n}\right) = 1$$

(Lehman 1960). The analytic form is unsolved, although MERTENS CONJECTURE that

$$|M(x)| < x^{1/2}$$

has been disproved.

Lehman (1960) gives an algorithm for computing $M(x)$ with $\mathcal{O}(x^{2/3+\epsilon})$ operations, while the Lagarias-Odlyzko (1987) algorithm for computing the PRIME COUNTING FUNCTION $\pi(x)$ can be modified to give $M(x)$ in $\mathcal{O}(x^{3/5+\epsilon})$ operations.

see also MERTENS CONJECTURE, MÖBIUS FUNCTION

References

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Sloane, N. J. A. Sequence A028442/M002321 in "An On-Line Version of the Encyclopedia of Integer Sequences."0102

Mertens Theorem

$$\lim_{x\to\infty} \frac{\prod_{\substack{2\leq p\leq x \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)}{\frac{e^{-\gamma}}{\ln x}} = 1,$$

where γ is the EULER-MASCHERONI CONSTANT and $e^{-\gamma} = 0.56145\ldots$.

References

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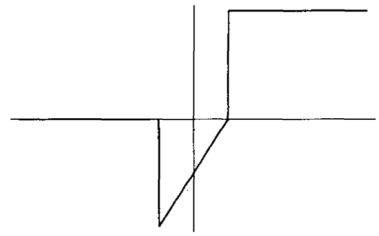
Mertz Apodization Function



An asymmetrical APODIZATION FUNCTION defined by

$$M(x,b,d) = \begin{cases} 0 & \text{for } x < -b \\ (x-b)/(2b) & \text{for } -b < x < b \\ 1 & \text{for } b < x < b+2d \\ 0 & \text{for } x < b+2d, \end{cases}$$

where the two-sided portion is $2b$ long (total) and the one-sided portion is $b+2d$ long (Schnopper and Thompson 1974, p. 508). The APPARATUS FUNCTION is

$$M_A(k,b,d) = \frac{\sin[2\pi k(b+2d)}{2\pi k} + i\left\{\frac{\cos[2\pi k(b+2d)]}{2\pi k} - \frac{\sin(2b)}{4\pi^2 k^2 b}\right\}.$$

References

Schnopper, H. W. and Thompson, R. I. "Fourier Spectrometers." In *Methods of Experimental Physics* **12A**. New York: Academic Press, pp. 491–529, 1974.

Mesh Size

When a CLOSED INTERVAL $[a, b]$ is partitioned by points $a < x_1 < x_2 < \ldots < x_{n-1} < b$, the lengths of the resulting intervals between the points are denoted Δx_1, Δx_2, ..., Δx_n, and the value $\max \Delta x_k$ is called the mesh size of the partition.

see also INTEGRAL, LOWER SUM, RIEMANN INTEGRAL, UPPER SUM

Mesokurtic

A distribution with zero KURTOSIS ($\gamma_2 = 0$).

see also KURTOSIS, LEPTOKURTIC

Metabiaugmented Dodecahedron

see JOHNSON SOLID

Metabiaugmented Hexagonal Prism

see JOHNSON SOLID

Metabiaugmented Truncated Dodecahedron

see JOHNSON SOLID

Metabidiminished Icosahedron

see JOHNSON SOLID

Metabidiminished Rhombicosidodecahedron

see JOHNSON SOLID

Metabigyrate Rhombicosidodecahedron

see JOHNSON SOLID

Metadrome

A metadrome is a number whose HEXADECIMAL digits are in strict ascending order. The first few are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, ... (Sloane's A023784).

see also HEXADECIMAL

References

Sloane, N. J. A. Sequence A023784 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Metagyrate Diminished Rhombicosidodecahedron

see JOHNSON SOLID

Metalogic

see METAMATHEMATICS

Metamathematics

The branch of LOGIC dealing with the study of the combination and application of mathematical symbols, sometimes called METALOGIC. Metamathematics is the study of MATHEMATICS itself, and one of its primary goals is to determine the nature of mathematical reasoning (Hofstadter 1989).

see also LOGIC, MATHEMATICS

References

Birkhoff, G. and Mac Lane, S. *A Survey of Modern Algebra, 3rd ed.* New York: Macmillan, p. 326, 1965.

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 23, 1989.

Method

A particular way of doing something, sometimes also called an ALGORITHM or PROCEDURE. (According to Petkovšek *et al.* (1996), "a method is a trick that has worked at least twice.")

see also ADAMS-BASHFORTH-MOULTON METHOD, ADAMS' METHOD, BACKUS-GILBERT METHOD, BADER-DEUFLHARD METHOD, BAILEY'S METHOD, BAIRSTOW'S METHOD, BRENT'S FACTORIZATION METHOD, BRENT'S METHOD, CIRCLE METHOD, CONJUGATE GRADIENT METHOD, CRISS-CROSS METHOD, CROUT'S METHOD, DE LA LOUBERE'S METHOD, DIXON'S FACTORIZATION METHOD, DIXON'S RANDOM SQUARES FACTORIZATION METHOD, ELLIPTIC CURVE FACTORIZATION METHOD, EULER'S FACTORIZATION METHOD, EXCLUDENT FACTORIZATION METHOD, EXHAUSTION METHOD, FALSE POSITION METHOD, FERMAT'S FACTORIZATION METHOD, FROBENIUS METHOD, GILL'S METHOD, GOSPER'S METHOD, GRAEFFE'S METHOD, GREENE'S METHOD, HALLEY'S METHOD, HORNER'S METHOD, HUTTON'S METHOD, JACOBI METHOD, KAPS-RENTROP METHODS, LAGUERRE'S METHOD, LAMBERT'S METHOD, LEGENDRE'S FACTORIZATION METHOD, LEHMER METHOD, LEHMER-SCHUR METHOD, LENSTRA ELLIPTIC CURVE METHOD, LIN'S METHOD, LOZENGE METHOD, LUX METHOD, MAPES' METHOD, MAXIMUM ENTROPY METHOD, MILNE'S METHOD, MULLER'S METHOD, NEWTON'S METHOD, NEWTON-RAPHSON METHOD, NUMBER FIELD SIEVE FACTORIZATION METHOD, OVERLAPPING RESONANCE METHOD, POLLARD MONTE CARLO FACTORIZATION METHOD, POLLARD ρ FACTORIZATION METHOD, POLLARD $p-1$ FACTORIZATION METHOD, PREDICTOR-CORRECTOR METHODS, QUADRATIC SIEVE FACTORIZATION METHOD, RESONANCE OVERLAP METHOD, ROSENBROCK METHODS, RUNGE-KUTTA METHOD, SCHRÖDER'S METHOD, SECANT METHOD, SIAMESE METHOD, SIMPLEX METHOD, SNAKE OIL METHOD, SQUARE ROOT METHOD, STEEPEST DESCENT METHOD, TANGENT HYPERBOLAS METHOD, UNDETERMINED COEFFICIENTS METHOD, WILLIAMS $p+1$ FACTORIZATION METHOD, WYNN'S EPSILON METHOD

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, p. 117, 1996.

Metric

A NONNEGATIVE function $g(x,y)$ describing the "DISTANCE" between neighboring points for a given SET. A metric satisfies the TRIANGLE INEQUALITY

$$g(x,y)+g(y,z)\geq g(x,z),$$

with equality IFF $x=y$, and is symmetric, so

$$g(x,y)=g(y,x).$$

A SET possessing a metric is called a METRIC SPACE. When viewed as a TENSOR, the metric is called a METRIC TENSOR.

see also CAYLEY-KLEIN-HILBERT METRIC, DISTANCE, FUNDAMENTAL FORMS, HYPERBOLIC METRIC, METRIC ENTROPY, METRIC EQUIVALENCE PROBLEM, METRIC SPACE, METRIC TENSOR, PART METRIC, RIEMANNIAN METRIC, ULTRAMETRIC

References

Gray, A. "Metrics on Surfaces." Ch. 13 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 251–265, 1993.

Metric Entropy

Also known as KOLMOGOROV ENTROPY, KOLMOGOROV-SINAI ENTROPY, or KS Entropy. The metric entropy is 0 for nonchaotic motion and > 0 for CHAOTIC motion.

References
Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 138, 1993.

Metric Equivalence Problem

1. Find a complete system of invariants, or
2. decide when two METRICS differ only by a coordinate transformation.

The most common statement of the problem is, "Given METRICS g and g', does there exist a coordinate transformation from one to the other?" Christoffel and Lipschitz (1870) showed how to decide this question for two RIEMANNIAN METRICS.

The solution by E. Cartan requires computation of the 10th order COVARIANT DERIVATIVES. The demonstration was simplified by A. Karlhede using the TETRAD formalism so that only seventh order COVARIANT DERIVATIVES need be computed. However, in many common cases, the first or second-order DERIVATIVES are SUFFICIENT to answer the question.

References
Karlhede, A. and Lindström, U. "Finding Space-Time Geometries without Using a Metric." *Gen. Relativity Gravitation* **15**, 597–610, 1983.

Metric Space

A SET S with a global distance FUNCTION (the METRIC g) which, for every two points x, y in S, gives the DISTANCE between them as a NONNEGATIVE REAL NUMBER $g(x, y)$. A metric space must also satisfy

1. $g(x, x) = 0$ IFF $x = y$,
2. $g(x, y) = g(y, x)$,
3. The TRIANGLE INEQUALITY $g(x, y) + g(y, z) \geq g(x, z)$.

References
Munkres, J. R. *Topology: A First Course.* Englewood Cliffs, NJ: Prentice-Hall, 1975.
Rudin, W. *Principles of Mathematical Analysis.* New York: McGraw-Hill, 1976.

Metric Tensor

A TENSOR, also called a RIEMANNIAN METRIC, which is symmetric and POSITIVE DEFINITE. Very roughly, the metric tensor g_{ij} is a function which tells how to compute the distance between any two points in a given SPACE. Its components can be viewed as multiplication factors which must be placed in front of the differential displacements dx_i in a generalized PYTHAGOREAN THEOREM

$$ds^2 = g_{11} dx_1{}^2 + g_{12}\, dx_1\, dx_2 + g_{22}\, dx_2{}^2 + \ldots. \tag{1}$$

In EUCLIDEAN SPACE, $g_{ij} = \delta_{ij}$ where δ is the KRONECKER DELTA (which is 0 for $i \neq j$ and 1 for $i = j$), reproducing the usual form of the PYTHAGOREAN THEOREM

$$ds^2 = dx_1{}^2 + dx_2{}^2 + \ldots. \tag{2}$$

The metric tensor is defined abstractly as an INNER PRODUCT of every TANGENT SPACE of a MANIFOLD such that the INNER PRODUCT is a symmetric, non-degenerate, bilinear form on a VECTOR SPACE. This means that it takes two VECTORS $\mathbf{v}, \mathbf{w}$ as arguments and produces a REAL NUMBER $\langle \mathbf{v}, \mathbf{w} \rangle$ such that

$$\langle k\mathbf{v}, w \rangle = k \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, k\mathbf{w} \rangle \tag{3}$$

$$\langle \mathbf{v} + \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{x} \rangle + \langle \mathbf{w}, \mathbf{x} \rangle \tag{4}$$

$$\langle \mathbf{v}, \mathbf{w} + \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{x} \rangle \tag{5}$$

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle \tag{6}$$

$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0, \tag{7}$$

with equality IFF $\mathbf{v} = 0$.

In coordinate NOTATION (with respect to the basis),

$$g^{\alpha\beta} = \vec{e}^{\alpha} \cdot \vec{e}^{\beta} \tag{8}$$

$$g_{\alpha\beta} = \vec{e}_{\alpha} \cdot \vec{e}_{\beta}. \tag{9}$$

$$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}, \tag{10}$$

where $\eta_{\alpha\beta}$ is the MINKOWSKI METRIC. This can also be written

$$g = D^{\mathrm{T}} \eta D, \tag{11}$$

where

$$D_{\alpha\mu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \tag{12}$$

$$D_{\alpha\mu}{}^{\mathrm{T}} \equiv D_{\mu\alpha}. \tag{13}$$

$$\frac{\partial}{\partial x^m} g_{il} g^{lk} = \frac{\partial}{\partial x^m} \delta_i^k \tag{14}$$

gives

$$g_{il} \frac{\partial g^{lk}}{\partial x^m} = -g^{lk} \frac{\partial g_{il}}{\partial x^m}. \tag{15}$$

The metric is POSITIVE DEFINITE, so a metric's DISCRIMINANT is POSITIVE. For a metric in 2-space,

$$g \equiv g_{11} g_{22} - g_{12}{}^2 > 0. \tag{16}$$

The ORTHOGONALITY of CONTRAVARIANT and COVARIANT metrics stipulated by

$$g_{ik} g^{ij} = \delta_k^j \tag{17}$$

for $i = 1, \ldots, n$ gives n linear equations relating the $2n$ quantities g_{ij} and g^{ij}. Therefore, if n metrics are known, the others can be determined.

In 2-space,

$$g^{11} = \frac{g_{22}}{g} \quad (18)$$

$$g^{12} = g^{21} = -\frac{g_{12}}{g} \quad (19)$$

$$g^{22} = \frac{g_{11}}{g}. \quad (20)$$

If g is symmetric, then

$$g_{\alpha\beta} = g_{\beta\alpha} \quad (21)$$

$$g^{\alpha\beta} = g^{\beta\alpha}. \quad (22)$$

In EUCLIDEAN SPACE (and all other symmetric SPACES),

$$g_\alpha^\beta = g_\alpha^\beta = \delta_\alpha^\beta, \quad (23)$$

so

$$g_{\alpha\alpha} = \frac{1}{g^{\alpha\alpha}}. \quad (24)$$

The ANGLE ϕ between two parametric curves is given by

$$\cos\phi = \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 = \frac{\mathbf{r}_1}{g_1} \cdot \frac{\mathbf{r}_2}{g_2} = \frac{g_{12}}{g_1 g_2}, \quad (25)$$

so

$$\sin\phi = \frac{\sqrt{g}}{g_1 g_2} \quad (26)$$

and

$$|\mathbf{r}_1 \times \mathbf{r}_2| = g_1 g_2 \sin\phi = \sqrt{g}. \quad (27)$$

The LINE ELEMENT can be written

$$ds^2 = dx_i\, dx_i = g_{ij}\, dq_i\, dq_j \quad (28)$$

where EINSTEIN SUMMATION has been used. But

$$dx_i = \frac{\partial x_i}{\partial q_1}\, dq_1 + \frac{\partial x_i}{\partial q_2}\, dq_2 + \frac{\partial x_i}{\partial q_3}\, dq_3 = \frac{\partial x_i}{\partial q_j}\, dq_j, \quad (29)$$

so

$$g_{ij} = \sum_k \frac{\partial^2 x_k}{\partial q_i \partial q_j}. \quad (30)$$

For ORTHOGONAL coordinate systems, $g_{ij} = 0$ for $i \neq j$, and the LINE ELEMENT becomes (for 3-space)

$$\begin{aligned} ds^2 &= g_{11}\, dq_1{}^2 + g_{22}\, dq_2{}^2 + g_{33}\, dq_3{}^2 \\ &= (h_1\, dq_1)^2 + (h_2\, dq_2)^2 + (h_3\, dq_3)^2, \end{aligned} \quad (31)$$

where $h_i \equiv \sqrt{g_{ii}}$ are called the SCALE FACTORS.

see also CURVILINEAR COORDINATES, DISCRIMINANT (METRIC), LICHNEROWICZ CONDITIONS, LINE ELEMENT, METRIC, METRIC EQUIVALENCE PROBLEM, MINKOWSKI SPACE, SCALE FACTOR, SPACE

Mex

The MINIMUM excluded value. The mex of a SET S of NONNEGATIVE INTEGERS is the least NONNEGATIVE INTEGER *not* in the set.

see also MEX SEQUENCE

References

Guy, R. K. "Max and Mex Sequences." §E27 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 227–228, 1994.

Mex Sequence

A sequence defined from a FINITE sequence $a_0, a_1, \ldots, a_n$ by defining $a_{n+1} = \text{mex}_i(a_i + a_{n-i})$, where mex is the MEX (minimum excluded value).

see also MAX SEQUENCE, MEX

References

Guy, R. K. "Max and Mex Sequences." §E27 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 227–228, 1994.

Mian-Chowla Sequence

The sequence produced by starting with $a_1 = 1$ and applying the GREEDY ALGORITHM in the following way: for each $k \geq 2$, let a_k be the least INTEGER exceeding a_{k-1} for which $a_j + a_k$ are all distinct, with $1 \leq j \leq k$. This procedure generates the sequence 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, 182, 204, 252, 290, ... (Sloane's A005282). The RECIPROCAL sum of the sequence,

$$S \equiv \sum_{i=1}^{\infty} \frac{1}{a_i},$$

satisfies

$$2.1568 \leq S \leq 2.1596.$$

see also A-SEQUENCE, B_2-SEQUENCE

References

Guy, R. K. "B_2-Sequences." §E28 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 228–229, 1994.

Sloane, N. J. A. Sequence A005282/M1094 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Mice Problem

n mice start at the corners of a regular n-gon of unit side length, each heading towards its closest neighboring mouse in a counterclockwise direction at constant speed. The mice each trace out a SPIRAL, meet in the center of the POLYGON, and travel a distance

$$d_n = \frac{1}{1 - \cos\left(\frac{2\pi}{m}\right)}.$$

The first few values for $n = 2, 3, \ldots$, are

$$\tfrac{1}{2}, \tfrac{2}{3}, 1, \tfrac{1}{5}(5 + \sqrt{5}), 2, \frac{1}{1 - \cos\left(\frac{2\pi}{7}\right)},$$

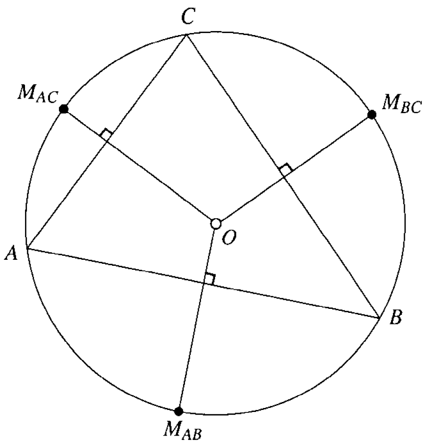
$$2 + \sqrt{2}, \frac{1}{1 - \cos\left(\frac{2\pi}{9}\right)}, 3 + \sqrt{5}, \ldots,$$

giving the numerical values 0.5, 0.666667, 1, 1.44721, 2, 2.65597, 3.41421, 4.27432, 5.23607, ...

see also APOLLONIUS PURSUIT PROBLEM, PURSUIT CURVE, SPIRAL, TRACTRIX

References
Bernhart, A. "Polygons of Pursuit." *Scripta Math.* **24**, 23–50, 1959.
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 201–204, 1979.

Mid-Arc Points



The mid-arc points M_{AB}, M_{AC}, and M_{BC} of a TRIANGLE ΔABC are the points on the CIRCUMCIRCLE of the triangle which lie half-way along each of the three ARCS determined by the vertices (Johnson 1929). These points arise in the definition of the FUHRMANN CIRCLE and FUHRMANN TRIANGLE, and lie on the extensions of the PERPENDICULAR BISECTORS of the triangle sides drawn from the CIRCUMCENTER O.

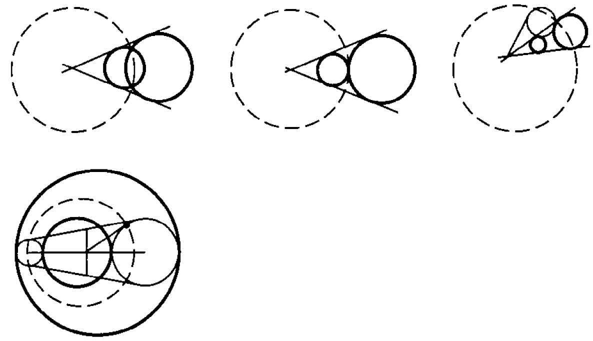
Kimberling (1988, 1994) and Kimberling and Veldkamp (1987) define the mid-arc points as the POINTS which have TRIANGLE CENTER FUNCTIONS

$$\alpha_1 = [\cos(\tfrac{1}{2}B) + \cos(\tfrac{1}{2}C)]\sec(\tfrac{1}{2}A)$$
$$\alpha_2 = [\cos(\tfrac{1}{2}B) + \cos(\tfrac{1}{2}C)]\csc(\tfrac{1}{2}A).$$

see also FUHRMANN CIRCLE, FUHRMANN TRIANGLE

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 228–229, 1929.
Kimberling, C. "Problem 804." *Nieuw Archief voor Wiskunde* **6**, 170, 1988.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. and Veldkamp, G. R. "Problem 1160 and Solution." *Crux Math.* **13**, 298–299, 1987.

Midcircle



The midcircle of two given CIRCLES is the CIRCLE which would INVERT the circles into each other. Dixon (1991) gives constructions for the midcircle for four of the five possible configurations. In the case of the two given CIRCLES tangent to each other, there are two midcircles.

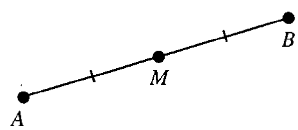
see also INVERSION, INVERSION CIRCLE

References
Dixon, R. *Mathographics.* New York: Dover, pp. 66–68, 1991.

Middlespoint

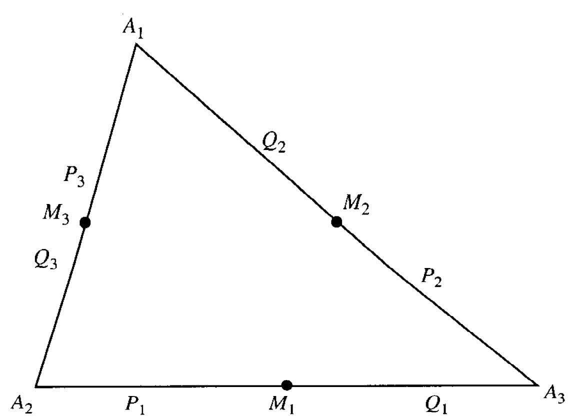
see MITTENPUNKT

Midpoint



The point on a LINE SEGMENT dividing it into two segments of equal length. The midpoint of a line segment is easy to locate by first constructing a LENS using circular arcs, then connecting the cusps of the LENS. The point where the cusp-connecting line intersects the segment is then the midpoint (Pedoe 1995, p. xii). It is more challenging to locate the midpoint using only a COMPASS, but Pedoe (1995, pp. xviii–xix) gives one solution.

In a RIGHT TRIANGLE, the midpoint of the HYPOTENUSE is equidistant from the three VERTICES (Dunham 1990).



Given a TRIANGLE $\Delta A_1A_2A_3$ with AREA Δ, locate the midpoints M_i. Now inscribe two triangles $\Delta P_1P_2P_3$ and $\Delta Q_1Q_2Q_3$ with VERTICES P_i and Q_i placed so that $\overline{P_iM_i} = \overline{Q_iM_i}$. Then $\Delta P_1P_2P_3$ and $\Delta Q_1Q_2Q_3$ have equal areas

$$\Delta_P = \Delta_Q = \Delta \left[1 - \left(\frac{m_1}{a_1} + \frac{m_2}{a_2} + \frac{m_3}{a_3}\right) + \frac{m_2 m_2}{a_2 a_3} + \frac{m_3 m_1}{a_3 a_1} + \frac{m_1 m_2}{a_1 a_2}\right],$$

where a_i are the sides of the original triangle and m_i are the lengths of the MEDIANS (Johnson 1929).

see also ARCHIMEDES' MIDPOINT THEOREM, BROCARD MIDPOINT, CIRCLE-POINT MIDPOINT THEOREM, LINE SEGMENT, MEDIAN (TRIANGLE), MIDPOINT ELLIPSE

References

Dunham, W. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 120–121, 1990.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 80, 1929.

Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., 1995.

Midpoint Ellipse

The unique ELLIPSE tangent to the MIDPOINTS of a TRIANGLE'S LEGS. The midpoint ellipse has the maximum AREA of any INSCRIBED ELLIPSE (Chakerian 1979). Under an AFFINE TRANSFORMATION, the midpoint ellipse can be transformed into the INCIRCLE of an EQUILATERAL TRIANGLE.

see also AFFINE TRANSFORMATION, ELLIPSE, INCIRCLE, MIDPOINT, TRIANGLE

References

Central Similarities. University of Minnesota College Geometry Project. Distributed by International Film Bureau, Inc.

Chakerian, G. D. "A Distorted View of Geometry." Ch. 7 in *Mathematical Plums* (Ed. R. Honsberger). Washington, DC: Math. Assoc. Amer., pp. 135–136 and 145–146, 1979.

Pedoe, D. "Thinking Geometrically." *Amer. Math. Monthly* **77**, 711–721, 1970.

Midradius

The RADIUS of the MIDSPHERE of a POLYHEDRON, also called the INTERRADIUS. For a REGULAR POLYHEDRON with SCHLÄFLI SYMBOL $\{q,p\}$, the DUAL POLYHEDRON is $\{p,q\}$. Denote the INRADIUS r, midradius ρ, and CIRCUMRADIUS R, and let the side length be a. Then

$$r^2 = \left[a \csc\left(\frac{\pi}{p}\right)\right]^2 + R^2 = a^2 + \rho^2 \qquad (1)$$

$$\rho^2 = \left[a \cot\left(\frac{\pi}{p}\right)\right]^2 + R^2. \qquad (2)$$

For REGULAR POLYHEDRA and UNIFORM POLYHEDRA, the DUAL POLYHEDRON has CIRCUMRADIUS ρ^2/r and INRADIUS ρ^2/R. Let θ be the ANGLE subtended by the EDGE of an ARCHIMEDEAN SOLID. Then

$$r = \tfrac{1}{2}a\cos(\tfrac{1}{2}\theta)\cot(\tfrac{1}{2}\theta) \qquad (3)$$

$$\rho = \tfrac{1}{2}a\cot(\tfrac{1}{2}\theta) \qquad (4)$$

$$R = \tfrac{1}{2}a\csc(\tfrac{1}{2}\theta), \qquad (5)$$

so

$$r:\rho:R = \cos(\tfrac{1}{2}\theta):1:\sec(\tfrac{1}{2}\theta) \qquad (6)$$

(Cundy and Rollett 1989). Expressing the midradius in terms of the INRADIUS r and CIRCUMRADIUS R gives

$$\rho = \tfrac{1}{2}\sqrt{2}\sqrt{r^2 + r\sqrt{r^2+a^2}} = \sqrt{R^2 - \tfrac{1}{4}a^2} \qquad (7)$$

for an ARCHIMEDEAN SOLID.

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 126–127, 1989.

Midrange

$$\text{midrange}[f(x)] \equiv \tfrac{1}{2}\{\max[f(x)] + \min[f(x)]\}.$$

see also MAXIMUM, MEAN, MEDIAN (STATISTICS), MINIMUM

Midsphere

The SPHERE with respect to which the VERTICES of a POLYHEDRON are the poles of the planes of the faces of the DUAL POLYHEDRON (and vice versa). It touches all EDGES of a SEMIREGULAR POLYHEDRON or REGULAR POLYHEDRON. It is also called the INTERSPHERE or RECIPROCATING SPHERE.

see also CIRCUMSPHERE, DUAL POLYHEDRON, INSPHERE

Midy's Theorem

If the period of a REPEATING DECIMAL for a/p has an EVEN number of digits, the sum of the two halves is a string of 9s, where p is PRIME and a/p is a REDUCED FRACTION.

see also DECIMAL EXPANSION, REPEATING DECIMAL

References

Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 158–160, 1957.

Mikusiński's Problem

Is it possible to cover completely the surface of a SPHERE with congruent, nonoverlapping arcs of GREAT CIRCLES? Conway and Croft (1964) proved that it can be covered with half-open arcs, but not with open arcs. They also showed that the PLANE can be covered with congruent closed and half-open segments, but not with open ones.

References

Conway, J. H. and Croft, H. T. "Covering a Sphere with Great-Circle Arcs." *Proc. Cambridge Phil. Soc.* **60**, 787–900, 1964.

Gardner, M. "Point Sets on the Sphere." Ch. 12 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, pp. 145–154, 1986.

Milin Conjecture

An INEQUALITY which IMPLIES the correctness of the ROBERTSON CONJECTURE (Milin 1971). de Branges (1985) proved this conjecture, which led to the proof of the full BIEBERBACH CONJECTURE.

see also BIEBERBACH CONJECTURE, ROBERTSON CONJECTURE

References

de Branges, L. "A Proof of the Bieberbach Conjecture." *Acta Math.* **154**, 137–152, 1985.

Milin, I. M. *Univalent Functions and Orthonormal Systems.* Providence, RI: Amer. Math. Soc., 1977.

Stewart, I. *From Here to Infinity: A Guide to Today's Mathematics.* Oxford, England: Oxford University Press, p. 165, 1996.

Mill

The n-roll mill curve is given by the equation

$$x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \ldots = a^n,$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 86, 1993.

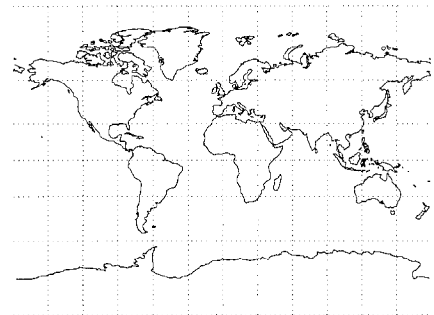
Miller's Algorithm

For a catastrophically unstable recurrence in one direction, any seed values for consecutive x_j and x_{j+1} will converge to the desired sequence of functions in the opposite direction times an unknown normalization factor.

Miller-Aškinuze Solid

see ELONGATED SQUARE GYROBICUPOLA

Miller Cylindrical Projection



A MAP PROJECTION given by the following transformation,

$$x = \lambda - \lambda_0 \tag{1}$$

$$y = \tfrac{5}{4} \ln[\tan(\tfrac{1}{4}\pi + \tfrac{2}{5}\phi)] \tag{2}$$

$$= \tfrac{5}{4} \sinh^{-1}[\tan(\tfrac{4}{5}\phi)]. \tag{3}$$

Here x and y are the plane coordinates of a projected point, λ is the longitude of a point on the globe, λ_0 is central longitude used for the projection, and ϕ is the latitude of the point on the globe. The inverse FORMULAS are

$$\phi = \tfrac{5}{2} \tan^{-1}(e^{4y/5}) - \tfrac{5}{8}\pi = \tfrac{5}{4} \tan^{-1}[\sinh(\tfrac{4}{5}y)] \tag{4}$$

$$\lambda = \lambda_0 + x. \tag{5}$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 86–89, 1987.

Miller's Primality Test

If a number fails this test, it is not a PRIME. If the number passes, it *may* be a PRIME. A number passing Miller's test is called a STRONG PSEUDOPRIME to base a. If a number n does not pass the test, then it is called a WITNESS for the COMPOSITENESS of n. If n is an ODD, POSITIVE COMPOSITE NUMBER, then n passes Miller's test for at most $(n-1)/4$ bases with $1 \leq a \leq -1$ (Long 1995). There is no analog of CARMICHAEL NUMBERS for STRONG PSEUDOPRIMES.

The only COMPOSITE NUMBER less than 2.5×10^{13} which does not have 2, 3, 5, or 7 as a WITNESS is 3215031751. Miller showed that any composite n has a WITNESS less than $70(\ln n)^2$ if the RIEMANN HYPOTHESIS is true.

see also ADLEMAN-POMERANCE-RUMELY PRIMALITY TEST, STRONG PSEUDOPRIME

References

Long, C. T. Th. 4.21 in *Elementary Introduction to Number Theory, 3rd ed.* Prospect Heights, IL: Waveland Press, 1995.

Miller's Solid

see ELONGATED SQUARE GYROBICUPOLA

Milliard

In British, French, and German usage, one milliard equals 10^9. American usage does not have a number called the milliard, instead using the term BILLION to denote 10^9.

see also BILLION, LARGE NUMBER, MILLION, TRILLION

Millin Series

The series with sum

$$S' \equiv \sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \tfrac{1}{2}(7 - \sqrt{5}\,),$$

where F_k is a FIBONACCI NUMBER (Honsberger 1985).

see also FIBONACCI NUMBER

References

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 135–137, 1985.

Million

The number $1,000,000 = 10^6$. While one million in America means the same thing as one million in Britain, the words BILLION, TRILLION, etc., refer to *different numbers* in the two naming systems. While Americans may say "Thanks a million" to express gratitude, Norwegians offer "Thanks a thousand" ("tusen takk").

see also BILLION, LARGE NUMBER, MILLIARD, THOUSAND, TRILLION

Mills' Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Mills (1947) proved the existence of a constant $\theta = 1.3064\ldots$ such that

$$\left\lfloor \theta^{3^n} \right\rfloor \tag{1}$$

is PRIME for all $n \geq 1$, where $\lfloor x \rfloor$ is the FLOOR FUNCTION. It is not, however, known if θ is IRRATIONAL. Mills' proof was based on the following theorem by Hoheisel (1930) and Ingham (1937). Let p_n be the nth PRIME, then there exists a constant K such that

$$p_{n+1} - p_n < K{p_n}^{5/8} \tag{2}$$

for all n. This has more recently been strengthened to

$$p_{n+1} - p_n < K{p_n}^{1051/1920} \tag{3}$$

(Mozzochi 1986). If the RIEMANN HYPOTHESIS is true, then Cramér (1937) showed that

$$p_{n+1} - p_n = \mathcal{O}(\ln p_n \sqrt{p_n}\,) \tag{4}$$

(Finch).

Hardy and Wright (1979) point out that, despite the beauty of such FORMULAS, they do not have any practical consequences. In fact, unless the exact value of θ is known, the PRIMES themselves must be known in advance to determine θ. A generalization of Mills' theorem to an arbitrary sequence of POSITIVE INTEGERS is given as an exercise by Ellison and Ellison (1985). Consequently, infinitely many values for θ other than the number $1.3064\ldots$ are possible.

References

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Mills, W. H. "A Prime-Representing Function." *Bull. Amer. Math. Soc.* **53**, 604, 1947.
Mozzochi, C. J. "On the Difference Between Consecutive Primes." *J. Number Th.* **24**, 181–187, 1986.
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, pp. 135 and 191–193, 1989.
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Milne's Method

A PREDICTOR-CORRECTOR METHOD for solution of ORDINARY DIFFERENTIAL EQUATIONS. The third-order equations for predictor and corrector are

$$y_{n+1} = y_{n-3} + \tfrac{4}{3}h(2y'_n - y'_{n-1} + 2y'_{n-2}) + \mathcal{O}(h^5)$$
$$y_{n+1} = y_{n-1} + \tfrac{1}{3}h(y'_{n-1} + 4y'_n + y'_{n+1}) + \mathcal{O}(h^5).$$

Abramowitz and Stegun (1972) also give the fifth order equations and formulas involving higher derivatives.

see also ADAMS' METHOD, GILL'S METHOD, PREDICTOR-CORRECTOR METHODS, RUNGE-KUTTA METHOD

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 896–897, 1972.

Milnor's Conjecture

The UNKNOTTING NUMBER for a TORUS KNOT (p, q) is $(p-1)(q-1)/2$. This 40-year-old CONJECTURE was proved (Adams 1994) in Kronheimer and Mrowka (1993, 1995).

see also TORUS KNOT, UNKNOTTING NUMBER

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, p. 113, 1994.
Kronheimer, P. B. and Mrowka, T. S. "Gauge Theory for Embedded Surfaces. I." *Topology* **32**, 773–826, 1993.
Kronheimer, P. B. and Mrowka, T. S. "Gauge Theory for Embedded Surfaces. II." *Topology* **34**, 37–97, 1995.

Milnor's Theorem

If a COMPACT MANIFOLD M has NONNEGATIVE RICCI CURVATURE, then its FUNDAMENTAL GROUP has at most POLYNOMIAL growth. On the other hand, if M has NEGATIVE curvature, then its FUNDAMENTAL GROUP has exponential growth in the sense that $n(\lambda)$ grows exponentially, where $n(\lambda)$ is (essentially) the number of different "words" of length λ which can be made in the FUNDAMENTAL GROUP.

References
Chavel, I. *Riemannian Geometry: A Modern Introduction.* New York: Cambridge University Press, 1994.

Minimal Cover

A minimal cover is a COVER for which removal of one member destroys the covering property. Let $\mu(n,k)$ be the number of minimal covers of $\{1,\ldots,n\}$ with k members. Then

$$\mu(n,k)=\frac{1}{k!}\sum_{m=k}^{\alpha_k}\binom{2^k-k-1}{m-k}m!s(n,m),$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, $s(n,m)$ is a STIRLING NUMBER OF THE SECOND KIND, and

$$\alpha_k=\min(n,2^k-1).$$

Special cases include $\mu(n,1)=1$ and $\mu(n,2)=s(n+1,3)$.

k	1	2	3	4	5	6	7
Sloane		000392	003468	016111			
n							
1	1						
2	1	1					
3	1	6	1				
4	1	25	22	1			
5	1	90	305	65	1		
6	1	301	3410	2540	171	1	
7	1	966	33621	77350	17066	420	1

see also COVER, LEW k-GRAM, STIRLING NUMBER OF THE SECOND KIND

References
Hearne, T. and Wagner, C. "Minimal Covers of Finite Sets." *Disc. Math.* **5**, 247–251, 1973.
Macula, A. J. "Lewis Carroll and the Enumeration of Minimal Covers." *Math. Mag.* **68**, 269–274, 1995.

Minimal Discriminant

see FREY CURVE

Minimal Matrix

A MATRIX with 0 DETERMINANT whose DETERMINANT becomes NONZERO when any element on or below the diagonal is changed from 0 to 1. An example is

$$\mathsf{M}=\begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 1 & 1 & 1 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

There are 2^{n-1} minimal SPECIAL MATRICES of size $n\times n$.

see also SPECIAL MATRIX

References
Knuth, D. E. "Problem 10470." *Amer. Math. Monthly* **102**, 655, 1995.

Minimal Residue

The value b or $b-m$, whichever is smaller in ABSOLUTE VALUE, where $a\equiv b \pmod m$.

see also RESIDUE (CONGRUENCE)

Minimal Set

A SET for which the dynamics can be generated by the dynamics on any subset.

Minimal Surface

Minimal surfaces are defined as surfaces with zero MEAN CURVATURE, and therefore satisfy LAGRANGE'S EQUATION

$$(1+f_y{}^2)f_{xx}+2f_xf_yf_{xy}+(1+f_x{}^2)f_{yy}=0.$$

Minimal surfaces may also be characterized as surfaces of minimal AREA for given boundary conditions. A PLANE is a trivial MINIMAL SURFACE, and the first nontrivial examples (the CATENOID and HELICOID) were found by Meusnier in 1776 (Meusnier 1785).

Euler proved that a minimal surface is planar IFF its GAUSSIAN CURVATURE is zero at every point so that it is locally SADDLE-shaped. The EXISTENCE of a solution to the general case was independently proven by Douglas (1931) and Radó (1933), although their analysis could not exclude the possibility of singularities. Osserman (1970) and Gulliver (1973) showed that a minimizing solution *cannot* have singularities.

The only known complete (boundaryless), embedded (no self-intersections) minimal surfaces of finite topology known for 200 years were the CATENOID, HELICOID, and PLANE. Hoffman discovered a three-ended GENUS 1 minimal embedded surface, and demonstrated the existence of an infinite number of such surfaces. A four-ended embedded minimal surface has also been found. L. Bers proved that any finite isolated SINGULARITY of a single-valued parameterized minimal surface is removable.

A surface can be parameterized using a ISOTHERMAL PARAMETERIZATION. Such a parameterization is minimal if the coordinate functions x_k are HARMONIC, i.e., $\phi_k(\zeta)$ are ANALYTIC. A minimal surface can therefore be defined by a triple of ANALYTIC FUNCTIONS such that $\phi_k\phi_k=0$. The REAL parameterization is then obtained as

$$x_k=\Re\int\phi_k(\zeta)\,d\zeta. \tag{1}$$

But, for an ANALYTIC FUNCTION f and a MEROMORPHIC function g, the triple of functions

$$\phi_1(\zeta) = f(1-g^2) \tag{2}$$
$$\phi_2(\zeta) = if(1+g^2) \tag{3}$$
$$\phi_3(\zeta) = 2fg \tag{4}$$

are ANALYTIC as long as f has a zero of order $\geq m$ at every POLE of g of order m. This gives a minimal surface in terms of the ENNEPER-WEIERSTRAß PARAMETERIZATION

$$\Re \int \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix} d\zeta. \tag{5}$$

see also BERNSTEIN MINIMAL SURFACE THEOREM, CALCULUS OF VARIATIONS, CATALAN'S SURFACE, CATENOID, COSTA MINIMAL SURFACE, ENNEPER-WEIERSTRAß PARAMETERIZATION, FLAT SURFACE, HENNEBERG'S MINIMAL SURFACE, HOFFMAN'S MINIMAL SURFACE, IMMERSED MINIMAL SURFACE, LICHTENFELS SURFACE, MAEDER'S OWL MINIMAL SURFACE, NIRENBERG'S CONJECTURE, PARAMETERIZATION, PLATEAU'S PROBLEM, SCHERK'S MINIMAL SURFACES, TRINOID, UNDULOID

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Osserman, R. "A Proof of the Regularity Everywhere of the Classical Solution to Plateau's Problem." *Ann. Math.* **91**, 550–569, 1970.

Radó, T. "On the Problem of Plateau." *Ergeben. d. Math. u. ihrer Grenzgebiete.* Berlin: Springer-Verlag, 1933.

Minimax Approximation

A minimization of the MAXIMUM error for a fixed number of terms.

Minimax Polynomial

The approximating POLYNOMIAL which has the smallest maximum deviation from the true function. It is closely approximated by the CHEBYSHEV POLYNOMIALS OF THE FIRST KIND.

Minimax Theorem

The fundamental theorem of GAME THEORY which states that every FINITE, ZERO-SUM, two-person GAME has optimal MIXED STRATEGIES. It was proved by John von Neumann in 1928.

Formally, let $\mathbf{X}$ and $\mathbf{Y}$ be MIXED STRATEGIES for players A and B. Let A be the PAYOFF MATRIX. Then

$$\max_X \min_Y \mathbf{X}^{\mathrm{T}} \mathsf{A} \mathbf{Y} = \min_Y \max_X \mathbf{X}^{\mathrm{T}} \mathsf{A} \mathbf{Y} = v,$$

where v is called the VALUE of the GAME and $\mathbf{X}$ and $\mathbf{Y}$ are called the solutions. It also turns out that if there is more than one optimal MIXED STRATEGY, there are infinitely many.

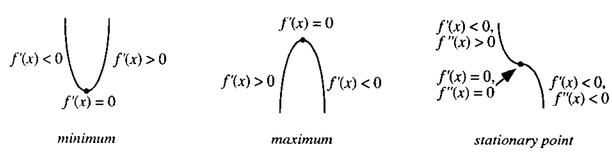
see also MIXED STRATEGY

References

Willem, M. *Minimax Theorem.* Boston, MA: Birkhäuser, 1996.

Minimum

The smallest value of a set, function, etc. The minimum value of a set of elements $A = \{a_i\}_{i=1}^N$ is denoted $\min A$ or $\min_i a_i$, and is equal to the first element of a sorted (i.e., ordered) version of A. For example, given the set $\{3, 5, 4, 1\}$, the sorted version is $\{1, 3, 4, 5\}$, so the minimum is 1. The MAXIMUM and minimum are the simplest ORDER STATISTICS.



A continuous FUNCTION may assume a minimum at a single point or may have minima at a number of points. A GLOBAL MINIMUM of a FUNCTION is the smallest value in the entire RANGE of the FUNCTION, while a LOCAL MINIMUM is the smallest value in some local neighborhood.

For a function $f(x)$ which is CONTINUOUS at a point x_0, a NECESSARY but not SUFFICIENT condition for $f(x)$ to have a RELATIVE MINIMUM at $x = x_0$ is that x_0 be a CRITICAL POINT (i.e., $f(x)$ is either not DIFFERENTIABLE at x_0 or x_0 is a STATIONARY POINT, in which case $f'(x_0) = 0$).

The FIRST DERIVATIVE TEST can be applied to CONTINUOUS FUNCTIONS to distinguish minima from MAXIMA. For twice differentiable functions of one variable, $f(x)$, or of two variables, $f(x,y)$, the SECOND DERIVATIVE TEST can sometimes also identify the nature of an EXTREMUM. For a function $f(x)$, the EXTREMUM TEST succeeds under more general conditions than the SECOND DERIVATIVE TEST.

see also CRITICAL POINT, EXTREMUM, FIRST DERIVATIVE TEST, GLOBAL MAXIMUM, INFLECTION POINT, LOCAL MAXIMUM, MAXIMUM, MIDRANGE, ORDER STATISTIC, SADDLE POINT (FUNCTION), SECOND DERIVATIVE TEST, STATIONARY POINT

References

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Minkowski-Bouligand Dimension

In many cases, the HAUSDORFF DIMENSION correctly describes the correction term for a resonator with FRACTAL PERIMETER in Lorentz's conjecture. However, in general, the proper dimension to use turns out to be the Minkowski-Bouligand dimension (Schroeder 1991).

Let $F(r)$ be the AREA traced out by a small CIRCLE with RADIUS r following a fractal curve. Then, providing the LIMIT exists,

$$D_M \equiv \lim_{r\to 0} \frac{\ln F(r)}{-\ln r} + 2$$

(Schroeder 1991). It is conjectured that for all strictly self-similar fractals, the Minkowski-Bouligand dimension is equal to the HAUSDORFF DIMENSION D; otherwise $D_M > D$.

see also HAUSDORFF DIMENSION

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Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, pp. 41–45, 1991.

Minkowski Convex Body Theorem

A bounded plane convex region symmetric about a LATTICE POINT and with AREA > 4 must contain at least three LATTICE POINTS in the interior. In n-D, the theorem can be generalized to a region with AREA $> 2^n$, which must contain at least three LATTICE POINTS. The theorem can be derived from BLICHFELDT'S THEOREM.

see also BLICHFELDT'S THEOREM

Minkowski Geometry

see MINKOWSKI SPACE

Minkowski-Hlawka Theorem

There exist lattices in n-D having HYPERSPHERE PACKING densities satisfying

$$\eta \geq \frac{\zeta(n)}{2^{n-1}},$$

where $\zeta(n)$ is the RIEMANN ZETA FUNCTION. However, the proof of this theorem is nonconstructive and it is still not known how to actually construct packings that are this dense.

see also HERMITE CONSTANTS, HYPERSPHERE PACKING

References

Conway, J. H. and Sloane, N. J. A. *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, pp. 14–16, 1993.

Minkowski Integral Inequality

If $p > 1$, then

$$\left[\int_a^b |f(x)+g(x)|^p\,dx\right]^{1/p} \leq \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} + \left[\int_a^b |g(x)|^p\,dx\right]^{1/p}.$$

see also MINKOWSKI SUM INEQUALITY

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 11, 1972.

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Minkowski, H. *Geometrie der Zahlen, Vol. 1.* Leipzig, Germany: pp. 115–117, 1896.

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Minkowski Measure

The Minkowski measure of a bounded, CLOSED SET is the same as its LEBESGUE MEASURE.

References

Ko, K.-I. "A Polynomial-Time Computable Curve whose Interior has a Nonrecursive Measure." *Theoret. Comput. Sci.* **145**, 241–270, 1995.

Minkowski Metric

In CARTESIAN COORDINATES,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

$$d\tau^2 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2, \tag{2}$$

and

$$g_{\alpha\beta} \equiv \eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

In SPHERICAL COORDINATES,

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \tag{4}$$

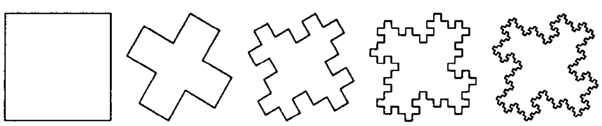
$$d\tau^2 = -c^2\,dt^2 + dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2, \tag{5}$$

and

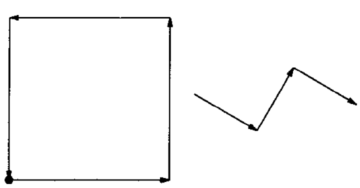
$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{6}$$

see also LORENTZ TRANSFORMATION, MINKOWSKI SPACE

Minkowski Sausage



A FRACTAL created from the base curve and motif illustrated below.



The number of segments after the nth iteration is

$$N_n = 8^n,$$

and

$$\epsilon_n = \left(\tfrac{1}{4}\right)^n,$$

so the CAPACITY DIMENSION is

$$D \equiv -\lim_{n\to\infty} \frac{\ln N_n}{\ln \epsilon_n} = -\lim_{n\to\infty} \frac{\ln 8^n}{\ln 4^n} = \frac{\ln 8}{\ln 4} = \frac{3\ln 2}{2\ln 2} = \frac{3}{2}.$$

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Weisstein, E. W. "Fractals." http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m.

Minkowski Space

A 4-D space with the MINKOWSKI METRIC. Alternatively, it can be considered to have a EUCLIDEAN METRIC, but with its VECTORS defined by

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ict \\ x \\ y \\ z \end{bmatrix}, \tag{1}$$

where c is the speed of light. The METRIC is DIAGONAL with

$$g_{\alpha\alpha} = \frac{1}{g_{\alpha\alpha}}, \tag{2}$$

so

$$\eta^{\beta\delta} = \eta_{\beta\delta}. \tag{3}$$

Let Λ be the TENSOR for a LORENTZ TRANSFORMATION. Then

$$\eta^{\beta\delta}\Lambda^{\gamma}{}_{\delta} = \Lambda^{\beta\gamma} \tag{4}$$

$$\eta_{\alpha\gamma}\Lambda^{\beta\gamma} = \Lambda_{\alpha}{}^{\beta} \tag{5}$$

$$\Lambda_{\alpha}{}^{\beta} = \eta_{\alpha\gamma}\Lambda^{\beta\gamma} = \eta_{\alpha\gamma}\eta^{\beta\delta}\Lambda^{\gamma}{}_{\delta}. \tag{6}$$

The NECESSARY and SUFFICIENT conditions for a metric $g_{\mu\nu}$ to be equivalent to the Minkowski metric $\eta_{\alpha\beta}$ are that the RIEMANN TENSOR vanishes everywhere ($R^{\lambda}{}_{\mu\nu\kappa} = 0$) and that at some point $g^{\mu\nu}$ has three POSITIVE and one NEGATIVE EIGENVALUES.

see also LORENTZ TRANSFORMATION, MINKOWSKI METRIC

References

Thompson, A. C. *Minkowski Geometry.* New York: Cambridge University Press, 1996.

Minkowski Sum

The sum of sets A and B in a VECTOR SPACE, equal to $\{a+b : a \in A, b \in B\}$.

Minkowski Sum Inequality

If $p > 1$ and a_k, $b_k > 0$, then

$$\left[\sum_{k=1}^{n} (a_k + b_k)^p\right]^{1/p} \leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{1/p} + \left(\sum_{k=1}^{n} {b_k}^p\right)^{1/p}.$$

Equality holds IFF the sequences a_1, a_2, ... and b_1, b_2, ... are proportional.

see also MINKOWSKI INTEGRAL INEQUALITY

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Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 11, 1972.
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Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 24–26, 1988.

Minor

The reduced DETERMINANT of a DETERMINANT EXPANSION, denoted M_{ij}, which is formed by omitting the ith row and jth column.

see also COFACTOR, DETERMINANT, DETERMINANT EXPANSION BY MINORS

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 169–170, 1985.

Minor Axis

see SEMIMINOR AXIS

Minor Graph

A "minor" is a sort of SUBGRAPH and is what Kuratowski means when he says "contain." It is roughly a small graph which can be mapped into the big one without merging VERTICES.

Minus

The operation of SUBTRACTION, i.e., a minus b. The operation is denoted $a-b$. The MINUS SIGN "$-$" is also used to denote a NEGATIVE number, i.e., $-x$.

see also MINUS SIGN, NEGATIVE, PLUS, PLUS OR MINUS, TIMES

Minus or Plus

see PLUS OR MINUS

Minus Sign

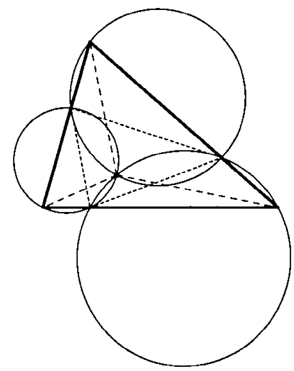
The symbol "$-$" which is used to denote a NEGATIVE number or SUBTRACTION.

see also MINUS, PLUS SIGN, SIGN, SUBTRACTION

Minute

see ARC MINUTE

Miquel Circles



For a TRIANGLE ΔABC and three points A', B', and C', one on each of its sides, the three Miquel circles are the circles passing through each VERTEX and its neighboring side points (i.e., $AC'B'$, $BA'C'$, and $CB'A'$). According to MIQUEL'S THEOREM, the Miquel circles are CONCURRENT in a point M known as the MIQUEL POINT. Similarly, there are n Miquel circles for n lines taken $(n-1)$ at a time.

see also MIQUEL POINT, MIQUEL'S THEOREM, MIQUEL TRIANGLE

Miquel Equation

$$\measuredangle A_2MA_3 = \measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3,$$

where $\measuredangle$ is a DIRECTED ANGLE.

see also DIRECTED ANGLE, MIQUEL'S THEOREM

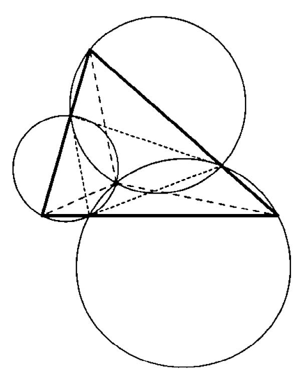
References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 131–144, 1929.

Miquel Point

The point of CONCURRENCE of the MIQUEL CIRCLES.

see also MIQUEL CIRCLES, MIQUEL'S THEOREM, MIQUEL TRIANGLE

Miquel's Theorem



If a point is marked on each side of a TRIANGLE ΔABC, then the three MIQUEL CIRCLES (each through a VERTEX and the two marked points on the adjacent sides) are CONCURRENT at a point M called the MIQUEL POINT. This result is a slight generalization of the so-called PIVOT THEOREM.

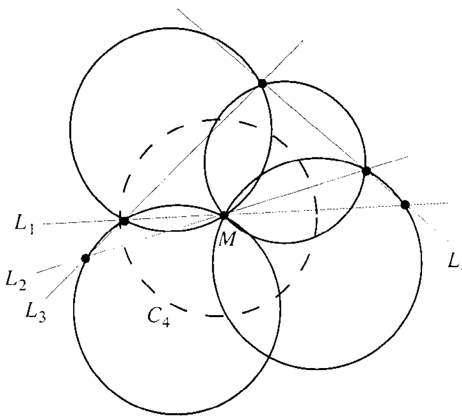
If M lies in the interior of the triangle, then it satisfies

$$\angle P_2MP_3 = 180^\circ - \alpha_1$$

$$\angle P_3MP_1 = 180^\circ - \alpha_2$$

$$\angle P_1MP_2 = 180^\circ - \alpha_3.$$

The lines from the MIQUEL POINT to the marked points make equal angles with the respective sides. (This is a by-product of the MIQUEL EQUATION.)



Given four lines $L_1, \ldots, L_4$ each intersecting the other three, the four MIQUEL CIRCLES passing through each subset of three intersection points of the lines meet in a point known as the 4-Miquel point M. Furthermore, the centers of these four MIQUEL CIRCLES lie on a CIRCLE C_4 (Johnson 1929, p. 139). The lines from M to given points on the sides make equal ANGLES with respect to the sides.

Similarly, given n lines taken by $(n-1)$s yield n MIQUEL CIRCLES like C_4 passing through a point P_n, and their centers lie on a CIRCLE C_{n+1}.

see also MIQUEL CIRCLES, MIQUEL EQUATION, MIQUEL TRIANGLE, NINE-POINT CIRCLE, PEDAL CIRCLE, PIVOT THEOREM

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 131–144, 1929.

Miquel Triangle

Given a point P and a triangle $\Delta A_1A_2A_3$, the Miquel triangle is the triangle connecting the side points P_1, P_2, and P_3 of $\Delta A_1A_2A_3$ with respect to which P is the MIQUEL POINT. All Miquel triangles of a given point M are directly similar, and M is the SIMILITUDE CENTER in every case.

Mira Fractal

A FRACTAL based on the map

$$F(x) = ax + \frac{2(1-a)x^2}{1+x^2}.$$

References

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, p. 136, 1991.

Mirimanoff's Congruence

If the first case of FERMAT'S LAST THEOREM is false for the PRIME exponent p, then $3^{p-1} \equiv 1 \pmod{p^2}$.

see also FERMAT'S LAST THEOREM

Mirror Image

An image of an object obtained by reflecting it in a mirror so that the signs of one of its coordinates are reversed.

see AMPHICHIRAL, CHIRAL, ENANTIOMER, HANDEDNESS

Mirror Plane

The SYMMETRY OPERATION $(x, y, z) \to (x, y, -z)$, etc., which is equivalent to $\bar{2}$, where the bar denotes an IMPROPER ROTATION.

Misère Form

A version of NIM-like GAMES in which the player taking the last piece is the loser. For most IMPARTIAL GAMES, this form is much harder to analyze, but it requires only a trivial modification for the game of NIM.

Mitchell Index

The statistical INDEX

$$P_M \equiv \frac{\sum p_n q_a}{\sum p_0 q_a},$$

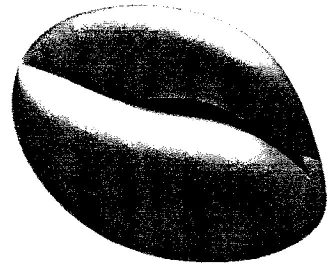
where p_n is the price per unit in period n and q_n is the quantity produced in period n.

see also INDEX

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 66–67, 1962.

Miter Surface



A QUARTIC SURFACE named after its resemblance to the liturgical headdress worn by bishops and given by the equation

$$4x^2(x^2+y^2+z^2)-y^2(1-y^2-z^2)=0.$$

see also QUARTIC SURFACE

References
Nordstrand, T. "Surfaces." http://www.uib.no/people/nfytn/surfaces.htm.

Mittag-Leffler Function

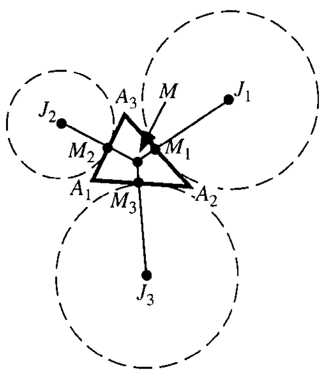
$$E_\gamma(x)\equiv\sum_{k=0}^\infty\frac{x^k}{\Gamma(\gamma k+1)}.$$

It is related to the GENERALIZED HYPERBOLIC FUNCTIONS by

$$F_{n,0}^1(x)=E_n(x^n).$$

References
Muldoon, M. E. and Ungar, A. A. "Beyond Sin and Cos." *Math. Mag.* **69**, 3–14, 1996.

Mittenpunkt



The LEMOINE POINT of the EXCENTRAL TRIANGLE, i.e., the point of concurrence M of the lines from the EXCENTERS J_i through the corresponding TRIANGLE side MIDPOINT M_i. It is also called the MIDDLESPOINT and has TRIANGLE CENTER FUNCTION

$$\alpha=b+c-a=\tfrac{1}{2}\cot A.$$

see also EXCENTER, EXCENTRAL TRIANGLE, NAGEL POINT

References
Baptist, P. *Die Entwicklung der Neueren Dreiecksgeometrie.* Mannheim: Wissenschaftsverlag, p. 72, 1992.
Eddy, R. H. "A Generalization of Nagel's Middlespoint." *Elem. Math.* **45**, 14–18, 1990.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. "Mittenpunkt." http://www.evansville.edu/~ck6/tcenters/class/mitten.html.

Mixed Partial Derivative

A PARTIAL DERIVATIVE of second or greater order with respect to two or more different variables, for example

$$f_{xy}=\frac{\partial^2 f}{\partial x\partial y}.$$

If the mixed partial derivatives exist and are continuous at a point $\mathbf{x}_0$, then they are equal at $\mathbf{x}_0$ regardless of the order in which they are taken.

see also PARTIAL DERIVATIVE

Mixed Strategy

A collection of moves together with a corresponding set of weights which are followed probabilistically in the playing of a GAME. The MINIMAX THEOREM of GAME THEORY states that every finite, zero-sum, two-person game has optimal mixed strategies.

see also GAME THEORY, MINIMAX THEOREM, STRATEGY

Mixed Tensor

A TENSOR having CONTRAVARIANT and COVARIANT indices.

see also CONTRAVARIANT TENSOR, COVARIANT TENSOR, TENSOR

Mnemonic

A mental device used to aid memorization. Common mnemonics for mathematical constants such as e and PI consist of sentences in which the number of letters in each word give successive digits.

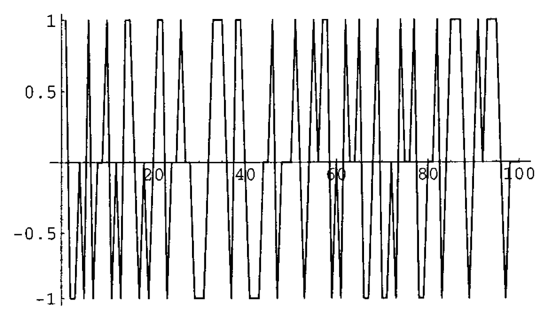
see also e, JOSEPHUS PROBLEM, PI

References
Luria, A. R. *The Mind of a Mnemonist: A Little Book about a Vast Memory.* Cambridge, MA: Harvard University Press, 1987.

Möbius Band

see MÖBIUS STRIP

Möbius Function



$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \end{cases}$$

so $mu(n) \neq 0$ indicates that n is SQUAREFREE. The first few values are 1, -1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, ... (Sloane's A008683).

The SUMMATORY FUNCTION of the Möbius function is called MERTENS FUNCTION.

see also BRAUN'S CONJECTURE, MERTENS FUNCTION, MÖBIUS INVERSION FORMULA, MÖBIUS PERIODIC FUNCTION, PRIME ZETA FUNCTION, RIEMANN FUNCTION, SQUAREFREE

References
Abramowitz, M. and Stegun, C. A. (Eds.). "The Möbius Function." §24.3.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 826, 1972.
Deléglise, M. and Rivat, J. "Computing the Summation of the Möbius Function." *Experiment. Math.* **5**, 291–295, 1996.
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford: Clarendon Press, p. 236, 1979.
Sloane, N. J. A. Sequence A008683 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, pp. 7–8 and 223–225, 1991.

Möbius Group

The equation

$$x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2 - 2x_0x_\infty = 0$$

represents an n-D HYPERSPHERE $\mathbb{S}^n$ as a quadratic hypersurface in an $(n+1)$-D real projective space $\mathbb{P}^{n+1}$, where x_a are homogeneous coordinates in $\mathbb{P}^{n+1}$. Then the GROUP $M(n)$ of projective transformations which leave $\mathbb{S}^n$ invariant is called the Möbius group.

References
Iyanaga, S. and Kawada, Y. (Eds.). "Möbius Geometry." §78A in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 265–266, 1980.

Möbius Inversion Formula

If $g(n) \equiv \sum_{d|n} f(d)$, then

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right),$$

where the sums are over all possible INTEGERS d that DIVIDE n and $\mu(d)$ is the MÖBIUS FUNCTION. The LOGARITHM of the CYCLOTOMIC POLYNOMIAL

$$\Phi_n(x) = \prod_{d|n} (1 - x^{n/d})^{\mu(d)}$$

is the Möbius inversion formula.

see also CYCLOTOMIC POLYNOMIAL, MÖBIUS FUNCTION

References
Hardy, G. H. and Wright, W. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Oxford University Press, pp. 91–93, 1979.
Schroeder, M. R. *Number Theory in Science and Communication, 3rd ed.* New York: Springer-Verlag, 1997.
Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, pp. 7–8 and 223–225, 1991.

Möbius Periodic Function

A function periodic with period 2π such that

$$p(\theta + \pi) = -p(\theta)$$

for all θ is said to be Möbius periodic.

Möbius Problem

Let $A = \{a_1, a_2, \ldots\}$ be a free Abelian SEMIGROUP, where a_1 is the unit element. Then do the following properties,

1. $a < b$ IMPLIES $ac < bc$ for $a, b, c \in A$, where A has the linear order $a_1 < a_2 < \ldots$,
2. $\mu(a_n) = \mu(n)$ for all n,

imply that

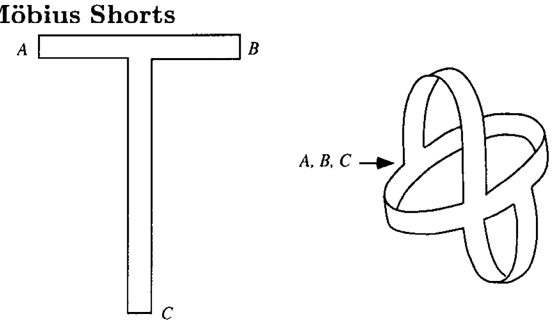
$$a_{mn} = a_m a_n$$

for all $m, n \geq 1$? The problem is known to be true for $mn \leq 74$ for all $n \leq 240$.

see also BRAUN'S CONJECTURE, MÖBIUS FUNCTION

References
Flath, A. and Zulauf, A. "Does the Möbius Function Determine Multiplicative Arithmetic?" *Amer. Math. Monthly* **102**, 354–256, 1995.

Möbius Shorts

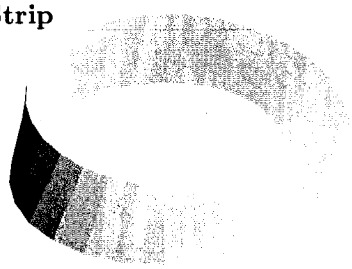


A one-sided surface reminiscent of the MÖBIUS STRIP.

see also MÖBIUS STRIP

References
Boas, R. P. Jr. "Möbius Shorts." *Math. Mag.* **68**, 127, 1995.

Möbius Strip



A one-sided surface obtained by cutting a band widthwise, giving it a half twist, and re-attaching the two ends. According to Madachy (1979), the B. F.Goodrich Company patented a conveyor belt in the form of a Möbius strip which lasts twice as long as conventional belts.

A Möbius strip can be represented parametrically by

$$\begin{aligned} x &= [R + s\cos(\tfrac{1}{2}\theta)]\cos\theta \\ y &= [R + s\cos(\tfrac{1}{2}\theta)]\sin\theta \\ z &= s\sin(\tfrac{1}{2}\theta), \end{aligned}$$

for $s \in [-1, 1]$ and $\theta \in [0, 2\pi)$. Cutting a Möbius strip, giving it extra twists, and reconnecting the ends produces unexpected figures called PARADROMIC RINGS (Listing and Tait 1847, Ball and Coxeter 1987) which are summarized in the table below.

half-twists	cuts	divs.	result
1	1	2	1 band, length 2
1	1	3	1 band, length 2
			1 Möbius strip, length 1
1	2	4	2 bands, length 2
1	2	5	2 bands, length 2
			1 Möbius strip, length 1
1	3	6	3 bands, length 2
1	3	7	3 bands, length 2
			1 Möbius strip, length 1
2	1	2	2 bands, length 1
2	2	3	3 bands, length 1
2	3	4	4 bands, length 1

A TORUS can be cut into a Möbius strip with an EVEN number of half-twists, and a KLEIN BOTTLE can be cut in half along its length to make two Möbius strips. In addition, two strips on top of each other, each with a half-twist, give a single strip with four twists when disentangled.

There are three possible SURFACES which can be obtained by sewing a Möbius strip to the edge of a DISK: the BOY SURFACE, CROSS-CAP, and ROMAN SURFACE.

The Möbius strip has EULER CHARACTERISTIC 1, and the HEAWOOD CONJECTURE therefore shows that any set of regions on it can be colored using six-colors only.

see also BOY SURFACE, CROSS-CAP, MAP COLORING, PARADROMIC RINGS, PRISMATIC RING, ROMAN SURFACE

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 127–128, 1987.

Bogomolny, A. "Möbius Strip." `http://www.cut-the-knot.com/do_you_know/moebius.html`.

Gardner, M. "Möbius Bands." Ch. 9 in *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions and Other Mathematical Sleight-of-Mind from Scientific American.* New York: Vintage, pp. 123–136, 1978.

Geometry Center. "The Klein Bottle." `http://www.geom.umn.edu/zoo/features/mobius/`.

Gray, A. "The Möbius Strip." §12.3 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 236–238, 1993.

Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, pp. 41–45, 1975.

Kraitchik, M. §8.4.3 in *Mathematical Recreations.* New York: W. W. Norton, pp. 212–213, 1942.

Listing and Tait. *Vorstudien zur Topologie, Göttinger Studien,* Pt. 10, 1847.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, p. 7, 1979.

Nordstrand, T. "Mobiusband." `http://www.uib.no/people/nfytn/moebtxt.htm`.

Pappas, T. "The Moebius Strip & the Klein Bottle," "A Twist to the Moebius Strip," "The 'Double' Moebius Strip." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 207, 1989.

Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, pp. 269–274, 1983.

Wagon, S. "Rotating Circles to Produce a Torus or Möbius Strip." §7.4 in *Mathematica in Action.* New York: W. H. Freeman, pp. 229–232, 1991.

Wang, P. "Renderings." `http://www.ugcs.caltech.edu/~peterw/portfolio/renderings/`.

Möbius Transformation

A transformation of the form

$$w = f(z) = \frac{az + b}{cz + d},$$

where a, b, c, $d \in \mathbb{C}$ and

$$ad - bc \neq 0,$$

is a CONFORMAL TRANSFORMATION and is called a Möbius transformation. It is linear in both w and z.

Every Möbius transformation except $f(z) = z$ has one or two FIXED POINTS. The Möbius transformation sends CIRCLES and lines to CIRCLES or lines. Möbius transformations preserve symmetry. The CROSS-RATIO is invariant under a Möbius transformation. A Möbius transformation is a composition of translations, rotations, magnifications, and inversions.

To determine a particular Möbius transformation, specify the map of three points which preserve orientation. A particular Möbius transformation is then uniquely determined. To determine a general Möbius transformation, pick two symmetric points α and α_S. Define $\beta \equiv f(\alpha)$, restricting β as required. Compute β_S. $f(\alpha_S)$ then equals β_S since the Möbius transformation preserves symmetry (the SYMMETRY PRINCIPLE). Plug in α and α_S into the general Möbius transformation and set equal to β and β_S. Without loss of generality, let $c = 1$ and solve for a and b in terms of β. Plug back into the general expression to obtain a Möbius transformation.

see also SYMMETRY PRINCIPLE

Möbius Triangles

SPHERICAL TRIANGLES into which a SPHERE is divided by the planes of symmetry of a UNIFORM POLYHEDRON.

see also SPHERICAL TRIANGLE, UNIFORM POLYHEDRON

Mock Theta Function

Ramanujan was the first to extensively study these THETA FUNCTION-like functions

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}.$$

see also q-SERIES, THETA FUNCTION

References

Bellman, R. E. *A Brief Introduction to Theta Functions.* New York: Holt, Rinehart, and Winston, 1961.

Mod

see CONGRUENCE

Mode

The most common value obtained in a set of observations.

see also MEAN, MEDIAN (STATISTICS), ORDER STATISTIC

Mode Locking

A phenomenon in which a system being forced at an IRRATIONAL period undergoes rational, periodic motion which persists for a finite range of forcing values. It may occur for strong couplings between natural and forcing oscillation frequencies.

The phenomenon can be exemplified in the CIRCLE MAP when, after q iterations of the map, the new angle differs from the initial value by a RATIONAL NUMBER

$$\theta_{n+q} = \theta_n + \frac{p}{q}.$$

This is the form of the unperturbed CIRCLE MAP with the WINDING NUMBER

$$\Omega = \frac{p}{q}.$$

For Ω not a RATIONAL NUMBER, the trajectory is QUASIPERIODIC.

see also CHAOS, QUASIPERIODIC FUNCTION

Model Completion

Model completion is a term employed when EXISTENTIAL CLOSURE is successful. The formation of the COMPLEX NUMBERS, and the move from affine to projective geometry, are successes of this kind. The theory of existential closure gives a theoretical basis of Hilbert's "method of ideal elements."

References

Manders, K. L. "Interpretations and the Model Theory of the Classical Geometries." In *Models and Sets.* Berlin: Springer-Verlag, pp. 297–330, 1984.
Manders, K. L. "Domain Extension and the Philosophy of Mathematics." *J. Philos.* **86**, 553–562, 1989.

Model Theory

Model theory is a general theory of interpretations of an AXIOMATIC SET THEORY. It is the branch of LOGIC studying mathematical structures by considering first-order sentences which are true of those structures and the sets which are definable in those structures by first-order FORMULAS (Marker 1996).

Mathematical structures obeying axioms in a system are called "models" of the system. The usual axioms of ANALYSIS are second order and are known to have the REAL NUMBERS as their unique model. Weakening the axioms to include only the first-order ones leads to a new type of model in what is called NONSTANDARD ANALYSIS.

see also KHOVANSKI'S THEOREM, NONSTANDARD ANALYSIS, WILKIE'S THEOREM

References

Doets, K. *Basic Model Theory.* New York: Cambridge University Press, 1996.
Marker, D. "Model Theory and Exponentiation." *Not. Amer. Math. Soc.* **43**, 753–759, 1996.
Stewart, I. "Non-Standard Analysis." In *From Here to Infinity: A Guide to Today's Mathematics.* Oxford, England: Oxford University Press, pp. 80–81, 1996.

Modified Bessel Differential Equation

The second-order ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0.$$

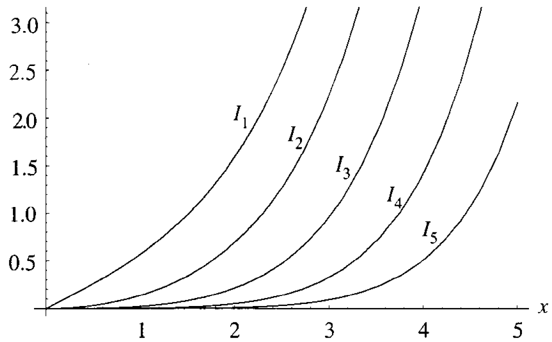
The solutions are the MODIFIED BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS. If $n = 0$, the modified Bessel differential equation becomes

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2y = 0,$$

which can also be written

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = xy.$$

Modified Bessel Function of the First Kind



A function $I_n(x)$ which is one of the solutions to the MODIFIED BESSEL DIFFERENTIAL EQUATION and is closely related to the BESSEL FUNCTION OF THE FIRST KIND $J_n(x)$. The above plot shows $I_n(x)$ for $n = 1, 2, \ldots, 5$. In terms of $J_n(x)$,

$$I_n(x) \equiv i^{-n} J_n(ix) = e^{-n\pi i/2} J_n(xe^{i\pi/2}). \qquad (1)$$

For a REAL NUMBER ν, the function can be computed using

$$I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}, \qquad (2)$$

where $\Gamma(z)$ is the GAMMA FUNCTION. An integral formula is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}\,dt, \qquad (3)$$

which simplifies for ν an INTEGER n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta \qquad (4)$$

(Abramowitz and Stegun 1972, p. 376).

A derivative identity for expressing higher order modified Bessel functions in terms of $I_0(x)$ is

$$I_n(x) = T_n\left(\frac{d}{dx}\right) I_0(x), \qquad (5)$$

where $T_n(x)$ is a CHEBYSHEV POLYNOMIAL OF THE FIRST KIND.

see also BESSEL FUNCTION OF THE FIRST KIND, MODIFIED BESSEL FUNCTION OF THE FIRST KIND, WEBER'S FORMULA

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Modified Bessel Functions I and K." §9.6 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 374–377, 1972.

Arfken, G. "Modified Bessel Functions, $I_\nu(x)$ and $K_\nu(x)$." §11.5 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 610–616, 1985.

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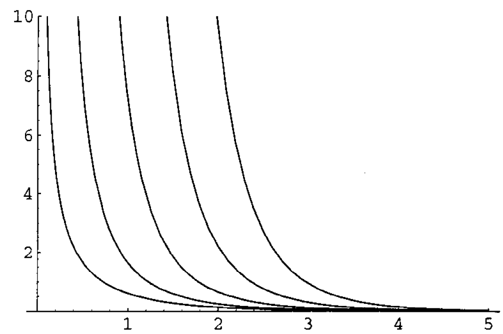
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Bessel Functions of Fractional Order, Airy Functions, Spherical Bessel Functions." §6.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 234–245, 1992.

Spanier, J. and Oldham, K. B. "The Hyperbolic Bessel Functions $I_0(x)$ and $I_1(x)$" and "The General Hyperbolic Bessel Function $I_\nu(x)$." Chs. 49–50 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 479–487 and 489–497, 1987.

Modified Bessel Function of the Second Kind



The function $K_n(x)$ which is one of the solutions to the MODIFIED BESSEL DIFFERENTIAL EQUATION. The above plot shows $K_n(x)$ for $n = 1, 2, \ldots, 5$. $K_n(x)$ is closely related to the MODIFIED BESSEL FUNCTION OF THE FIRST KIND $I_n(x)$ and HANKEL FUNCTION $H_n(x)$,

$$K_n(x) \equiv \tfrac{1}{2}\pi i^{n+1} H_n^{(1)}(ix) \qquad (1)$$

$$= \tfrac{1}{2}\pi i^{n+1}[J_n(ix) + iN_n(ix)] \qquad (2)$$

$$= \frac{\pi}{2}\frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)} \qquad (3)$$

(Watson 1966, p. 185). A sum formula for K_n is

$$K_n(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\tfrac{1}{4}z^2)^k + (-1)^{n+1}\ln(\tfrac{1}{2}z)I_n(z) + (-1)^n\tfrac{1}{2}(\tfrac{1}{2}z)^n\sum_{k=0}^{\infty}[\psi(k+1)+\psi(n+k+1)]\frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}, \qquad (4)$$

where ψ is the DIGAMMA FUNCTION (Abramowitz and Stegun 1972). An integral formula is

$$K_\nu(z) = \frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\sqrt{\pi}}\int_0^\infty \frac{\cos t\,dt}{(t^2+z^2)^{\nu+1/2}} \qquad (5)$$

which, for $\nu = 0$, simplifies to

$$K_0(x) = \int_0^\infty \cos(x \sinh t)\, dt = \int_0^\infty \frac{\cos(xt)\, dt}{\sqrt{t^2+1}}. \quad (6)$$

Other identities are

$$K_n(z) = \frac{\sqrt{\pi}}{(n-\frac{1}{2})!}(\tfrac{1}{2}z)^n \int_1^\infty e^{-zx}(x^2-1)^{n-1/2}\, dx \quad (7)$$

for $n > -1/2$ and

$$K_n(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{(n-\frac{1}{2})!} \int_0^\infty e^{-t} t^{n-1/2} \left(1 - \frac{t}{2z}\right)^{n-1/2} dt \quad (8)$$

$$= \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{(n-\frac{1}{2})!} \sum_{r=0}^\infty \frac{(n-\frac{1}{2})!}{r!(n-r-\frac{1}{2})!}(2z)^{-r} \times \int_0^\infty e^{-t} t^{n+r-1/2}\, dt. \quad (9)$$

The modified Bessel function of the second kind is sometimes called the BASSET FUNCTION.

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Modified Bessel Functions I and K." §9.6 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 374–377, 1972.
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Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Modified Bessel Functions of Integral Order" and "Bessel Functions of Fractional Order, Airy Functions, Spherical Bessel Functions." §6.6 and 6.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 229–245, 1992.
Spanier, J. and Oldham, K. B. "The Basset $K_\nu(x)$." Ch. 51 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 499–507, 1987.
Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1966.

Modified Spherical Bessel Differential Equation

The SPHERICAL BESSEL DIFFERENTIAL EQUATION with a NEGATIVE separation constant, given by

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - [k^2r^2 + n(n+1)]R = 0.$$

The solutions are called MODIFIED SPHERICAL BESSEL FUNCTIONS.

Modified Spherical Bessel Function

Solutions to the MODIFIED SPHERICAL BESSEL DIFFERENTIAL EQUATION, given by

$$i_n(x) \equiv \sqrt{\frac{\pi}{2x}}\, I_{n+1/2}(x) \quad (1)$$

$$i_0(x) = \frac{\sinh(x)}{x} \quad (2)$$

$$k_n(x) \equiv \sqrt{\frac{2\pi}{x}}\, K_{n+1/2}(x) \quad (3)$$

$$k_0(x) = \frac{e^{-x}}{x}, \quad (4)$$

where $I_n(x)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND and $K_n(x)$ is a MODIFIED BESSEL FUNCTION OF THE SECOND KIND.

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Modified Spherical Bessel Functions." §10.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 443–445, 1972.

Modified Struve Function

$$\mathcal{L}_\nu(z) = \left(\tfrac{1}{2}z\right)^{\nu+1} \sum_{k=0}^\infty \frac{\left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

$$= \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^{\pi/2} \sinh(z\cos\theta)\sin^{2\nu}\theta\, d\theta,$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

see also ANGER FUNCTION, STRUVE FUNCTION, WEBER FUNCTIONS

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Modified Struve Function $\mathbf{L}_\nu(x)$." §12.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 498, 1972.

Modular Angle

Given a MODULUS k in an ELLIPTIC INTEGRAL, the modular angle is defined by $k \equiv \sin\alpha$. An ELLIPTIC INTEGRAL is written $I(\phi|m)$ when the PARAMETER is used, $I(\phi, k)$ when the MODULUS is used, and $I(\phi\backslash\alpha)$ when the modular angle is used.

see also AMPLITUDE, CHARACTERISTIC (ELLIPTIC INTEGRAL), ELLIPTIC INTEGRAL, MODULUS (ELLIPTIC INTEGRAL), NOME, PARAMETER

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 590, 1972.

Modular Equation

The modular equation of degree n gives an algebraic connection of the form

$$\frac{K'(l)}{K(l)} = n\frac{K'(k)}{K(k)} \tag{1}$$

between the TRANSCENDENTAL COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND with moduli k and l. When k and l satisfy a modular equation, a relationship of the form

$$\frac{M(l,k)\,dy}{\sqrt{(1-y^2)(1-l^2y^2)}} = \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2}$$

exists, and M is called the MODULAR FUNCTION MULTIPLIER. In general, if p is an ODD PRIME, then the modular equation is given by

$$\Omega_p(u,v) = (v-u_0)(v-u_1)\cdots(v-u_p), \tag{3}$$

where

$$u_p \equiv (-1)^{(p^2-1)/8}[\lambda(q^p)]^{1/8} \equiv (-1)^{(p^2-1)/8}u(q^p), \tag{4}$$

λ is a ELLIPTIC LAMBDA FUNCTION, and

$$q \equiv e^{i\pi t} \tag{5}$$

(Borwein and Borwein 1987, p. 126). An ELLIPTIC INTEGRAL identity gives

$$\frac{K'(k)}{K(k)} = 2\frac{K'\left(\frac{2\sqrt{k}}{1+k}\right)}{K\left(\frac{2\sqrt{k}}{1+k}\right)}, \tag{6}$$

so the modular equation of degree 2 is

$$l = \frac{2\sqrt{k}}{1+k} \tag{7}$$

which can be written as

$$l^2(1+k)^2 = 4k. \tag{8}$$

A few low order modular equations written in terms of k and l are

$$\Omega_2 = l^2(1+k)^2 - 4k = 0 \tag{9}$$

$$\Omega_7 = (kl)^{1/4} + (k'l')^{1/4} - 1 = 0 \tag{10}$$

$$\Omega_{23} = (kl)^{1/4} + (k'l')^{1/4} + 2^{2/3}(klk'l')^{1/12} - 1 = 0. \tag{11}$$

In terms of u and v,

$$\Omega_3(u,v) = u^4 - v^4 + 2uv(1-u^2v^2) = 0 \tag{12}$$

$$\begin{aligned}\Omega_5(u,v) &= v^6 - u^6 + 5u^2v^2(v^2-u^2) + 4uv(u^4v^4-1)\\ &= \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right) = 0\end{aligned} \tag{13}$$

$$\Omega_7(u,v) = (1-u^8)(1-v^8) - (1-uv)^8 = 0, \tag{14}$$

where

$$u^2 \equiv \sqrt{k} = \frac{\vartheta_2(q)}{\vartheta_3(q)} \tag{15}$$

and

$$v^2 \equiv \sqrt{l} = \frac{\vartheta_2(q^p)}{\vartheta_3(q^p)}. \tag{16}$$

Here, ϑ_i are THETA FUNCTIONS.

A modular equation of degree 2^r for $r \geq 2$ can be obtained by iterating the equation for 2^{r-1}. Modular equations for PRIME p from 3 to 23 are given in Borwein and Borwein (1987).

Quadratic modular identities include

$$\frac{\vartheta_3(q)}{\vartheta_3(q^4)} - 1 = \left[\frac{{\vartheta_3}^2(q^2)}{{\vartheta_3}^2(q^4)} - 1\right]^{1/2}. \tag{17}$$

Cubic identities include

$$\left[3\frac{\vartheta_2(q^9)}{\vartheta_2(q)} - 1\right]^3 = 9\frac{{\vartheta_2}^4(q^3)}{{\vartheta_2}^4(q)} - 1 \tag{18}$$

$$\left[3\frac{\vartheta_3(q^9)}{\vartheta_3(q)} - 1\right]^3 = 9\frac{{\vartheta_3}^4(q^3)}{{\vartheta_3}^4(q)} - 1 \tag{19}$$

$$\left[3\frac{\vartheta_4(q^9)}{\vartheta_4(q)} - 1\right]^3 = 9\frac{{\vartheta_4}^4(q^3)}{{\vartheta_4}^4(q)} - 1. \tag{20}$$

A seventh-order identity is

$$\sqrt{\vartheta_3(q)\vartheta_3(q^7)} - \sqrt{\vartheta_4(q)\vartheta_4(q^7)} = \sqrt{\vartheta_2(q)\vartheta_2(q^7)}. \tag{21}$$

From Ramanujan (1913–1914),

$$(1+q)(1+q^3)(1+q^5)\cdots = 2^{1/6}q^{1/24}(kk')^{-1/12} \tag{22}$$

$$(1-q)(1-q^3)(1-q^5)\cdots = 2^{1/6}q^{1/24}k^{-1/12}k'^{1/6}. \tag{23}$$

see also SCHLÄFLI'S MODULAR FORM

References

Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, pp. 127–132, 1987.

Hanna, M. "The Modular Equations." *Proc. London Math. Soc.* **28**, 46–52, 1928.

Ramanujan, S. "Modular Equations and Approximations to π." *Quart. J. Pure. Appl. Math.* **45**, 350–372, 1913–1914.

Modular Form

A modular form is a function in the COMPLEX PLANE with rather spectacular and special properties resulting from a surprising array of internal symmetries. If

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^2F(z),$$

then $F(z)$ is said to be a modular form of weight 2 and level N. If it is correctly parameterized, a modular form is ANALYTIC and vanishes at the cusps, so it is called a CUSP FORM. It is also an eigenform under a certain HECKE ALGEBRA.

A remarkable connection between rational ELLIPTIC CURVES and modular forms is given by the TANIYAMA-SHIMURA CONJECTURE, which states that any rational ELLIPTIC CURVE is a modular form in disguise. This result was the one proved by Andrew Wiles in his celebrated proof of FERMAT'S LAST THEOREM.

see also CUSP FORM, ELLIPTIC CURVE, ELLIPTIC FUNCTION, FERMAT'S LAST THEOREM, HECKE ALGEBRA, MODULAR FUNCTION, MODULAR FUNCTION MULTIPLIER, SCHLÄFLI'S MODULAR FORM, TANIYAMA-SHIMURA CONJECTURE

References
Knopp, M. I. *Modular Functions, 2nd ed.* New York: Chelsea, 1993.
Koblitz, N. *Introduction to Elliptic Curves and Modular Forms.* New York: Springer-Verlag, 1993.
Rankin, R. A. *Modular Forms and Functions.* Cambridge, England: Cambridge University Press, 1977.
Sarnack, P. *Some Applications of Modular Forms.* Cambridge, England: Cambridge University Press, 1993.

Modular Function

f is a modular function of level N on the upper half H of the COMPLEX PLANE if it is MEROMORPHIC (even at the CUSPS), $ad-bc=1$ for all a, b, c, d, and $N|c$.

see also ELLIPTIC FUNCTION, ELLIPTIC MODULAR FUNCTION, MODULAR FORM

References
Apostol, T. M. *Modular Functions and Dirichlet Series in Number Theory.* New York: Springer-Verlag, 1976.
Askey, R. In *Ramanujan International Symposium* (Ed. N. K Thakare). pp. 1–83.
Borwein, J. M. and Borwein, P. B. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, 1987.
Rankin, R. A. *Modular Forms and Functions.* Cambridge, England: Cambridge University Press, 1977.
Schoeneberg, B. *Elliptic Modular Functions: An Introduction.* Berlin: New York: Springer-Verlag, 1974.

Modular Function Multiplier

When k and l satisfy a MODULAR EQUATION, a relationship of the form

$$\frac{M(l,k)\,dy}{\sqrt{(1-y^2)(1-l^2y^2)}}=\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\qquad(1)$$

exists, and M is called the multiplier. The multiplier of degree n can be given by

$$M_n(l,k)\equiv\frac{{\vartheta_3}^2(q)}{{\vartheta_3}^2(q^{1/p})}=\frac{K(k)}{K(l)},\qquad(2)$$

where ϑ_i is a THETA FUNCTION and $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND.

The first few multipliers in terms of l and k are

$$M_2(l,k)=\frac{1}{1+k}=\frac{1+l'}{2}\qquad(3)$$

$$M_3(l,k)=\frac{1-\sqrt{\frac{l^3}{k}}}{1-\sqrt{\frac{k^3}{l}}}.\qquad(4)$$

In terms of the u and v defined for MODULAR EQUATIONS,

$$M_3=\frac{v}{v+2u^3}=\frac{2v^3-u}{3u}\qquad(5)$$

$$M_5=\frac{v(1-uv^3)}{v-u^5}=\frac{u+v^5}{5u(1+u^3v)}\qquad(6)$$

$$\begin{aligned}M_7&=\frac{v(1-uv)[1-uv+(uv)^2]}{v-u^7}\\&=\frac{v^7-u}{7u(1-uv)[1-uv+(uv)^2]}.\end{aligned}\qquad(7)$$

Modular Gamma Function

The GAMMA GROUP Γ is the set of all transformations w of the form

$$w(t)=\frac{at+b}{ct+d},$$

where a, b, c, and d are INTEGERS and $ad-bc=1$. Γ-modular functions are then defined as in Borwein and Borwein (1987, p. 114).

see also KLEIN'S ABSOLUTE INVARIANT, LAMBDA GROUP, THETA FUNCTION

References
Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, pp. 127–132, 1987.

Modular Group

The GROUP of all MÖBIUS TRANSFORMATIONS having INTEGER coefficients and DETERMINANT equal to 1.

Modular Lambda Function

see ELLIPTIC LAMBDA FUNCTION

Modular Lattice

A LATTICE which satisfies the identity

$$(x\wedge y)\vee(x\wedge z)=x\wedge(y\vee(x\wedge z))$$

is said to be modular.

see also DISTRIBUTIVE LATTICE

References
Grätzer, G. *Lattice Theory: First Concepts and Distributive Lattices.* San Francisco, CA: W. H. Freeman, pp. 35–36, 1971.

Modular System

A set M of all POLYNOMIALS in s variables, $x_1, \ldots, x_s$ such that if P, P_1, and P_2 are members, then so are $P_1 + P_2$ and QP, where Q is any POLYNOMIAL in $x_1, \ldots, x_s$.

see also HILBERT'S THEOREM, MODULAR SYSTEM BASIS

Modular System Basis

A basis of a MODULAR SYSTEM M is any set of POLYNOMIALS $B_1, B_2, \ldots$ of M such that every POLYNOMIAL of M is expressible in the form

$$R_1B_1 + R_2B_2 + \ldots,$$

where $R_1, R_2, \ldots$ are POLYNOMIALS.

Modular Transformation

see MODULAR EQUATION

Modulation Theorem

The important property of FOURIER TRANSFORMS that $\mathcal{F}[cos(2\pi k_0 x)f(x)]$ can be expressed in terms of $\mathcal{F}[f(x)] = F(k)$ as follows,

$$\mathcal{F}[cos(2\pi k_0 x)f(x)] = \tfrac{1}{2}[F(k - k_0) + F(k + k_0)].$$

see also FOURIER TRANSFORM

References

Bracewell, R. "Modulation Theorem." *The Fourier Transform and Its Applications.* New York: McGraw-Hill, p. 108, 1965.

Module

A mathematical object in which things can be added together COMMUTATIVELY by multiplying COEFFICIENTS and in which most of the rules of manipulating VECTORS hold. A module is abstractly very similar to a VECTOR SPACE, although modules have COEFFICIENTS in much more general algebraic objects and use RINGS as the COEFFICIENTS instead of FIELDS.

The additive submodule of the INTEGERS is a set of quantities closed under ADDITION and SUBTRACTION (although it is SUFFICIENT to require closure under SUBTRACTION). Numbers of the form $n\alpha \pm m\alpha$ for $n, m \in \mathbb{Z}$ form a module since,

$$n\alpha \pm m\alpha = (n \pm m)\alpha.$$

Given two INTEGERS a and b, the smallest module containing a and b is $\mathrm{GCD}(a, b)$.

References

Foote, D. and Dummit, D. *Abstract Algebra.* Englewood Cliffs, NJ: Prentice-Hall, 1990.

Modulo

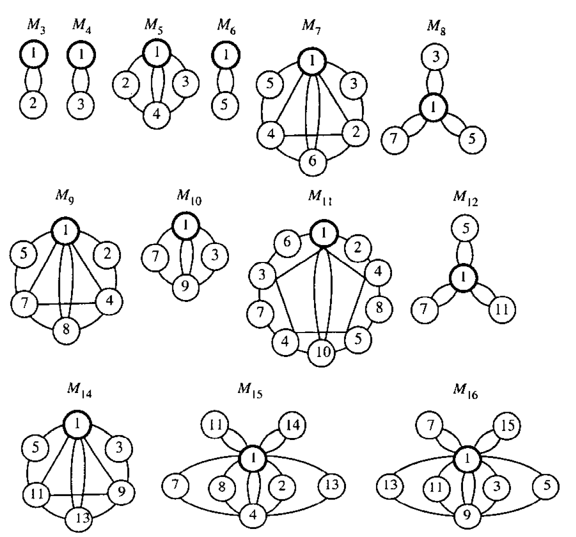
see CONGRUENCE

Modulo Multiplication Group

A FINITE GROUP M_m of RESIDUE CLASSES prime to m under multiplication mod m. M_m is ABELIAN of ORDER $\phi(m)$, where $\phi(m)$ is the TOTIENT FUNCTION. The following table gives the modulo multiplication groups of small orders.

M_m	Group	$\phi(m)$	Elements
M_2	$\langle e \rangle$	1	1
M_3	Z_2	2	1, 2
M_4	Z_2	2	1, 3
M_5	Z_4	4	1, 2, 3, 4
M_6	Z_2	2	1, 5
M_7	Z_6	6	1, 2, 3, 4, 5, 6
M_8	$Z_2 \otimes Z_2$	4	1, 3, 5, 7
M_9	Z_6	6	1, 2, 4, 5, 7, 8
M_{10}	Z_4	4	1, 3, 7, 9
M_{11}	Z_{10}	10	1, 2, 3, 4, 5, 6, 7, 8, 9, 10
M_{12}	$Z_2 \otimes Z_2$	4	1, 5, 7, 11
M_{13}	Z_{12}	12	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
M_{14}	Z_6	6	1, 3, 5, 9, 11, 13
M_{15}	$Z_2 \otimes Z_4$	8	1, 2, 4, 7, 8, 11, 13, 14
M_{16}	$Z_2 \otimes Z_4$	8	1, 3, 5, 7, 9, 11, 13, 15
M_{17}	Z_{16}	16	1, 2, 3, ..., 16
M_{18}	Z_6	6	1, 5, 7, 11, 13, 17
M_{19}	Z_{18}	18	1, 2, 3, ..., 18
M_{20}	$Z_2 \otimes Z_4$	8	1, 3, 7, 9, 11, 13, 17, 19
M_{21}	$Z_2 \otimes Z_6$	12	1, 2, 4, 5, 7, 8, 10, 11, 13, 16, 17, 19
M_{22}	Z_{10}	10	1, 3, 5, 7, 9, 13, 15, 17, 19, 21
M_{23}	Z_{22}	22	1, 2, 3, ..., 22
M_{24}	$Z_2 \otimes Z_2 \otimes Z_2$	8	1, 5, 7, 11, 13, 17, 19, 23

M_m is a CYCLIC GROUP (which occurs exactly when m has a PRIMITIVE ROOT) IFF m is of one of the forms $m = 2$, 4, p^n, or $2p^n$, where p is an ODD PRIME and $n \geq 1$ (Shanks 1993, p. 92).



ISOMORPHIC modulo multiplication groups can be determined using a particular type of factorization of $\phi(m)$ as described by Shanks (1993, pp. 92–93). To perform this factorization (denoted ϕ_m), factor m in the standard form

$$m = p_1{}^{a_1} p_2{}^{a_2} \cdots p_n{}^{a_n}. \tag{1}$$

Now write the factorization of the TOTIENT FUNCTION involving each power of an ODD PRIME

$$\phi(p_i{}^{a_i}) = (p_i - 1) p_i{}^{a_i - 1} \tag{2}$$

as

$$\phi(p_i{}^{a_i}) = \left\langle q_1{}^{b_1} \right\rangle \left\langle q_2{}^{b_2} \right\rangle \cdots \left\langle q_s{}^{b_s} \right\rangle \left\langle p_i{}^{a_i-1} \right\rangle, \tag{3}$$

where

$$p_i - 1 = q_1{}^{b_1} q_2{}^{b_2} \cdots q_s{}^{b_s}, \tag{4}$$

$\left\langle q^b \right\rangle$ denotes the explicit expansion of q^b (i.e., $5^2 = 25$), and the last term is omitted if $a_i = 1$. If $p_1 = 2$, write

$$\phi(2^{a_1}) = \begin{cases} 2 & \text{for } a_1 = 2 \\ 2 \left\langle 2^{a_1 - 2} \right\rangle & \text{for } a_1 > 2. \end{cases} \tag{5}$$

Now combine terms from the odd and even primes. For example, consider $m = 104 = 2^3 \cdot 13$. The only odd prime factor is 13, so factoring gives $13 - 1 = 12 = \left\langle 2^2 \right\rangle \langle 3 \rangle = 3 \cdot 4$. The rule for the powers of 2 gives $2^3 = 2 \left\langle 2^{3-2} \right\rangle = 2 \langle 2 \rangle = 2 \cdot 2$. Combining these two gives $\phi_{104} = 2 \cdot 2 \cdot 3 \cdot 4$. Other explicit values of ϕ_m are given below.

$$\begin{aligned} \phi_3 &= 2 \\ \phi_4 &= 2 \\ \phi_5 &= 4 \\ \phi_6 &= 2 \\ \phi_{15} &= 2 \cdot 4 \\ \phi_{16} &= 2 \cdot 4 \\ \phi_{17} &= 16 \\ \phi_{104} &= 2 \cdot 2 \cdot 3 \cdot 4 \\ \phi_{105} &= 2 \cdot 2 \cdot 3 \cdot 4. \end{aligned}$$

M_m and M_n are isomorphic IFF ϕ_m and ϕ_n are identical. More specifically, the abstract GROUP corresponding to a given M_m can be determined explicitly in terms of a DIRECT PRODUCT of CYCLIC GROUPS of the so-called CHARACTERISTIC FACTORS, whose product is denoted Φ_n. This representation is obtained from ϕ_m as the set of products of largest powers of each factor of ϕ_m. For example, for ϕ_{104}, the largest power of 2 is $4 = 2^2$ and the largest power of 3 is $3 = 3^1$, so the first characteristic factor is $4 \times 3 = 12$, leaving $2 \cdot 2$ (i.e., only powers of two). The largest power remaining is $2 = 2^1$, so the second CHARACTERISTIC FACTOR is 2, leaving 2, which is the third and last CHARACTERISTIC FACTOR. Therefore, $\Phi_{104} = 2 \cdot 2 \cdot 4$, and the group M_m is isomorphic to $Z_2 \otimes Z_2 \otimes Z_4$.

The following table summarizes the isomorphic modulo multiplication groups M_n for the first few n and identifies the corresponding abstract GROUP. No M_m is ISOMORPHIC to Z_8, Q_8, or D_4. However, every finite ABELIAN GROUP is isomorphic to a SUBGROUP of M_m for infinitely many different values of m (Shanks 1993, p. 96). CYCLE GRAPHS corresponding to M_n for small n are illustrated above, and more complicated CYCLE GRAPHS are illustrated by Shanks (1993, pp. 87–92).

Group	Isomorphic M_m
$\langle e \rangle$	M_2
Z_2	M_3, M_4, M_6
Z_4	M_5, M_{10}
$Z_2 \otimes Z_2$	M_8, M_{12}
Z_6	M_7, M_9, M_{14}, M_{18}
$Z_2 \otimes Z_4$	M_{15}, M_{16}, M_{20}, M_{30}
$Z_2 \otimes Z_2 \otimes Z_2$	M_{24}
Z_{10}	M_{11}, M_{22}
Z_{12}	M_{13}, M_{26}
$Z_2 \otimes Z_6$	M_{21}, M_{28}, M_{36}, M_{42}
Z_{16}	M_{17}, M_{34}
$Z_2 \otimes Z_8$	M_{32}
$Z_2 \otimes Z_2 \otimes Z_4$	M_{40}, M_{48}, M_{60}
Z_{18}	M_{19}, M_{27}, M_{38}, M_{54}
Z_{20}	M_{25}, M_{50}
$Z_2 \otimes Z_{10}$	M_{33}, M_{44}, M_{66}
Z_{22}	M_{23}, M_{46}
$Z_2 \otimes Z_{12}$	M_{35}, M_{39}, M_{45}, M_{52}, M_{70}, M_{78}, M_{90}
Z_{28}	M_{29}, M_{58}
Z_{30}	M_{31}, M_{62}
Z_{36}	M_{37}, M_{74}

The number of CHARACTERISTIC FACTORS r of M_m for $m = 1, 2, \ldots$ are 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2, ... (Sloane's A046072). The number of QUADRATIC RESIDUES in M_m for $m > 2$ are given by $\phi(m)/2^r$ (Shanks 1993, p. 95). The first few for $m = 1, 2, \ldots$ are 0, 1, 1, 1, 2, 1, 3, 1, 3, 2, 5, 1, 6, ... (Sloane's A046073).

In the table below, $\phi(n)$ is the TOTIENT FUNCTION (Sloane's A000010) factored into CHARACTERISTIC FACTORS, $\lambda(n)$ is the CARMICHAEL FUNCTION (Sloane's A011773), and g_i are the smallest generators of the group M_n (of which there is a number equal to the number of CHARACTERISTIC FACTORS).

n	$\phi(n)$	$\lambda(n)$	g_i	n	$\phi(n)$	$\lambda(n)$	g_i
3	2	2	2	27	18	18	2
4	2	2	3	28	$2 \cdot 6$	6	13, 3
5	4	2	2	29	28	28	2
6	2	2	5	30	$2 \cdot 4$	4	11, 7
7	6	6	3	31	30	30	3
8	$2 \cdot 2$	2	7, 3	32	$2 \cdot 8$	8	31, 3
9	6	6	2	33	$2 \cdot 10$	10	10, 2
10	4	4	3	34	16	16	3
11	10	10	2	35	$2 \cdot 12$	12	6, 2
12	$2 \cdot 2$	2	5, 7	36	$2 \cdot 6$	6	19,5
13	12	12	2	37	36	36	2
14	6	6	3	38	18	18	3
15	$2 \cdot 4$	4	14, 2	39	$2 \cdot 12$	12	38, 2
16	$2 \cdot 4$	4	15, 3	40	$2 \cdot 2 \cdot 4$	4	39, 11, 3
17	16	16	3	41	40	40	6
18	6	6	5	42	$2 \cdot 6$	6	13, 5
19	18	18	2	43	42	42	3
20	$2 \cdot 4$	4	19, 3	44	$2 \cdot 10$	10	43, 3
21	$2 \cdot 6$	6	20, 2	45	$2 \cdot 12$	12	44, 2
22	10	10	7	46	22	22	5
23	22	22	5	47	46	46	5
24	$2 \cdot 2 \cdot 2$	2	5, 7, 13	48	$2 \cdot 2 \cdot 4$	4	47, 7, 5
25	20	20	2	49	42	42	3
26	12	12	7	50	20	20	3

see also CHARACTERISTIC FACTOR, CYCLE GRAPH, FINITE GROUP, RESIDUE CLASS

References

Riesel, H. "The Structure of the Group M_n." *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, pp. 270–272, 1994.

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 61–62 and 92, 1993.

Sloane, N. J. A. Sequences A011773, A046072, A046073, and A000010/M0299 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Weisstein, E. W. "Groups." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Groups.m`.

Modulus (Complex Number)

The modulus of a COMPLEX NUMBER z is denoted $|z|$.

$$|x+iy| \equiv \sqrt{x^2+y^2} \tag{1}$$

$$|re^{i\phi}| = |r|. \tag{2}$$

Let $c_1 \equiv Ae^{i\phi_1}$ and $c_2 \equiv Be^{i\phi_2}$ be two COMPLEX NUMBERS. Then

$$\left|\frac{c_1}{c_2}\right| = \left|\frac{Ae^{i\phi_1}}{Be^{i\phi_2}}\right| = \frac{A}{B}|e^{i(\phi_1-\phi_2)}| = \frac{A}{B} \tag{3}$$

$$\frac{|c_1|}{|c_2|} = \frac{|Ae^{i\phi_1}|}{|Be^{i\phi_2}|} = \frac{A}{B}\frac{|e^{i\phi_1}|}{|e^{i\phi_2}|} = \frac{A}{B}, \tag{4}$$

so

$$\left|\frac{c_1}{c_2}\right| = \frac{|c_1|}{|c_2|}. \tag{5}$$

Also,

$$|c_1c_2| = |(Ae^{i\phi_1})(Be^{i\phi_2})| = AB|e^{i(\phi_1+\phi_2)}| = AB \tag{6}$$

$$|c_1|\,|c_2| = |Ae^{i\phi_1}|\,|Be^{i\phi_2}| = AB|e^{i\phi_1}|\,|e^{i\phi_2}| = AB, \tag{7}$$

so

$$|c_1c_2| = |c_1|\,|c_2| \tag{8}$$

and, by extension,

$$|z^n| = |z|^n. \tag{9}$$

The only functions satisfying identities of the form

$$|f(x+iy)| = |f(x)+f(iy)| \tag{10}$$

are $f(z) = Az$, $f(z) = A\sin(bz)$, and $f(z) = A\sinh(bz)$ (Robinson 1957).

see also ABSOLUTE SQUARE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Robinson, R. M. "A Curious Mathematical Identity." *Amer. Math. Monthly* **64**, 83–85, 1957.

Modulus (Congruence)

see CONGRUENCE

Modulus (Elliptic Integral)

A parameter k used in ELLIPTIC INTEGRALS and ELLIPTIC FUNCTIONS defined to be $k \equiv \sqrt{m}$, where m is the PARAMETER. An ELLIPTIC INTEGRAL is written $I(\phi, k)$ when the modulus is used. It can be computed explicitly in terms of THETA FUNCTIONS of zero argument:

$$k = \frac{\vartheta_2{}^2(0|\tau)}{\vartheta_3{}^2(0|\tau)}. \tag{1}$$

The REAL period $K(k)$ and IMAGINARY period $K'(k) = K(k') = K(\sqrt{1-k^2}\,)$ are given by

$$4K(k) = 2\pi\vartheta_3{}^2(0|\tau) \tag{2}$$

$$2iK'(k) = \pi\tau\vartheta_3{}^2(0|\tau), \tag{3}$$

where $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND and the complementary modulus is defined by

$$k'^2 \equiv 1-k^2, \tag{4}$$

with k the modulus.

see also AMPLITUDE, CHARACTERISTIC (ELLIPTIC INTEGRAL), ELLIPTIC FUNCTION, ELLIPTIC INTEGRAL, ELLIPTIC INTEGRAL SINGULAR VALUE, MODULAR ANGLE, NOME, PARAMETER, THETA FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 590, 1972.

Modulus (Quadratic Invariants)

The quantity $ps - rq$ obtained by letting

$$x = pX + qY \tag{1}$$
$$y = rX + sY \tag{2}$$

in

$$ax^2 + 2bxy + cy^2 \tag{3}$$

so that

$$A = ap^2 + 2bpr + cr^2 \tag{4}$$
$$B = apq + b(ps + qr) + crs \tag{5}$$
$$C = aq^2 + 2bqs + cs^2 \tag{6}$$

and

$$B^2 - AC = (ps - rq)^2(b^2 - ac), \tag{7}$$

is called the modulus.

Modulus (Set)

The name for the SET of INTEGERS modulo m, denoted $\mathbb{Z}\backslash m\mathbb{Z}$. If m is a PRIME p, then the modulus is a FINITE FIELD $\mathbb{F}_p = \mathbb{Z}\backslash p\mathbb{Z}$.

Moessner's Theorem

Write down the POSITIVE INTEGERS in row one, cross out every k_1th number, and write the partial sums of the remaining numbers in the row below. Now cross off every k_2th number and write the partial sums of the remaining numbers in the row below. Continue. For every POSITIVE INTEGER $k > 1$, if every kth number is ignored in row 1, every $(k-1)$th number in row 2, and every $(k+1-i)$th number in row i, then the kth row of partial sums will be the kth POWERS 1^k, 2^k, 3^k, ...

References

Conway, J. H. and Guy, R. K. "Moessner's Magic." In *The Book of Numbers*. New York: Springer-Verlag, pp. 63–65, 1996.

Honsberger, R. *More Mathematical Morsels*. Washington, DC: Math. Assoc. Amer., pp. 268–277, 1991.

Long, C. T. "On the Moessner Theorem on Integral Powers." *Amer. Math. Monthly* **73**, 846–851, 1966.

Long, C. T. "Strike it Out—Add it Up." *Math. Mag.* **66**, 273–277, 1982.

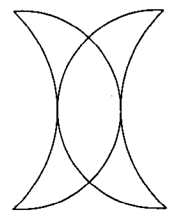
Moessner, A. "Eine Bemerkung über die Potenzen der natürlichen Zahlen." *S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.* **29**, 1952.

Paasche, I. "Ein neuer Beweis des moessnerischen Satzes." *S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.* **1952**, 1–5, 1953.

Paasche, I. "Ein zahlentheoretische-logarithmischer 'Rechenstab'." *Math. Naturwiss. Unterr.* **6**, 26–28, 1953–54.

Paasche, I. "Eine Verallgemeinerung des moessnerschen Satzes." *Compositio Math.* **12**, 263–270, 1956.

Mohammed Sign



A curve consisting of two mirror-reversed intersecting crescents. This curve can be traced UNICURSALLY.

see also UNICURSAL CIRCUIT

Møiré Pattern

An interference pattern produced by overlaying similar but slightly offset templates. Møiré patterns can also be created be plotting series of curves on a computer screen. Here, the interference is provided by the discretization of the finite-sized pixels.

see also CIRCLES-AND-SQUARES FRACTAL

References

Cassin, C. *Visual Illusions in Motion with Møiré Screens: 60 Designs and 3 Plastic Screens*. New York: Dover, 1997.

Grafton, C. B. *Optical Designs in Motion with Møiré Overlays*. New York: Dover, 1976.

Mollweide's Formulas

Let a TRIANGLE have side lengths a, b, and c with opposite angles A, B, and C. Then

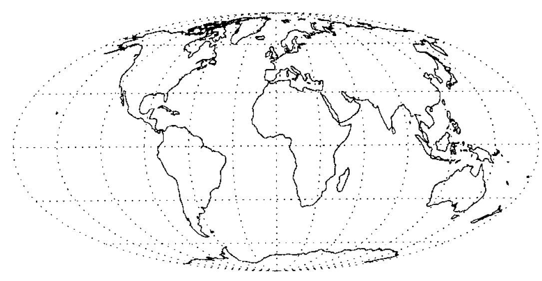
$$\frac{b-c}{a} = \frac{\sin[\frac{1}{2}(B-C)]}{\cos(\frac{1}{2}A)}$$
$$\frac{c-a}{b} = \frac{\sin[\frac{1}{2}(C-A)]}{\cos(\frac{1}{2}B)}$$
$$\frac{a-b}{c} = \frac{\sin[\frac{1}{2}(A-B)]}{\cos(\frac{1}{2}C)}.$$

see also NEWTON'S FORMULAS, TRIANGLE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 146, 1987.

Mollweide Projection



A MAP PROJECTION also called the ELLIPTICAL PROJECTION or HOMOLOGRAPHIC EQUAL AREA PROJECTION. The forward transformation is

$$x = \frac{2\sqrt{2}\,(\lambda - \lambda_0)\cos\theta}{\pi} \tag{1}$$

$$y = 2^{1/2}\sin\theta, \tag{2}$$

where θ is given by

$$2\theta + \sin(2\theta) = \pi\sin\phi. \tag{3}$$

NEWTON'S METHOD can then be used to compute θ' iteratively from

$$\Delta\theta' = -\frac{\theta' + \sin\theta' - \pi\sin\phi}{1 + \cos\theta'}, \tag{4}$$

where

$$\theta' = \tfrac{1}{2}\theta' \tag{5}$$

or, better yet,

$$\theta' = 2\sin^{-1}\left(\frac{2\phi}{\pi}\right) \tag{6}$$

can be used as a first guess.

The inverse FORMULAS are

$$\phi = \sin^{-1}\left[\frac{2\theta + \sin(2\theta)}{\pi}\right] \tag{7}$$

$$\lambda = \lambda_0 + \frac{\pi x}{2\sqrt{2}\,\cos\theta}, \tag{8}$$

where

$$\theta = \sin^{-1}\left(\frac{y}{\sqrt{2}}\right). \tag{9}$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 249–252, 1987.

Moment

The nth moment of a distribution about zero μ'_n is defined by

$$\mu'_n = \langle x^n \rangle, \tag{1}$$

where

$$\langle f(x) \rangle = \begin{cases} \sum f(x)P(x) & \text{discrete distribution} \\ \int f(x)P(x)\,dx & \text{continuous distribution.} \end{cases} \tag{2}$$

μ'_1, the MEAN, is usually simply denoted $\mu = \mu_1$. If the moment is instead taken about a point a,

$$\mu_n(a) = \langle (x-a)^n \rangle = \sum (x-a)^n P(x). \tag{3}$$

The moments are most commonly taken about the MEAN. These moments are denoted μ_n and are defined by

$$\mu_n \equiv \langle (x-\mu)^n \rangle, \tag{4}$$

with $\mu_1 = 0$. The moments about zero and about the MEAN are related by

$$\mu_2 = \mu'_2 - (\mu'_1)^2 \tag{5}$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 \tag{6}$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4. \tag{7}$$

The second moment about the MEAN is equal to the VARIANCE

$$\mu_2 = \sigma^2, \tag{8}$$

where $\sigma = \sqrt{\mu_2}$ is called the STANDARD DEVIATION.

The related CHARACTERISTIC FUNCTION is defined by

$$\phi^{(n)}(0) \equiv \left[\frac{d^n\phi}{dt^n}\right]_{t=0} = i^n \mu_n(0). \tag{9}$$

The moments may be simply computed using the MOMENT-GENERATING FUNCTION,

$$\mu'_n = M^{(n)}(0). \tag{10}$$

A DISTRIBUTION is not uniquely specified by its moments, although it is by its CHARACTERISTIC FUNCTION.

see also CHARACTERISTIC FUNCTION, CHARLIER'S CHECK, CUMULANT-GENERATING FUNCTION, FACTORIAL MOMENT, KURTOSIS, MEAN, MOMENT-GENERATING FUNCTION, SKEWNESS, STANDARD DEVIATION, STANDARDIZED MOMENT, VARIANCE

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Moments of a Distribution: Mean, Variance, Skewness, and So Forth." §14.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 604–609, 1992.

Moment-Generating Function

Given a RANDOM VARIABLE $x \in R$, if there exists an $h > 0$ such that

$$\begin{aligned} M(t) &\equiv \langle e^{tx} \rangle \\ &= \begin{cases} \sum_R e^{tx}P(x) & \text{for a discrete distribution} \\ \int_{-\infty}^{\infty} e^{tx}P(x)\,dx & \text{for a continuous distribution} \end{cases} \end{aligned} \tag{1}$$

for $|t| < h$, then

$$M(t) \equiv \langle e^{tx} \rangle \tag{2}$$

is the moment-generating function.

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty} (1 + tx + \tfrac{1}{2!}t^2x^2 + \ldots)P(x)\,dx \\ &= 1 + tm_1 + \tfrac{1}{2!}t^2 m_2 + \ldots, \end{aligned} \tag{3}$$

where m_r is the rth MOMENT about zero. The moment-generating function satisfies

$$\begin{aligned} M_{x+y}(t) &= \langle e^{t(x+y)} \rangle = \langle e^{tx} e^{ty} \rangle \\ &= \langle e^{tx} \rangle \langle e^{ty} \rangle = M_x(t) M_y(t). \end{aligned} \tag{4}$$

If $M(t)$ is differentiable at zero, then the nth MOMENTS about the ORIGIN are given by $M^n(0)$

$$M(t) = \langle e^{tx} \rangle \qquad M(0) = 1 \tag{5}$$

$$M'(t) = \langle x e^{tx} \rangle \qquad M'(0) = \langle x \rangle \tag{6}$$

$$M''(t) = \langle x^2 e^{tx} \rangle \qquad M''(0) = \langle x^2 \rangle \tag{7}$$

$$M^{(n)}(t) = \langle x^n e^{tx} \rangle \qquad M^{(n)}(0) = \langle x^n \rangle. \tag{8}$$

The MEAN and VARIANCE are therefore

$$\mu \equiv \langle x \rangle = M'(0) \tag{9}$$

$$\sigma^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = M''(0) - [M'(0)]^2. \tag{10}$$

It is also true that

$$\mu_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mu_j' (\mu_1')^{n-j}, \tag{11}$$

where $\mu_0' = 1$ and μ_j' is the jth moment about the origin.

It is sometimes simpler to work with the LOGARITHM of the moment-generating function, which is also called the CUMULANT-GENERATING FUNCTION, and is defined by

$$R(t) \equiv \ln[M(t)] \tag{12}$$

$$R'(t) = \frac{M'(t)}{M(t)} \tag{13}$$

$$R''(t) = \frac{M(t)M''(t) - [M'(t)]^2}{[M(t)]^2}. \tag{14}$$

But $M(0) = \langle 1 \rangle = 1$, so

$$\mu = M'(0) = R'(0) \tag{15}$$

$$\sigma^2 = M''(0) - [M'(0)]^2 = R''(0). \tag{16}$$

see also CHARACTERISTIC FUNCTION, CUMULANT, CUMULANT-GENERATING FUNCTION, MOMENT

References
Kenney, J. F. and Keeping, E. S. "Moment-Generating and Characteristic Functions," "Some Examples of Moment-Generating Functions," and "Uniqueness Theorem for Characteristic Functions." §4.6–4.8 in *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 72–77, 1951.

Momental Skewness

$$\alpha^{(m)} \equiv \tfrac{1}{2}\gamma_1 = \frac{\mu_3}{2\sigma^3},$$

where γ_1 is the FISHER SKEWNESS.

see also FISHER SKEWNESS, SKEWNESS

Monad

A mathematical object which consists of a set of a single element. The YIN-YANG is also known as the monad.

see also HEXAD, QUARTET, QUINTET, TETRAD, TRIAD, YIN-YANG

Money-Changing Problem

see COIN PROBLEM

Monge-Ampère Differential Equation

A second-order PARTIAL DIFFERENTIAL EQUATION of the form

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0,$$

where H, K, L, M, and N are functions of x, y, z, p, and q, and r, s, t, p, and q are defined by

$$\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} \\ s &= \frac{\partial^2 z}{\partial x \partial y} \\ t &= \frac{\partial^2 z}{\partial y^2} \\ p &= \frac{\partial z}{\partial x} \\ q &= \frac{\partial z}{\partial y}. \end{aligned}$$

The solutions are given by a system of differential equations given by Iyanaga and Kawada (1980).

References
Iyanaga, S. and Kawada, Y. (Eds.). "Monge-Ampère Equations." §276 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 879–880, 1980.

Monge's Chordal Theorem

see RADICAL CENTER

Monge's Form

A surface given by the form $z = F(x, y)$.

see also MONGE PATCH

Monge Patch

A Monge patch is a PATCH $\mathbf{x} : U \to \mathbb{R}^3$ of the form

$$\mathbf{x}(u, v) = (u, v, h(u, v)), \tag{1}$$

where U is an OPEN SET in $\mathbb{R}^2$ and $h : U \to \mathbb{R}$ is a differentiable function. The coefficients of the first FUNDAMENTAL FORM are given by

$$E = 1 + {h_u}^2 \tag{2}$$

$$F = h_u h_v \tag{3}$$

$$G = 1 + {h_v}^2 \tag{4}$$

and the second FUNDAMENTAL FORM by

$$e = \frac{h_{uu}}{\sqrt{1+{h_u}^2+{h_v}^2}} \tag{5}$$

$$f = \frac{h_{uv}}{\sqrt{1+{h_u}^2+{h_v}^2}} \tag{6}$$

$$g = \frac{g_{vv}}{\sqrt{1+{h_u}^2+{h_v}^2}}. \tag{7}$$

For a Monge patch, the GAUSSIAN CURVATURE and MEAN CURVATURE are

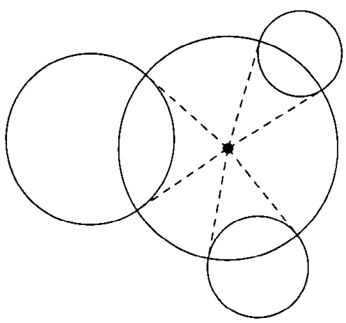
$$K = \frac{h_{uu}h_{vv} - {h_{uv}}^2}{(1+{h_u}^2+{h_v}^2)^2} \tag{8}$$

$$H = \frac{(1+{h_v}^2)h_{uu} - 2h_u h_v h_{uv} + (1+{h_u}^2)h_{vv}}{(1+{h_u}^2+{h_v}^2)^{3/2}}. \tag{9}$$

see also MONGE'S FORM, PATCH

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 305–306, 1993.

Monge's Problem



Draw a CIRCLE that cuts three given CIRCLES PERPENDICULARLY. The solution is obtained by drawing the RADICAL CENTER R of the given three CIRCLES. If it lies outside the three CIRCLES, then the CIRCLE with center R and RADIUS formed by the tangent from R to one of the given CIRCLES intersects the given CIRCLES perpendicularly. Otherwise, if R lies inside one of the circles, the problem is unsolvable.

see also CIRCLE TANGENTS, RADICAL CENTER

References
Dörrie, H. "Monge's Problem." §31 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 151–154, 1965.

Monge's Shuffle

A SHUFFLE in which CARDS from the top of the deck in the left hand are alternatively moved to the bottom and top of the deck in the right hand. If the deck is shuffled m times, the final position x_m and initial position x_0 of a card are related by

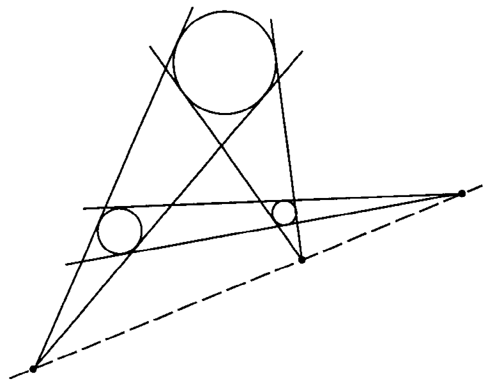
$$2^{m+1}x_m = (4p+1)[2^{m-1}+(-1)^{m-1}(2^{m-2}+\ldots+2+1)] +(-1)^{m-1}2x_0+2^m+(-1)^{m-1}$$

for a deck of $2p$ cards (Kraitchik 1942).

see also CARDS, SHUFFLE

References
Conway, J. H. and Guy, R. K. "Fractions Cycle into Decimals." In *The Book of Numbers.* New York: Springer-Verlag, pp. 157–163, 1996.
Kraitchik, M. "Monge's Shuffle." §12.2.14 in *Mathematical Recreations.* New York: W. W. Norton, pp. 321–323, 1942.

Monge's Theorem



Draw three nonintersecting CIRCLES in the plane, and the common tangent line for each pair of two. The points of intersection of the three pairs of tangent lines lie on a straight line.

References
Coxeter, H. S. M. "The Problem of Apollonius." *Amer. Math. Monthly* **75**, 5–15, 1968.
Graham, L. A. Problem 62 in *Ingenious Mathematical Problems and Methods.* New York: Dover, 1959. Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 115–117, 1990.
Walker, W. "Monge's Theorem in Many Dimensions." *Math. Gaz.* **60**, 185–188, 1976.

Monic Polynomial

A POLYNOMIAL in which the COEFFICIENT of the highest ORDER term is 1.

see also MONOMIAL

Monica Set

The nth Monica set M_n is defined as the set of COMPOSITE NUMBERS x for which $n|S(x)-S_p(x)$, where

$$x = a_0 + a_1(10^1)+\ldots+a_d(10^d) = p_1p_2\cdots p_n, \tag{1}$$

and

$$S(x) = \sum_{j=0}^d a_j \tag{2}$$

$$S_p(x) = \sum_{i=1}^m S(p_i). \tag{3}$$

Every Monica set has an infinite number of elements. The Monica set M_n is a subset of the SUZANNE SET S_n.

If x is a SMITH NUMBER, then it is a member of the Monica set M_n for all $n \in \mathbb{N}$. For any INTEGER $k > 1$, if x is a k-SMITH NUMBER, then $x \in M_{k-1}$.

see also SUZANNE SET

References

Smith, M. "Cousins of Smith Numbers: Monica and Suzanne Sets." *Fib. Quart.* **34**, 102–104, 1996.

Monkey and Coconut Problem

A DIOPHANTINE problem (i.e., one whose solution must be given in terms of INTEGERS) which seeks a solution to the following problem. Given n men and a pile of coconuts, each man in sequence takes $(1/n)$th of the coconuts and gives the m coconuts which do not divide equally to a monkey. When all n men have so divided, they divide the remaining coconuts five ways, and give the m coconuts which are left-over to the monkey. How many coconuts N were there originally? The solution is equivalent to solving the $n+1$ DIOPHANTINE EQUATIONS

$$\begin{aligned} N &= nA + m \\ (n-1)A &= nB + m \\ (n-1)B &= nC + m \\ &\vdots \\ (n-1)X &= nY + m \\ (n-1)Y &= nZ + m, \end{aligned}$$

and is given by

$$N = kn^{n+1} - m(n-1),$$

where k is an an arbitrary INTEGER (Gardner 1961).

For the particular case of $n = 5$ men and $m = 1$ left over coconuts, the 6 equations can be combined into the single DIOPHANTINE EQUATION

$$1,024N = 15,625F + 11,529,$$

where F is the number given to each man in the last division. The smallest POSITIVE solution in this case is $N = 15,621$ coconuts, corresponding to $k = 1$ and $F = 1,023$ (Gardner 1961). The following table shows how this rather large number of coconuts is divided under the scheme described above.

Removed	Given to Monkey	Left
		15,621
3,124	1	12,496
2,499	1	9,996
1,999	1	7,996
1,599	1	6,396
1,279	1	5,116
5 × 1023	1	0

If no coconuts are left for the monkey after the final n-way division (Williams 1926), then the original number of coconuts is

$$\begin{cases} (1+nk)n^n - (n-1) & n \text{ odd} \\ (n-1+nk)n^n - (n-1) & n \text{ even.} \end{cases}$$

The smallest POSITIVE solution for case $n = 5$ and $m = 1$ is $N = 3,121$ coconuts, corresponding to $k = 1$ and 1,020 coconuts in the final division (Gardner 1961). The following table shows how these coconuts are divided.

Removed	Given to Monkey	Left
		3,121
624	1	2,496
499	1	1,996
399	1	1,596
319	1	1,276
255	1	1,020
5 × 204	0	0

A different version of the problem having a solution of 79 coconuts is considered by Pappas (1989).

see also DIOPHANTINE EQUATION—LINEAR, PELL EQUATION

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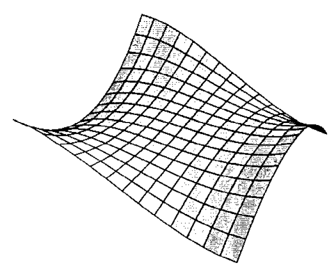
Williams, B. A. "Coconuts." *The Saturday Evening Post,* Oct. 9, 1926.

Monkey Saddle



A SURFACE which a monkey can straddle with both his two legs and his tail. A simply Cartesian equation for such a surface is

$$z = x(x^2 - 3y^2), \tag{1}$$

which can also be given by the parametric equations

$$x(u, v) = u \tag{2}$$
$$y(u, v) = v \tag{3}$$
$$z(u, v) = u^3 - 3uv^2. \tag{4}$$

The coefficients of the first and second FUNDAMENTAL FORMS of the monkey saddle are given by

$$e = \frac{6u}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}} \tag{5}$$
$$f = -\frac{6v}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}} \tag{6}$$
$$g = -\frac{6u}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}} \tag{7}$$
$$E = 1 + 9(u^2 - v^2)^2 \tag{8}$$
$$F = -18uv(u^2 - v^2) \tag{9}$$
$$G = 1 + 36u^2v^2, \tag{10}$$

giving RIEMANNIAN METRIC

$$ds^2 = [1 + (3u^2 - 3v^2)^2]\,du^2 - 2[18uv(u^2 - v^2)]\,du\,dv + (1 + 36u^2v^2)\,dv^2, \tag{11}$$

AREA ELEMENT

$$dA = \sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}\,du \wedge dv, \tag{12}$$

and GAUSSIAN and MEAN CURVATURES

$$K = -\frac{36(u^2 + v^2)}{(1 + 9u^4 + 18u^2v^2 + 9v^4)^2} \tag{13}$$
$$H = \frac{27u(-u^4 + 2u^2v^2 + 3v^4)}{(1 + 9u^4 + 18u^2v^2 + 9v^4)^{3/2}} \tag{14}$$

(Gray 1993). Every point of the monkey saddle except the origin has NEGATIVE GAUSSIAN CURVATURE.

see also CROSSED TROUGH, PARTIAL DERIVATIVE

References

Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, p. 365, 1969.

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Monochromatic Forced Triangle

Given a COMPLETE GRAPH K_n which is two-colored, the number of forced monochromatic TRIANGLES is at least

$$\begin{cases} \frac{1}{3}u(u-1)(u-2) & \text{for } n = 2u \\ \frac{2}{3}u(u-1)(4u+1) & \text{for } n = 4u+1 \\ \frac{2}{3}u(u+1)(4u-1) & \text{for } n = 4u+3. \end{cases}$$

The first few numbers of monochromatic forced triangles are 0, 0, 0, 0, 0, 2, 4, 8, 12, 20, 28, 40, ... (Sloane's A014557).

see also COMPLETE GRAPH, EXTREMAL GRAPH

References

Goodman, A. W. "On Sets of Acquaintances and Strangers at Any Party." *Amer. Math. Monthly* **66**, 778–783, 1959.

Sloane, N. J. A. Sequence A014553 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Monodromy

A general concept in CATEGORY THEORY involving the globalization of local MORPHISMS.

see also HOLONOMY

Monodromy Group

A technically defined GROUP characterizing a system of linear differential equations

$$y'_j = \sum_{k=1}^{n} a_{jk}(x) y_k$$

for $j = 1, \ldots, n$, where a_{jk} are COMPLEX ANALYTIC FUNCTIONS of x in a given COMPLEX DOMAIN.

see also HILBERT'S 21ST PROBLEM, RIEMANN P-SERIES

References

Iyanaga, S. and Kawada, Y. (Eds.). "Monodromy Groups." §253B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 793, 1980.

Monodromy Theorem

If a COMPLEX function f is ANALYTIC in a DISK contained in a simply connected DOMAIN D and f can be ANALYTICALLY CONTINUED along every polygonal arc in D, then f can be ANALYTICALLY CONTINUED to a single-valued ANALYTIC FUNCTION on all of D!

see also ANALYTIC CONTINUATION

Monogenic Function

If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

is the same for all paths in the COMPLEX PLANE, then $f(z)$ is said to be monogenic at z_0. Monogenic therefore essentially means having a single DERIVATIVE at a point. Functions are either monogenic or have infinitely many DERIVATIVES (in which case they are called POLYGENIC); intermediate cases are not possible.

see also POLYGENIC FUNCTION

References

Newman, J. R. *The World of Mathematics, Vol. 3.* New York: Simon & Schuster, p. 2003, 1956.

Monohedral Tiling

A TILING is which all tiles are congruent.

see also ANISOHEDRAL TILING, ISOHEDRAL TILING

References

Berglund, J. "Is There a k-Anisohedral Tile for $k \geq 5$?" *Amer. Math. Monthly* **100**, 585-588, 1993.

Grünbaum, B. and Shephard, G. C. "The 81 Types of Isohedral Tilings of the Plane." *Math. Proc. Cambridge Philos. Soc.* **82**, 177-196, 1977.

Monoid

A GROUP-like object which fails to be a GROUP because elements need not have an inverse within the object. A monoid S must also be ASSOCIATIVE and an IDENTITY ELEMENT $I \in S$ such that for all $a \in S$, $1a = a1 = a$. A monoid is therefore a SEMIGROUP with an identity element. A monoid must contain at least one element.

The numbers of free idempotent monoids on n letters are 1, 2, 7, 160, 332381, ... (Sloane's A005345).

see also BINARY OPERATOR, GROUP, SEMIGROUP

References

Rosenfeld, A. *An Introduction to Algebraic Structures.* New York: Holden-Day, 1968.

Sloane, N. J. A. Sequence A005345/M1820 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Monomial

A POLYNOMIAL consisting of a single term.

see also BINOMIAL, MONIC POLYNOMIAL, POLYNOMIAL, TRINOMIAL

Monomino

The unique 1-POLYOMINO, consisting of a single SQUARE.

see also DOMINO, TRIOMINO

References

Gardner, M. "Polyominoes." Ch. 13 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, pp. 124–140, 1959.

Monomorph

An INTEGER which is expressible in only one way in the form $x^2 + Dy^2$ or $x^2 - Dy^2$ where x^2 is RELATIVELY PRIME to Dy^2. If the INTEGER is expressible in more than one way, it is called a POLYMORPH.

see also ANTIMORPH, IDONEAL NUMBER, POLYMORPH

Monomorphism

An INJECTIVE MORPHISM.

Monotone

Another word for monotonic.

see also MONOTONIC FUNCTION, MONOTONIC SEQUENCE, MONOTONIC VOTING

Monotone Decreasing

Always decreasing; never remaining constant or increasing.

Monotone Increasing

Always increasing; never remaining constant or decreasing.

Monotonic Function

A function which is either entirely nonincreasing or nondecreasing. A function is monotonic if its first DERIVATIVE (which need not be continuous) does not change sign.

Monotonic Sequence

A SEQUENCE $\{a_n\}$ such that either (1) $a_{i+1} \geq a_i$ for every $i \geq 1$, or (2) $a_{i+1} \leq a_i$ for every $i \geq 1$.

Monotonic Voting

A term in SOCIAL CHOICE THEORY meaning a change favorable for X does not hurt X.

see also ANONYMOUS, DUAL VOTING

Monster Group

The highest order SPORADIC GROUP M. It has ORDER

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

and is also called the FRIENDLY GIANT GROUP. It was constructed in 1982 by Robert Griess as a GROUP of ROTATIONS in 196,883-D space.

see also BABY MONSTER GROUP, BIMONSTER, LEECH LATTICE

References

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. viii, 1985.

Conway, J. H. and Norton, S. P. "Monstrous Moonshine." *Bull. London Math. Soc.* **11**, 308–339, 1979.

Conway, J. H. and Sloane, N. J. A. "The Monster Group and its 196884-Dimensional Space" and "A Monster Lie Algebra?" Chs. 29–30 in *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, pp. 554–571, 1993.

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/M.html`.

Monte Carlo Integration

In order to integrate a function over a complicated DOMAIN D, Monte Carlo integration picks random points over some simple DOMAIN D' which is a superset of D, checks whether each point is within D, and estimates the AREA of D (VOLUME, n-D CONTENT, etc.) as the AREA of D' multiplied by the fraction of points falling within D'.

An estimate of the uncertainty produced by this technique is given by

$$\int f\,dV \approx V\langle f\rangle \pm \sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{N}}.$$

see also MONTE CARLO METHOD

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Simple Monte Carlo Integration" and "Adaptive and Recursive Monte Carlo Methods." §7.6 and 7.8 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 295–299 and 306–319, 1992.

Monte Carlo Method

Any method which solves a problem by generating suitable random numbers and observing that fraction of the numbers obeying some property or properties. The method is useful for obtaining numerical solutions to problems which are too complicated to solve analytically. The most common application of the Monte Carlo method is MONTE CARLO INTEGRATION.

see also MONTE CARLO INTEGRATION

References

Sobol, I. M. *A Primer for the Monte Carlo Method.* Boca Raton, FL: CRC Press, 1994.

Montel's Theorem

Let $f(z)$ be an analytic function of z, regular in the half-strip S defined by $a < x < b$ and $y > 0$. If $f(z)$ is bounded in S and tends to a limit l as $y \to \infty$ for a certain fixed value ξ of x between a and b, then $f(z)$ tends to this limit l on every line $x = x_0$ in S, and $f(z) \to l$ uniformly for $a + \delta \leq x_0 \leq b - \delta$.

see also VITALI'S CONVERGENCE THEOREM

References

Titchmarsh, E. C. *The Theory of Functions, 2nd ed.* Oxford, England: Oxford University Press, p. 170, 1960.

Monty Hall Dilemma

see MONTY HALL PROBLEM

Monty Hall Problem

The Monty Hall problem is named for its similarity to the *Let's Make a Deal* television game show hosted by Monty Hall. The problem is stated as follows. Assume that a room is equipped with three doors. Behind two are goats, and behind the third is a shiny new car. You are asked to pick a door, and will win whatever is behind it. Let's say you pick door 1. Before the door is opened, however, someone who knows what's behind the doors (Monty Hall) opens *one of the other* two doors, revealing a goat, and asks you if you wish to change your selection to the third door (i.e., the door which neither you picked nor he opened). The Monty Hall problem is deciding whether you do.

The correct answer is that you *do* want to switch. If you do not switch, you have the expected 1/3 chance of winning the car, since no matter whether you initially picked the correct door, Monty will show you a door with a goat. But after Monty has eliminated one of the doors for you, you obviously do not improve your chances of winning to better than 1/3 by sticking with your original choice. If you now switch doors, however, there is a 2/3 chance you will win the car (counterintuitive though it seems).

d_1	d_2	Winning Probability
pick	stick	1/3
pick	switch	2/3

The problem can be generalized to four doors as follows. Let one door conceal the car, with goats behind the other three. Pick a door d_1. Then the host will open one of the nonwinners and give you the option of switching. Call your new choice (which could be the same as d_1 if you don't switch) d_2. The host will then open a second nonwinner, and you must decide for choice d_3 if you want to stick to d_2 or switch to the remaining door. The probabilities of winning are shown below for the four possible strategies.

d_1	d_2	d_3	Winning Probability
pick	stick	stick	4/8
pick	switch	stick	3/8
pick	stick	switch	6/8
pick	switch	switch	5/8

The above results are characteristic of the best strategy for the n-stage Monty Hall problem: stick until the last choice, then switch.

see also ALIAS' PARADOX

References

Barbeau, E. "The Problem of the Car and Goats." *CMJ* **24**, 149, 1993.

Bogomolny, A. "Monty Hall Dilemma." `http://www.cut-the-knot.com/hall.html`.

Dewdney, A. K. *200% of Nothing.* New York: Wiley, 1993.

Donovan, D. "The WWW Tackles the Monty Hall Problem." `http://math.rice.edu/~ddonovan/montyurl.html`.

Ellis, K. M. "The Monty Hall Problem." `http://www.io.com/~kmellis/monty.html`.

Gardner, M. *Aha! Gotcha: Paradoxes to Puzzle and Delight.* New York: W. H. Freeman, 1982.
Gillman, L. "The Car and the Goats." *Amer. Math. Monthly* **99**, 3, 1992.
Selvin, S. "A Problem in Probability." *Amer. Stat.* **29**, 67, 1975.
vos Savant, M. *The Power of Logical Thinking.* New York: St. Martin's Press, 1996.

Moore Graph

A GRAPH with DIAMETER d and GIRTH $2d+1$. Moore graphs have DIAMETER of at most 2. Every Moore graph is both REGULAR and distance regular. Hoffman and Singleton (1960) show that k-regular Moore graphs with DIAMETER 2 have $k \in \{2, 3, 7, 57\}$.

References
Godsil, C. D. "Problems in Algebraic Combinatorics." *Electronic J. Combinatorics* **2**, F1, 1–20, 1995. http://www.combinatorics.org/Volume_2/volume2.html#F1.
Hoffman, A. J. and Singleton, R. R. "On Moore Graphs of Diameter Two and Three." *IBM J. Res. Develop.* **4**, 497–504, 1960.

Moore-Penrose Generalized Matrix Inverse

Given an $m \times n$ MATRIX B, the Moore-Penrose generalized MATRIX INVERSE is a unique $n \times m$ MATRIX B^+ which satisfies

$$\mathsf{BB}^+\mathsf{B} = \mathsf{B} \tag{1}$$
$$\mathsf{B}^+\mathsf{BB}^+ = \mathsf{B}^+ \tag{2}$$
$$(\mathsf{BB}^+)^{\mathrm{T}} = \mathsf{BB}^+ \tag{3}$$
$$(\mathsf{B}^+\mathsf{B})^{\mathrm{T}} = \mathsf{B}^+\mathsf{B}. \tag{4}$$

It is also true that

$$\mathbf{z} = \mathsf{B}^+\mathbf{c} \tag{5}$$

is the shortest length LEAST SQUARES solution to the problem

$$\mathsf{B}\mathbf{z} = \mathbf{c}. \tag{6}$$

If the inverse of $(\mathsf{B}^{\mathrm{T}}\mathsf{B})$ exists, then

$$\mathsf{B}^+ = (\mathsf{B}^{\mathrm{T}}\mathsf{B})^{-1}\mathsf{B}^{\mathrm{T}}, \tag{7}$$

where B^{T} is the MATRIX TRANSPOSE, as can be seen by premultiplying both sides of (7) by B^{T} to create a SQUARE MATRIX which can then be inverted,

$$\mathsf{B}^{\mathrm{T}}\mathsf{B}\mathbf{z} = \mathsf{B}^{\mathrm{T}}\mathbf{c}, \tag{8}$$

giving

$$\begin{aligned} \mathbf{z} &= (\mathsf{B}^{\mathrm{T}}\mathsf{B})^{-1}\mathsf{B}^{\mathrm{T}}\mathbf{c} \\ &\equiv \mathsf{B}^+\mathbf{c}. \end{aligned} \tag{9}$$

see also LEAST SQUARES FITTING, MATRIX INVERSE

References
Ben-Israel, A. and Greville, T. N. E. *Generalized Inverses: Theory and Applications.* New York: Wiley, 1977.
Lawson, C. and Hanson, R. *Solving Least Squares Problems.* Englewood Cliffs, NJ: Prentice-Hall, 1974.
Penrose, R. "A Generalized Inverse for Matrices." *Proc. Cambridge Phil. Soc.* **51**, 406–413, 1955.

Mordell Conjecture

DIOPHANTINE EQUATIONS that give rise to surfaces with two or more holes have only finite many solutions in GAUSSIAN INTEGERS with no common factors. Fermat's equation has $(n-1)(n-2)/2$ HOLES, so the Mordell conjecture implies that for each INTEGER $n \geq 3$, the FERMAT EQUATION has at most a finite number of solutions. This conjecture was proved by Faltings (1984).

see also FERMAT EQUATION, FERMAT'S LAST THEOREM, SAFAREVICH CONJECTURE, SHIMURA-TANIYAMA CONJECTURE

References
Faltings, G. "Die Vermutungen von Tate und Mordell." *Jahresber. Deutsch. Math.-Verein* **86**, 1–13, 1984.
Ireland, K. and Rosen, M. "The Mordell Conjecture." §20.3 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 340–342, 1990.

Mordell Integral

The integral

$$\phi(t,u) = \int \frac{e^{\pi itx^2 + 2\pi iux}}{e^{2\pi ix} - 1}\,dx$$

which is related to the THETA FUNCTIONS, MOCK THETA FUNCTIONS, and RIEMANN ZETA FUNCTION.

Mordell-Weil Theorem

For ELLIPTIC CURVES over the RATIONALS, $\mathbb{Q}$, the number of generators of the set of RATIONAL POINTS is always finite. This theorem was proved by Mordell in 1921 and extended by Weil in 1928 to ABELIAN VARIETIES over NUMBER FIELDS.

References
Ireland, K. and Rosen, M. "The Mordell-Weil Theorem." Ch. 19 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 319–338, 1990.

Morera's Theorem

If $f(z)$ is continuous in a simply connected region D and satisfies

$$\int_\gamma f\,dz = 0$$

for all closed CONTOURS γ in D, then $f(z)$ is ANALYTIC in D.

see also CAUCHY INTEGRAL THEOREM

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 373–374, 1985.

Morgado Identity

An identity satisfied by w GENERALIZED FIBONACCI NUMBERS:

$$4w_nw_{n+1}w_{n+2}w_{n+4}w_{n+5}w_{n+6} + e^2q^{2n}(w_nU_4U_5 - w_{n+1}U_2U_6 - w_nU_1U_8)^2 = (w_{n+1}w_{n+2}w_{n+6} + w_nw_{n+4}w_{n+5})^2, \quad (1)$$

where

$$e \equiv pab - qa^2 - b^2 \quad (2)$$

$$U_n \equiv w_n(0,1;p,q). \quad (3)$$

see also GENERALIZED FIBONACCI NUMBER

References

Morgado, J. "Note on Some Results of A. F. Horadam and A. G. Shannon Concerning a Catalan's Identity on Fibonacci Numbers." *Portugaliae Math.* **44**, 243–252, 1987.

Morgan-Voyce Polynomial

Polynomials related to the BRAHMAGUPTA POLYNOMIALS. They are defined by the RECURRENCE RELATIONS

$$b_n(x) = xB_{n-1}(x) + b_{n-1}(x) \quad (1)$$

$$B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x) \quad (2)$$

for $n \geq 1$, with

$$b_0(x) = B_0(x) = 1. \quad (3)$$

Alternative recurrences are

$$B_{n+1}B_{n-1} - {B_n}^2 = -1 \quad (4)$$

$$b_{n+1}b_{n-1} - {b_n}^2 = x. \quad (5)$$

The polynomials can be given explicitly by the sums

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k-1}{n-k} \quad (6)$$

$$b_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k}. \quad (7)$$

Defining the MATRIX

$$\mathsf{Q} = \begin{bmatrix} x+2 & -1 \\ 1 & 0 \end{bmatrix} \quad (8)$$

gives the identities

$$\mathsf{Q}^n = \begin{bmatrix} B_n & -B_{n-1} \\ B_{n-1} & -B_{n-2} \end{bmatrix} \quad (9)$$

$$\mathsf{Q}^n - \mathsf{Q}^{n-1} = \begin{bmatrix} b_n & -b_{n-1} \\ b_{n-1} & -b_{n-2} \end{bmatrix}. \quad (10)$$

Defining

$$\cos\theta = \tfrac{1}{2}(x+2) \quad (11)$$

$$\cosh\phi = \tfrac{1}{2}(x+2) \quad (12)$$

gives

$$B_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta} \quad (13)$$

$$B_n(x) = \frac{\sinh[(n+1)\phi]}{\sinh\phi} \quad (14)$$

and

$$b_n(x) = \frac{\cos[\frac{1}{2}(2n+1)\theta]}{\cos(\frac{1}{2}\theta)} \quad (15)$$

$$b_n(x) = \frac{\cosh[\frac{1}{2}(2n+1)\phi]}{\cosh(\frac{1}{2}\theta)}. \quad (16)$$

The Morgan-Voyce polynomials are related to the FIBONACCI POLYNOMIALS $F_n(x)$ by

$$b_n(x^2) = F_{2n+1}(x) \quad (17)$$

$$B_n(x^2) = \frac{1}{x}F_{2n+2}(x) \quad (18)$$

(Swamy 1968).

$B_n(x)$ satisfies the ORDINARY DIFFERENTIAL EQUATION

$$x(x+4)y'' + 3(x+2)y' - n(n+2)y = 0, \quad (19)$$

and $b_n(x)$ the equation

$$x(x+4)y'' + 2(x+1)y' - n(n+1)y = 0. \quad (20)$$

These and several other identities involving derivatives and integrals of the polynomials are given by Swamy (1968).

see also BRAHMAGUPTA POLYNOMIAL, FIBONACCI POLYNOMIAL

References

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Morgan-Voyce, A. M. "Ladder Network Analysis Using Fibonacci Numbers." *IRE Trans. Circuit Th.* **CT-6**, 321–322, Sep. 1959.

Swamy, M. N. S. "Properties of the Polynomials Defined by Morgan-Voyce." *Fib. Quart.* **4**, 73–81, 1966.

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Swamy, M. N. S. "Further Properties of Morgan-Voyce Polynomials." *Fib. Quart.* **6**, 167–175, 1968.

Morley Centers

The CENTROID of MORLEY'S TRIANGLE is called Morley's first center. It has TRIANGLE CENTER FUNCTION

$$\alpha = \cos(\tfrac{1}{3}A) + 2\cos(\tfrac{1}{3}B)\cos(\tfrac{1}{3}C).$$

The PERSPECTIVE CENTER of MORLEY'S TRIANGLE with reference TRIANGLE ABC is called Morley's second center. The TRIANGLE CENTER FUNCTION is

$$\alpha = \sec(\tfrac{1}{3}A).$$

see also CENTROID (GEOMETRIC), MORLEY'S THEOREM, PERSPECTIVE CENTER

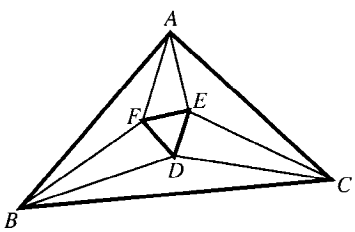
References
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. "1st and 2nd Morley Centers." `http://www.evansville.edu/~ck6/tcenters/recent/morley.html`.
Oakley, C. O. and Baker, J. C. "The Morley Trisector Theorem." *Amer. Math. Monthly* **85**, 737–745, 1978.

Morley's Formula

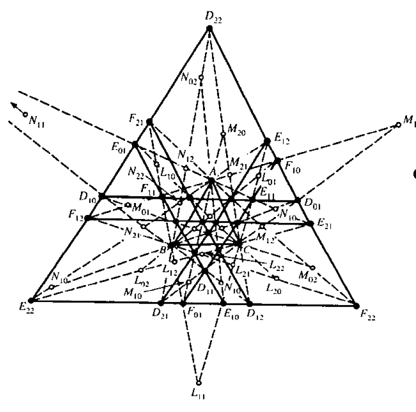
$$\sum_{k=0}^{\infty}\binom{m}{k}^3 = 1 + \left(\frac{m}{1}\right)^3 + \left[\frac{m(m+1)}{1\cdot 2}\right]^3 + \ldots$$
$$= \frac{\Gamma(1-\frac{3}{2}m)}{[\Gamma(1-\frac{1}{2}m)]^3}\cos(\tfrac{1}{2}m\pi),$$

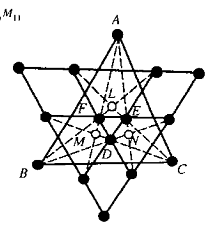
where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT and $\Gamma(z)$ is the GAMMA FUNCTION.

Morley's Theorem

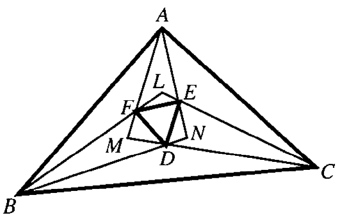


The points of intersection of the adjacent TRISECTORS of the ANGLES of any TRIANGLE ΔABC are the VERTICES of an EQUILATERAL TRIANGLE ΔDEF known as MORLEY'S TRIANGLE. Taylor and Marr (1914) give two geometric proofs and one trigonometric proof.





A generalization of MORLEY'S THEOREM was discovered by Morley in 1900 but first published by Taylor and Marr (1914). Each ANGLE of a TRIANGLE ΔABC has six trisectors, since each interior angle trisector has two associated lines making angles of 120° with it. The generalization of Morley's theorem states that these trisectors intersect in 27 points (denoted D_{ij}, E_{ij}, F_{ij}, for $i, j = 0$, 1, 2) which lie six by six on nine lines. Furthermore, these lines are in three triples of PARALLEL lines, $(D_{22}E_{22}, E_{12}D_{21}, F_{10}F_{01})$, $(D_{22}F_{22}, F_{21}D_{12}, E_{01}E_{10})$, and $(E_{22}F_{22}, F_{12}E_{21}, D_{10}D_{01})$, making ANGLES of 60° with one another (Taylor and Marr 1914, Johnson 1929, p. 254).



Let L, M, and N be the other trisector-trisector intersections, and let the 27 points L_{ij}, M_{ij}, N_{ij} for $i, j = 0$, 1, 2 be the ISOGONAL CONJUGATES of D, E, and F. Then these points lie 6 by 6 on 9 CONICS through ΔABC. In addition, these CONICS meet 3 by 3 on the CIRCUMCIRCLE, and the three meeting points form an EQUILATERAL TRIANGLE whose sides are PARALLEL to those of ΔDEF.

see also CONIC SECTION, MORLEY CENTERS, TRISECTION

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Gardner, M. *Martin Gardner's New Mathematical Diversions from Scientific American.* New York: Simon and Schuster, pp. 198 and 206, 1966.
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Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 253–256, 1929.
Kimberling, C. "Hofstadter Points." *Nieuw Arch. Wiskunder* **12**, 109–114, 1994.
Marr, W. L. "Morley's Trisection Theorem: An Extension and Its Relation to the Circles of Apollonius." *Proc. Edinburgh Math. Soc.* **32**, 136–150, 1914.
Oakley, C. O. and Baker, J. C. "The Morley Trisector Theorem." *Amer. Math. Monthly* **85**, 737–745, 1978.
Pappas, T. "Trisecting & the Equilateral Triangle." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 174, 1989.
Taylor, F. G. "The Relation of Morley's Theorem to the Hessian Axis and Circumcentre." *Proc. Edinburgh Math. Soc.* **32**, 132–135, 1914.
Taylor, F. G. and Marr, W. L. "The Six Trisectors of Each of the Angles of a Triangle." *Proc. Edinburgh Math. Soc.* **32**, 119–131, 1914.

Morley's Triangle

An EQUILATERAL TRIANGLE considered by MORLEY'S THEOREM with side lengths

$$8R\sin(\tfrac{1}{3}A)\sin(\tfrac{1}{3}B)\sin(\tfrac{1}{3}C),$$

where R is the CIRCUMRADIUS of the original TRIANGLE.

Morphism

A map between two objects in an abstract CATEGORY.

1. A general morphism is called a HOMOMORPHISM,
2. An injective morphism is called a MONOMORPHISM,
3. A surjective morphism is an EPIMORPHISM,
4. A bijective morphism is called an ISOMORPHISM (if there is an isomorphism between two objects, then we say they are isomorphic),
5. A surjective morphism from an object to itself is called an ENDOMORPHISM, and
6. An ISOMORPHISM between an object and itself is called an AUTOMORPHISM.

see also AUTOMORPHISM, EPIMORPHISM, HOMEOMORPHISM, HOMOMORPHISM, ISOMORPHISM, MONOMORPHISM, OBJECT

Morrie's Law

$$\cos(20^\circ)\cos(40^\circ)\cos(80^\circ)=\tfrac{1}{8}.$$

This identity was referred to by Feynman (Gleick 1992). It is a special case of the general identity

$$2^k\prod_{j=0}^{k-1}\cos(2^j a)=\frac{\sin(2^k a)}{\sin a},$$

with $k=3$ and $a=20^\circ$ (Beyer *et al.* 1996).

References

Anderson, E. C. "Morrie's Law and Experimental Mathematics." To appear in *J. Recr. Math.*

Beyer, W. A.; Louck, J. D.; Zeilberger, D. "A Generalization of a Curiosity that Feynman Remembered All His Life." *Math. Mag.* **69**, 43–44, 1996.

Gleick, J. *Genius: The Life and Science of Richard Feynman.* New York: Pantheon Books, p. 47, 1992.

Morse Inequalities

Topological lower bounds in terms of BETTI NUMBERS for the number of critical points form a smooth function on a smooth MANIFOLD.

Morse Theory

"CALCULUS OF VARIATIONS in the large" which uses nonlinear techniques to address problems in the CALCULUS OF VARIATIONS. Morse theory applied to a FUNCTION g on a MANIFOLD W with $g(M)=0$ and $g(M')=1$ shows that every COBORDISM can be realized as a finite sequence of SURGERIES. Conversely, a sequence of SURGERIES gives a COBORDISM.

see also CALCULUS OF VARIATIONS, COBORDISM, SURGERY

Morse-Thue Sequence

see THUE-MORSE SEQUENCE

Mortal

A nonempty finite set of $n \times n$ MATRICES with INTEGER entries for which there exists some product of the MATRICES in the set which is equal to the zero MATRIX.

Mortality Problem

For a given n, is the problem of determining if a set is MORTAL solvable? $n=1$ is solvable, $n=2$ is unknown, and $n\geq 3$ is unsolvable.

see also LIFE EXPECTANCY

Morton-Franks-Williams Inequality

Let E be the largest and e the smallest POWER of ℓ in the HOMFLY POLYNOMIAL of an oriented LINK, and i be the BRAID INDEX. Then the MORTON-FRANKS-WILLIAMS INEQUALITY holds,

$$i\geq \tfrac{1}{2}(E-e)+1$$

(Franks and Williams 1985, Morton 1985). The inequality is sharp for all PRIME KNOTS up to 10 crossings with the exceptions of 09_{042}, 09_{049}, 10_{132}, 10_{150}, and 10_{156}.

see also BRAID INDEX

References

Franks, J. and Williams, R. F. "Braids and the Jones Polynomial." *Trans. Amer. Math. Soc.* **303**, 97–108, 1987.

Mosaic

see TESSELLATION

Moser

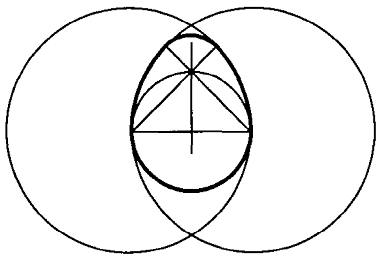
The very LARGE NUMBER consisting of the number 2 inside a MEGA-gon.

see also MEGA, MEGISTRON

Moser's Circle Problem

see CIRCLE CUTTING

Moss's Egg

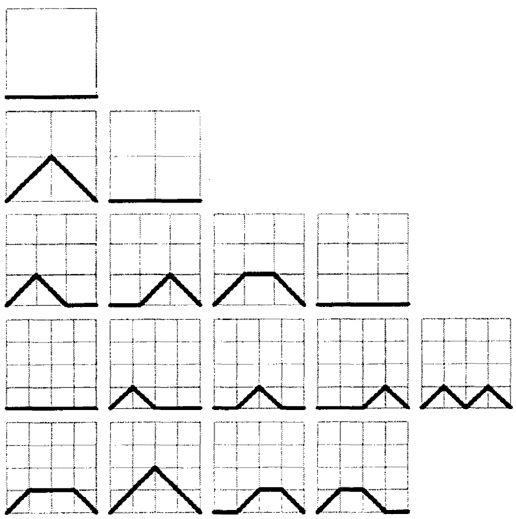


An OVAL whose construction is illustrated in the above diagram.

see also EGG, OVAL

References
Dixon, R. *Mathographics.* New York: Dover, p. 5, 1991.

Motzkin Number



The Motzkin numbers enumerate various combinatorial objects. Donaghey and Shapiro (1977) give 14 different manifestations of these numbers. In particular, they give the number of paths from (0, 0) to (n, 0) which never dip below $y = 0$ and are made up only of the steps (1, 0), (1, 1), and (1, −1), i.e., $\rightarrow$, $\nearrow$, and $\searrow$. The first are 1, 2, 4, 9, 21, 51, ... (Sloane's A001006). The Motzkin number GENERATING FUNCTION $M(z)$ satisfies

$$M = 1 + xM + x^2 M^2 \tag{1}$$

and is given by

$$\begin{aligned} M(x) &= \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \\ &= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + \ldots, \end{aligned} \tag{2}$$

or by the RECURRENCE RELATION

$$M_n = a_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k} \tag{3}$$

with $M_0 = 1$. The Motzkin number M_n is also given by

$$M_n = -\frac{1}{2} \sum_{\substack{a+b=n+2 \\ a\geq 0, b\geq 0}} (-3)^a \binom{\frac{1}{2}}{a} \binom{\frac{1}{2}}{b} \tag{4}$$

$$= \frac{(-1)^{n+1}}{2^{2n+5}} \sum_{\substack{a+b=n+2 \\ a\geq 0, b\geq 0}} \frac{(-3)^a}{(2a-1)(2b-1)} \binom{2a}{a} \binom{2b}{b}, \tag{5}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

see also CATALAN NUMBER, KING WALK, SCHRÖDER NUMBER

References
Barcucci, E.; Pinzani, R.; and Sprugnoli, R. "The Motzkin Family." *Pure Math. Appl. Ser. A* **2**, 249–279, 1991.
Donaghey, R. "Restricted Plane Tree Representations of Four Motzkin-Catalan Equations." *J. Combin. Th. Ser. B* **22**, 114–121, 1977.
Donaghey, R. and Shapiro, L. W. "Motzkin Numbers." *J. Combin. Th. Ser. A* **23**, 291–301, 1977.
Kuznetsov, A.; Pak, I.; and Postnikov, A. "Trees Associated with the Motzkin Numbers." *J. Combin. Th. Ser. A* **76**, 145–147, 1996.
Motzkin, T. "Relations Between Hypersurface Cross Ratios, and a Combinatorial Formula for Partitions of a Polygon, for Permanent Preponderance, and for Nonassociative Products." *Bull. Amer. Math. Soc.* **54**, 352–360, 1948.
Sloane, N. J. A. Sequence A001006/M1184 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Moufang Plane

A PROJECTIVE PLANE in which every line is a translation line is called a Moufang plane.

References
Colbourn, C. J. and Dinitz, J. H. (Eds.) *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, p. 710, 1996.

Mousetrap

A PERMUTATION problem invented by Cayley.

References
Guy, R. K. "Mousetrap." §E37 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 237–238, 1994.

Mouth

A PRINCIPAL VERTEX x_i of a SIMPLE POLYGON P is called a mouth if the diagonal $[x_{i-1}, x_{i+1}]$ is an extremal diagonal (i.e., the interior of $[x_{i-1}, x_{i+1}]$ lies in the exterior of P).

see also ANTHROPOMORPHIC POLYGON, EAR, ONE-MOUTH THEOREM

References
Toussaint, G. "Anthropomorphic Polygons." *Amer. Math. Monthly* **122**, 31–35, 1991.

Moving Average

Given a SEQUENCE $\{a_i\}_{i=1}^N$, an n-moving average is a new sequence $\{s_i\}_{i=1}^{N-n+1}$ defined from the a_i by taking the AVERAGE of subsequences of n terms,

$$s_i = \frac{1}{n}\sum_{j=i}^{i+n-1} a_j.$$

see also AVERAGE, SPENCER'S 15-POINT MOVING AVERAGE

References

Kenney, J. F. and Keeping, E. S. "Moving Averages." §14.2 in *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 221–223, 1962.

Moving Ladder Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

What is the longest ladder which can be moved around a right-angled hallway of unit width? For a straight, rigid ladder, the answer is $2\sqrt{2}$. For a smoothly-shaped ladder, the largest diameter is $\geq 2(1+\sqrt{2})$ (Finch).

see also MOVING SOFA CONSTANT, PIANO MOVER'S PROBLEM

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/sofa/sofa.html`.

Moving Sofa Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

What is the sofa of greatest AREA S which can be moved around a right-angled hallway of unit width? Hammersley (Croft *et al.* 1994) showed that

$$S \geq \frac{\pi}{2} + \frac{2}{\pi} = 2.2074\ldots. \tag{1}$$

Gerver (1992) found a sofa with larger AREA and provided arguments indicating that it is either optimal or close to it. The boundary of Gerver's sofa is a complicated shape composed of 18 ARCS. Its AREA can be given by defining the constants A, B, ϕ, and θ by solving

$$A(\cos\theta - \cos\phi) - 2B\sin\phi + (\theta - \phi - 1)\cos\theta - \sin\theta + \cos\phi + \sin\phi = 0 \tag{2}$$

$$A(3\sin\theta + \sin\phi) - 2B\cos\phi + 3(\theta - \phi - 1)\sin\theta + 3\cos\theta - \sin\phi + \cos\phi = 0 \tag{3}$$

$$A\cos\phi - (\sin\phi + \tfrac{1}{2} - \tfrac{1}{2}\cos\phi + B\sin\phi) = 0 \tag{4}$$

$$(A + \tfrac{1}{2}\pi - \phi - \theta) - [B - \tfrac{1}{2}(\theta - \phi)(1+A) - \tfrac{1}{4}(\theta-\phi)^2] = 0. \tag{5}$$

This gives

$$A = 0.094426560843653\ldots \tag{6}$$

$$B = 1.399203727333547\ldots \tag{7}$$

$$\phi = 0.039177364790084\ldots \tag{8}$$

$$\theta = 0.681301509382725\ldots. \tag{9}$$

Now define

$$r(\alpha) \equiv \begin{cases} \tfrac{1}{2} & \text{for } 0 \leq \alpha < \phi \\ \tfrac{1}{2}(1 + A + \alpha - \phi) & \text{for } \phi \leq \alpha < \theta \\ A + \alpha - \phi & \text{for } \theta \leq \alpha < \tfrac{1}{2}\pi - \theta \\ B - \tfrac{1}{2}(\tfrac{1}{2}\pi - \alpha - \phi)(1+A) - \tfrac{1}{4}(\tfrac{1}{2}\pi - \alpha - \phi)^2, & \text{for } \tfrac{1}{2}\pi - \theta \leq \alpha < \tfrac{1}{2}\pi - \phi, \end{cases} \tag{10}$$

where

$$s(\alpha) \equiv 1 - r(\alpha) \tag{11}$$

$$u(\alpha) \equiv \begin{cases} B - \tfrac{1}{2}(\alpha - \phi)(1+A) - \tfrac{1}{4}(\alpha-\phi)^2 & \text{for } \phi \leq \alpha < \theta \\ A + \tfrac{1}{2}\pi - \phi - \alpha & \text{for } \theta \leq \alpha < \tfrac{1}{4}\pi \end{cases} \tag{12}$$

$$D_u(\alpha) = \frac{du}{d\alpha} = \begin{cases} -\tfrac{1}{2}(1+A) - \tfrac{1}{2}(\alpha - \phi) & \text{for } \phi \leq \alpha \leq \theta \\ -1 & \text{if } \theta \leq \alpha < \tfrac{1}{4}\pi. \end{cases} \tag{13}$$

Finally, define the functions

$$y_1(\alpha) \equiv 1 - \int_0^\alpha r(t)\sin t\,dt \tag{14}$$

$$y_2(\alpha) \equiv 1 - \int_0^\alpha s(t)\sin t\,dt \tag{15}$$

$$y_3(\alpha) \equiv 1 - \int_0^\alpha s(t)\sin t\,dt - u(\alpha)\sin\alpha. \tag{16}$$

The AREA of the optimal sofa is given by

$$\begin{aligned} A = 2\int_0^{\pi/2-\phi} y_1(\alpha) r(\alpha)\cos\alpha\,d\alpha &+ 2\int_0^\theta y_2(\alpha)s(\alpha)\cos\alpha\,d\alpha \\ + 2\int_\phi^{\pi/4} y_3(\alpha)[u(\alpha)\sin\alpha - D_u(\alpha)\cos\alpha - s(\alpha)\cos\alpha]\,d\alpha \\ = 2.21953166887197\ldots \end{aligned} \tag{17}$$

(Finch).

see also PIANO MOVER'S PROBLEM

References

Croft, H. T.; Falconer, K. J.; and Guy, R. K. *Unsolved Problems in Geometry.* New York: Springer-Verlag, 1994.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/sofa/sofa.html`.

Gerver, J. L. "On Moving a Sofa Around a Corner." *Geometriae Dedicata* **42**, 267–283, 1992.

Stewart, I. *Another Fine Math You've Got Me Into....* New York: W. H. Freeman, 1992.

Mrs. Perkins' Quilt

The DISSECTION of a SQUARE of side n into a number S_n of smaller squares. Unlike a PERFECT SQUARE DISSECTION, however, the smaller SQUARES need not be all different sizes. In addition, only prime dissections are considered so that patterns which can be dissected on lower order SQUARES are not permitted. The following table gives the smallest number of coprime dissections of an $n \times n$ quilt (Sloane's A005670).

n	S_n
1	1
2	4
3	6
4	7
5	8
6–7	9
8–9	10
10–13	11
14–17	12
18–23	13
24–29	14
30-39	15
40	16
41	15
42–100	[17, 19]

see also PERFECT SQUARE DISSECTION

References

Conway, J. H. "Mrs. Perkins's Quilt." *Proc. Cambridge Phil. Soc.* **60**, 363–368, 1964.

Dudeney, H. E. Problem 173 in *Amusements in Mathematics.* New York: Dover, 1917.

Dudeney, H. E. Problem 177 in *536 Puzzles & Curious Problems.* New York: Scribner, 1967.

Gardner, M. "Mrs. Perkins' Quilt and Other Square-Packing Problems." Ch. 11 in *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage, 1977.

Sloane, N. J. A. Sequence A005670/M3267 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Trustrum, G. B. "Mrs. Perkins's Quilt." *Proc. Cambridge Phil. Soc.* **61**, 7–11, 1965.

Mu Function

$$\mu(x,\beta) \equiv \int_0^\infty \frac{x^t t^\beta\,dt}{\Gamma(\beta+1)\Gamma(t+1)}$$

$$\mu(x,\beta,\alpha) \equiv \int_0^\infty \frac{x^{\alpha+t} t^\beta\,dt}{\Gamma(\beta+1)\Gamma(\alpha+t+1)},$$

where $\Gamma(z)$ is the GAMMA FUNCTION (Gradshteyn and Ryzhik 1980, p. 1079).

see also LAMBDA FUNCTION, NU FUNCTION

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, 1979.

μ Molecule

see MANDELBROT SET

Much Greater

A strong INEQUALITY in which a is not only GREATER than b, but *much* greater (by some convention), is denoted $a \gg b$. For an astronomer, "much" may mean by a factor of 100 (or even 10), while for a mathematician, it might mean by a factor of 10^4 (or even much more).

see also GREATER, MUCH LESS

Much Less

A strong INEQUALITY in which a is not only LESS than b, but *much* less (by some convention) is denoted $a \ll b$.

see also LESS, MUCH GREATER

Muirhead's Theorem

A NECESSARY and SUFFICIENT condition that $[\alpha']$ should be comparable with $[\alpha]$ for all POSITIVE values of the a is that one of (α') and (α) should be majorized by the other. If $(\alpha') \prec (\alpha)$, then

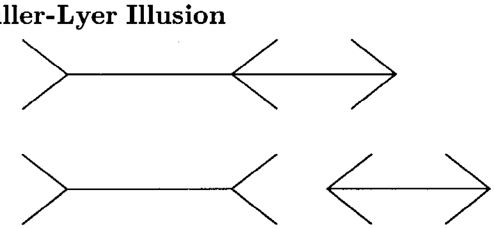
$$[\alpha'] \leq [\alpha],$$

with equality only when (α') and (α) are identical or when all the a are equal. See Hardy *et al.* (1988) for a definition of notation.

References

Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 44–48, 1988.

Müller-Lyer Illusion



An optical ILLUSION in which the orientation of arrowheads makes one LINE SEGMENT look longer than another. In the above figure, the LINE SEGMENTS on the left and right are of equal length in both cases.

see also ILLUSION, POGGENDORFF ILLUSION, PONZO'S ILLUSION, VERTICAL-HORIZONTAL ILLUSION

References

Fineman, M. *The Nature of Visual Illusion.* New York: Dover, p. 153, 1996.

Luckiesh, M. *Visual Illusions: Their Causes, Characteristics & Applications.* New York: Dover, p. 93, 1965.

Muller's Method

Generalizes the SECANT METHOD of root finding by using quadratic 3-point interpolation

$$q \equiv \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}. \quad (1)$$

Then define

$$A \equiv qP(x_n) - q(1+q)P(x_{n-1}) + q^2P(x_{n-2}) \quad (2)$$

$$B \equiv (2q+1)P(x_n) - (1+q)^2P(x_{n-1}) + q^2P(x_{n-2}) \quad (3)$$

$$C \equiv (1+q)P(x_n), \quad (4)$$

and the next iteration is

$$x_{n+1} = x_n - (x_n - x_{n-1})\frac{2C}{\max(B \pm \sqrt{B^2 - 4AC})}. \quad (5)$$

This method can also be used to find COMPLEX zeros of ANALYTIC FUNCTIONS.

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 364, 1992.

Mulliken Symbols

Symbols used to identify irreducible representations of GROUPS:

A = singly degenerate state which is symmetric with respect to ROTATION about the principal C_n axis,

B = singly DEGENERATE state which is antisymmetric with respect to ROTATION about the principal C_n axis,

E = doubly DEGENERATE,

T = triply DEGENERATE,

X_g = (gerade, symmetric) the sign of the wavefunction does not change on INVERSION through the center of the atom,

X_u = (ungerade, antisymmetric) the sign of the wavefunction changes on INVERSION through the center of the atom,

X_1 = (on a or b) the sign of the wavefunction does not change upon ROTATION about the center of the atom,

X_2 = (on a or b) the sign of the wavefunction changes upon ROTATION about the center of the atom,

' = symmetric with respect to a horizontal symmetry plane σ_h,

" = antisymmetric with respect to a horizontal symmetry plane σ_h.

see also GROUP THEORY

Multiamicable Numbers

Two integers n and $m < n$ are (α, β)-multiamicable if

$$\sigma(m) - m = \alpha n$$

and

$$\sigma(n) - n = \beta m,$$

where $\sigma(n)$ is the DIVISOR FUNCTION and α, β are POSITIVE integers. If $\alpha = \beta = 1$, (m, n) is an AMICABLE PAIR.

m cannot have just one distinct prime factor, and if it has precisely two prime factors, then $\alpha = 1$ and m is EVEN. Small multiamicable numbers for small α, β are given by Cohen *et al.* (1995). Several of these numbers are reproduced in the below table.

α	β	m	n
1	6	76455288	183102192
1	7	52920	152280
1	7	16225560	40580280
1	7	90863136	227249568
1	7	16225560	40580280
1	7	70821324288	177124806144
1	7	199615613902848	499240550375424

see also AMICABLE PAIR, DIVISOR FUNCTION

References

Cohen, G. L; Gretton, S.; and Hagis, P. Jr. "Multiamicable Numbers." *Math. Comput.* **64**, 1743–1753, 1995.

Multifactorial

A generalization of the FACTORIAL and DOUBLE FACTORIAL,

$$n! = n(n-1)(n-2)\cdots 2\cdot 1$$

$$n!! = n(n-2)(n-4)\cdots$$

$$n!!! = n(n-3)(n-6)\cdots,$$

etc., where the product runs through positive integers. The FACTORIALS $n!$ for $n = 1, 2, \ldots$, are 1, 2, 6, 24, 120, 720, ... (Sloane's A000142); the DOUBLE FACTORIALS $n!!$ are 1, 2, 3, 8, 15, 48, 105, ... (Sloane's A006882); the triple factorials $n!!!$ are 1, 2, 3, 4, 10, 18, 28, 80, 162, 280, ... (Sloane's A007661); and the quadruple factorials $n!!!!$ are 1, 2, 3, 4, 5, 12, 21, 32, 45, 120, ... (Sloane's A007662).

see also FACTORIAL, GAMMA FUNCTION

References

Sloane, N. J. A. Sequences A000142/M1675, A006882/M0876, A007661/M0596, and A007662/M0534 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Multifractal Measure

A MEASURE for which the q-DIMENSION D_q varies with q.

References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, 1993.

Multigrade Equation

A (k,l)-multigrade equation is a DIOPHANTINE EQUATION of the form

$$\sum_{i=1}^{l} n_i^j = \sum_{i=1}^{l} m_i^j$$

for $j = 1, \ldots, k$, where $\mathbf{m}$ and $\mathbf{n}$ are l-VECTORS. Multigrade identities remain valid if a constant is added to each element of $\mathbf{m}$ and $\mathbf{n}$ (Madachy 1979), so multigrades can always be put in a form where the minimum component of one of the vectors is 1.

Small-order examples are the (2, 3)-multigrade with $\mathbf{m} = \{1,6,8\}$ and $\mathbf{n} = \{2,4,9\}$:

$$\sum_{i=1}^{3} m_i^1 = \sum_{i=1}^{3} n_i^1 = 15$$
$$\sum_{i=1}^{3} m_i^2 = \sum_{i=1}^{3} n_i^2 = 101,$$

the (3, 4)-multigrade with $\mathbf{m} = \{1,5,8,12\}$ and $\mathbf{n} = \{2,3,10,11\}$:

$$\sum_{i=1}^{4} m_i^1 = \sum_{i=1}^{4} n_i^1 = 26$$
$$\sum_{i=1}^{4} m_i^2 = \sum_{i=1}^{4} n_i^2 = 234$$
$$\sum_{i=1}^{4} m_i^3 = \sum_{i=1}^{4} n_i^3 = 2366,$$

and the (4, 6)-multigrade with $\mathbf{m} = \{1,5,8,12,18,19\}$ and $\mathbf{n} = \{2,3,9,13,16,20\}$:

$$\sum_{i=1}^{6} m_i^1 = \sum_{i=1}^{6} n_i^1 = 63$$
$$\sum_{i=1}^{6} m_i^2 = \sum_{i=1}^{6} n_i^2 = 919$$
$$\sum_{i=1}^{6} m_i^3 = \sum_{i=1}^{6} n_i^3 = 15057$$
$$\sum_{i=1}^{6} m_i^4 = \sum_{i=1}^{6} n_i^4 = 260755$$

(Madachy 1979).

A spectacular example with $k = 9$ and $l = 10$ is given by $\mathbf{n} = \{\pm 12, \pm 11881, \pm 20231, \pm 20885, \pm 23738\}$ and $\mathbf{m} = \{\pm 436, \pm 11857, \pm 20449, \pm 20667, \pm 23750\}$ (Guy 1994), which has sums

$$\sum_{i=1}^{9} m_i^1 = \sum_{i=1}^{9} n_i^1 = 0$$
$$\sum_{i=1}^{9} m_i^2 = \sum_{i=1}^{9} n_i^2 = 3100255070$$
$$\sum_{i=1}^{9} m_i^3 = \sum_{i=1}^{9} n_i^3 = 0$$
$$\sum_{i=1}^{9} m_i^4 = \sum_{i=1}^{9} n_i^4 = 1390452894778220678$$
$$\sum_{i=1}^{9} m_i^5 = \sum_{i=1}^{9} n_i^5 = 0$$
$$\sum_{i=1}^{9} m_i^6 = \sum_{i=1}^{9} n_i^6 = 666573454337853049941719510$$
$$\sum_{i=1}^{9} m_i^7 = \sum_{i=1}^{9} n_i^7 = 0$$
$$\sum_{i=1}^{9} m_i^8 = \sum_{i=1}^{9} n_i^8$$
$$= 330958142560259813821203262692838598$$
$$\sum_{i=1}^{9} m_i^9 = \sum_{i=1}^{9} n_i^9 = 0.$$

see also DIOPHANTINE EQUATION

References

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Multilinear

A function, form, etc., in two or more variables is said to be multilinear if it is linear in each variable separately.

see also BILINEAR, LINEAR OPERATOR

Multimagic Series

n numbers form a p-multimagic series if the sum of their kth powers is the MAGIC CONSTANT of degree k for every $k = 1, \ldots, p$. The following table gives the number of p-multimagic series N_p of given orders n (Kraitchik 1942).

n	N_1	N_2	N_3
2	2		
3	8		
4	86	2	2
5	1,394	8	2
6	0	98	0
7	0	1,844	0
8	0	38,039	115
9	0	0	41
10	0	0	0
11	0	0	961

References
Kraitchik, M. "Multimagic Squares." §7.10 in *Mathematical Recreations.* New York: W. W. Norton, pp. 176–178, 1942.

Multimagic Square

A MAGIC SQUARE is p-multimagic if the square formed by replacing each element by its kth power for $k = 1, 2, \ldots, p$ is also magic. A 2-multimagic square is called a BIMAGIC SQUARE, and a 3-multimagic square is called a TRIMAGIC SQUARE.

see also BIMAGIC SQUARE, MAGIC SQUARE, TRIMAGIC SQUARE

References
Kraitchik, M. "Multimagic Squares." §7.10 in *Mathematical Recreations.* New York: W. W. Norton, pp. 176–178, 1942.

Multinomial Coefficient

The multinomial COEFFICIENTS

$$(x_1, x_2, \ldots) = \frac{x_1 + x_2 + \ldots}{x_1! x_2! \cdots}$$

are the terms in the MULTINOMIAL SERIES expansion. They satisfy

$$\begin{aligned}(x_1, x_2, x_3, \ldots) &= (x_1 + x_2, x_3, \ldots)(x_1, x_2)\\ &= (x_1 + x_2 + x_3, \ldots)(x_1, x_2, x_3) = \ldots\end{aligned}$$

(Beeler *et al.* 1972, Item 44).

see also BINOMIAL COEFFICIENT, MULTINOMIAL SERIES

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Multinomial Coefficients." §24.1.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 823–824, 1972.
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.
Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, p. 113, 1992.

Multinomial Distribution

Let a set of random variates $X_1, X_2, \ldots, X_n$ have a probability function

$$P(X_1 = x_1, \ldots, X_n = x_n) = \frac{N!}{\prod_{i=1}^n x_i!} \prod_{i=1}^n {\theta_i}^{x_i} \tag{1}$$

where x_i are POSITIVE INTEGERS, $\theta_i > 0$, and

$$\sum_{i=1}^n \theta_i = 1 \tag{2}$$

$$\sum_{i=1}^n x_i = N. \tag{3}$$

Then the joint distribution of $X_1, \ldots, X_n$ is a multinomial distribution and $P(X_1 = x_1, \ldots, X_n = x_n)$ is given by the corresponding coefficient of the MULTINOMIAL SERIES

$$(\theta_1 + \theta_2 + \ldots + \theta_n)^N. \tag{4}$$

The MEAN and VARIANCE of X_i are

$$\mu_i = N\theta_i \tag{5}$$

$${\sigma_i}^2 = N\theta_i(1 - \theta_i). \tag{6}$$

The COVARIANCE of X_i and X_j is

$${\sigma_{ij}}^2 = -N\theta_i\theta_j. \tag{7}$$

see also BINOMIAL DISTRIBUTION

References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 532, 1987.

Multinomial Series

A generalization of the BINOMIAL SERIES discovered by Johann Bernoulli and Leibniz.

$$(a_1 + a_2 + \ldots + a_k)^n = \sum_{n_1, n_2, \ldots, n_k}^{n} \frac{n!}{n_1! n_2! \cdots n_k!} {a_1}^{n_1} {a_2}^{n_2} \cdots {a_k}^{n_k},$$

where $n \equiv n_1 + n_2 + \ldots + n_k$. The multinomial series arises in a generalization of the BINOMIAL DISTRIBUTION called the MULTINOMIAL DISTRIBUTION.

see also BINOMIAL SERIES, MULTINOMIAL DISTRIBUTION

Multinomial Theorem

see MULTINOMIAL SERIES

Multiperfect Number

A number n is k-multiperfect (also called a k-MULTIPLY PERFECT NUMBER or k-PLUPERFECT NUMBER) if

$$\sigma(n) = kn$$

for some INTEGER $k > 2$, where $\sigma(n)$ is the DIVISOR FUNCTION. The value of k is called the CLASS. The special case $k = 2$ corresponds to PERFECT NUMBERS P_2, which are intimately connected with MERSENNE PRIMES (Sloane's A000396). The number 120 was long known to be 3-multiply perfect (P_3) since

$$\sigma(120) = 3 \cdot 120.$$

The following table gives the first few P_n for $n = 2, 3, \ldots, 6$.

n	Sloane	P_n
2	000396	6, 28, 496, 8128, ...,
3	005820	120, 672, 523776, 459818240, ...
4	027687	30240, 32760, 2178540, 23569920, ...
5	046060	14182439040, 31998395520, ...
6	046061	154345556085770649600, ...

In 1900–1901, Lehmer proved that P_3 has at least three distinct PRIME factors, P_4 has at least four, P_5 at least six, P_6 at least nine, and P_7 at least 14.

As of of 1911, 251 pluperfect numbers were known (Carmichael and Mason 1911). As of 1929, 334 pluperfect numbers were known, many of them found by Poulet. Franqui and García (1953) found 63 additional ones (five P_5s, 29 P_6s, and 29 P_7s), several of which were known to Poulet but had not been published, bringing the total to 397. Brown (1954) discovered 110 pluperfects, including 31 discovered but not published by Poulet and 25 previously published by Franqui and García (1953), for a total of 482. Franqui and García (1954) subsequently discovered 57 additional pluperfects (3 P_6s, 52 P_7s, and 2 P_8s), increasing the total known to 539.

An outdated database is maintained by R. Schroeppel, who lists 2,094 multiperfects, and an up-to-date list by J. L. Moxham (1998). It is believed that all multiperfect numbers of index 3, 4, 5, 6, and 7 are known. The number of known n-multiperfect numbers are 1, 37, 6, 36, 65, 245, 516, 1101, 1129, 46, 0, 0, ...

If n is a P_5 number such that $3\nmid n$, then $3n$ is a P_4 number. If $3n$ is a P_{4k} number such that $3\nmid n$, then n is a P_{3k} number. If n is a P_3 number such that 3 (but not 5 and 9) DIVIDES n, then $45n$ is a P_4 number.

see also e-MULTIPERFECT NUMBER, FRIENDLY PAIR, HYPERPERFECT NUMBER, INFINARY MULTIPERFECT NUMBER, MERSENNE PRIME, PERFECT NUMBER, UNITARY MULTIPERFECT NUMBER

References

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Franqui, B. and García, M. "Some New Multiply Perfect Numbers." *Amer. Math. Monthly* **60**, 459–462, 1953.

Franqui, B. and García, M. "57 New Multiply Perfect Numbers." *Scripta Math.* **20**, 169–171, 1954.

Guy, R. K. "Almost Perfect, Quasi-Perfect, Pseudoperfect, Harmonic, Weird, Multiperfect and Hyperperfect Numbers." §B2 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 45–53, 1994.

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Moxham, J. L. "13 New MPFN's." math-fun@cs.arizona.edu posting, Aug 13, 1998.

Poulet, P. *La Chasse aux nombres,* Vol. 1. Brussels, pp. 9–27, 1929.

Schroeppel, R. "Multiperfect Numbers–Multiply Perfect Numbers–Pluperfect Numbers–MPFNs." Rev. Dec. 13, 1995. `ftp://ftp.cs.arizona.edu/xkernel/rcs/mpfn.html`.

Schroeppel, R. (moderator). mpfn mailing list. e-mail `rcs@cs.arizona.edu` to subscribe.

Sloane, N. J. A. Sequences A000396/M4186 and A005820/M5376 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Multiple Integral

A repeated integral over $n > 1$ variables

$$\underbrace{\int \cdots \int}_{n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

is called a multiple integral. An nth order integral corresponds, in general, to an n-D VOLUME (CONTENT), with $n = 2$ corresponding to an AREA. In an indefinite multiple integral, the order in which the integrals are carried out can be varied at will; for definite multiple integrals, care must be taken to correctly transform the limits if the order is changed.

see also INTEGRAL, MONTE CARLO INTEGRATION

References

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Multiple Regression

A REGRESSION giving conditional expectation values of a given variable in terms of two or more other variables.

see also LEAST SQUARES FITTING, MULTIVARIATE ANALYSIS, NONLINEAR LEAST SQUARES FITTING

References

Edwards, A. L. *Multiple Regression and the Analysis of Variance and Covariance.* San Francisco, CA: W. H. Freeman, 1979.

Multiplication

In simple algebra, multiplication is the process of calculating the result when a number a is taken b times. The result of a multiplication is called the PRODUCT of a and b. It is denoted $a \times b$, $a \cdot b$, $(a)(b)$, or simply ab. The symbol $\times$ is known as the MULTIPLICATION SIGN. Normal multiplication is ASSOCIATIVE, COMMUTATIVE, and DISTRIBUTIVE.

More generally, multiplication can also be defined for other mathematical objects such as GROUPS, MATRICES, SETS, and TENSORS.

Karatsuba and Ofman (1962) discovered that multiplication of two n digit numbers can be done with a BIT COMPLEXITY of less than n^2 using an algorithm now known as KARATSUBA MULTIPLICATION.

see also ADDITION, BIT COMPLEXITY, COMPLEX MULTIPLICATION, DIVISION, KARATSUBA MULTIPLICATION, MATRIX MULTIPLICATION, PRODUCT, RUSSIAN MULTIPLICATION, SUBTRACTION, TIMES

References

Karatsuba, A. and Ofman, Yu. "Multiplication of Many-Digital Numbers by Automatic Computers." *Doklady Akad. Nauk SSSR* **145**, 293–294, 1962. Translation in *Physics-Doklady* **7**, 595–596, 1963.

Multiplication Magic Square

128	1	32
4	16	64
8	256	2

A square which is magic under multiplication instead of addition (the operation used to define a conventional MAGIC SQUARE) is called a multiplication magic square. Unlike (normal) MAGIC SQUARES, the n^2 entries for an nth order multiplicative magic square are not required to be consecutive. The above multiplication magic square has a multiplicative magic constant of 4,096.

see also ADDITION-MULTIPLICATION MAGIC SQUARE, MAGIC SQUARE

References

Hunter, J. A. H. and Madachy, J. S. "Mystic Arrays." Ch. 3 in *Mathematical Diversions.* New York: Dover, pp. 30–31, 1975.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 89–91, 1979.

Multiplication Principle

If one event can occur in m ways and a second can occur independently of the first in n ways, then the two events can occur in mn ways.

Multiplication Sign

The symbol $\times$ used to denote MULTIPLICATION, i.e., $a \times b$ denotes a times b.

Multiplication Table

A multiplication table is an array showing the result of applying a BINARY OPERATOR to elements of a given set S.

$\times$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

see also BINARY OPERATOR, TRUTH TABLE

Multiplicative Character

see CHARACTER (MULTIPLICATIVE)

Multiplicative Digital Root

Consider the process of taking a number, multiplying its DIGITS, then multiplying the DIGITS of numbers derived from it, etc., until the remaining number has only one DIGIT. The number of multiplications required to obtain a single DIGIT from a number n is called the MULTIPLICATIVE PERSISTENCE of n, and the DIGIT obtained is called the multiplicative digital root of n.

For example, the sequence obtained from the starting number 9876 is (9876, 3024, 0), so 9876 has a MULTIPLICATIVE PERSISTENCE of two and a multiplicative digital root of 0. The multiplicative digital roots of the first few positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 2, 4, 6, 8, 0, 2, 4, 6, 8, 0, 3, 6, 9, 2, 5, 8, 2, ... (Sloane's A031347).

see also ADDITIVE PERSISTENCE, DIGITADITION, DIGITAL ROOT, MULTIPLICATIVE PERSISTENCE

References

Sloane, N. J. A. Sequence A031347 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Multiplicative Function

A function $f(m)$ is called multiplicative if $(m, m') = 1$ (i.e., the statement that m and m' are RELATIVELY PRIME) implies

$$f(mm') = f(m)f(m').$$

see also QUADRATIC RESIDUE, TOTIENT FUNCTION

Multiplicative Inverse

The multiplicative of a REAL or COMPLEX NUMBER z is its RECIPROCAL $1/z$. For complex $z = x + iy$,

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2-y^2} - i\frac{y}{x^2-y^2}.$$

Multiplicative Perfect Number

A number n for which the PRODUCT of DIVISORS is equal to n^2. The first few are 1, 6, 8, 10, 14, 15, 21, 22, ... (Sloane's A007422).

see also PERFECT NUMBER

References

Sloane, N. J. A. Sequence A007422/M4068 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Multiplicative Persistence

Multiply all the digits of a number n by each other, repeating with the product until a single DIGIT is obtained. The number of steps required is known as the multiplicative persistence, and the final DIGIT obtained is called the MULTIPLICATIVE DIGITAL ROOT of n.

For example, the sequence obtained from the starting number 9876 is (9876, 3024, 0), so 9876 has an multiplicative persistence of two and a MULTIPLICATIVE DIGITAL ROOT of 0. The multiplicative persistences of the first few positive integers are 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 1, 1, ... (Sloane's A031346). The smallest numbers having multiplicative persistences of 1, 2, ... are 10, 25, 39, 77 679 6788 68889 267889 26888999 3778888999 277777788888899 (Sloane's A003001). There is no number $< 10^{50}$ with multiplicative persistence > 11.

The multiplicative persistence of an n-DIGIT number is also called its LENGTH. The maximum lengths for $n =$ 2-, 3-, ..., digit numbers are 4, 5, 6, 7, 7, 8, 9, 9, 10, 10, 10, ... (Sloane's A014553; (Beeler *et al.* 1972, Item 56; Gottlieb 1969–1970).

The concept of multiplicative persistence can be generalized to multiplying the kth powers of the digits of a number and iterating until the result remains constant. All numbers other than REPUNITS, which converge to 1, converge to 0. The number of iterations required for the kth powers of a number's digits to converge to 0 is called its k-multiplicative persistence. The following table gives the n-multiplicative persistences for the first few positive integers.

n	Sloane	n-Persistences
2	031348	0, 7, 6, 6, 3, 5, 5, 4, 5, 1, ...
3	031349	0, 4, 5, 4, 3, 4, 4, 3, 3, 1, ...
4	031350	0, 4, 3, 3, 3, 3, 2, 2, 3, 1, ...
5	031351	0, 4, 4, 2, 3, 3, 2, 3, 2, 1, ...
6	031352	0, 3, 3, 2, 3, 3, 3, 3, 3, 1, ...
7	031353	0, 4, 3, 3, 3, 3, 3, 2, 3, 1, ...
8	031354	0, 3, 3, 3, 2, 4, 2, 3, 2, 1, ...
9	031355	0, 3, 3, 3, 3, 2, 2, 3, 2, 1, ...
10	031356	0, 2, 2, 2, 3, 2, 3, 2, 2, 1, ...

see also 196-ALGORITHM, ADDITIVE PERSISTENCE, DIGITADITION, DIGITAL ROOT, KAPREKAR NUMBER, LENGTH (NUMBER), MULTIPLICATIVE DIGITAL ROOT, NARCISSISTIC NUMBER, RECURRING DIGITAL INVARIANT

References

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Gottlieb, A. J. Problems 28–29 in "Bridge, Group Theory, and a Jigsaw Puzzle." *Techn. Rev.* **72**, unpaginated, Dec. 1969.

Gottlieb, A. J. Problem 29 in "Integral Solutions, Ladders, and Pentagons." *Techn. Rev.* **72**, unpaginated, Apr. 1970.

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Multiplicative Primitive Residue Class Group

see MODULO MULTIPLICATION GROUP

Multiplicity

The word multiplicity is a general term meaning "the number of values for which a given condition holds." The most common use of the word is as the value of the TOTIENT VALENCE FUNCTION.

see also DEGENERATE, NOETHER'S FUNDAMENTAL THEOREM, TOTIENT VALENCE FUNCTION

Multiplier

see MODULAR FUNCTION MULTIPLIER

Multiply Connected

A set which is CONNECTED but not SIMPLY CONNECTED is called multiply connected. A SPACE is n-MULTIPLY CONNECTED if it is $(n-1)$-connected and if every MAP from the n-SPHERE into it extends continuously over the $(n+1)$-DISK

A theorem of Whitehead says that a SPACE is infinitely connected IFF it is contractible.

see also CONNECTIVITY, LOCALLY PATHWISE-CONNECTED SPACE, PATHWISE-CONNECTED, SIMPLY CONNECTED

Multiply Perfect Number

see MULTIPERFECT NUMBER

Multisection

see SERIES MULTISECTION

Multivalued Function

A FUNCTION which assumes two or more distinct values at one or more points in its DOMAIN.

see also BRANCH CUT, BRANCH POINT

References

Morse, P. M. and Feshbach, H. "Multivalued Functions." §4.4 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 398–408, 1953.

Multivariate Analysis

The study of random distributions involving more than one variable.

see also GAUSSIAN JOINT VARIABLE THEOREM, MULTIPLE REGRESSION, MULTIVARIATE FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 927–928, 1972.

Feinstein, A. R. *Multivariable Analysis.* New Haven, CT: Yale University Press, 1996.

Hair, J. F. Jr. *Multivariate Data Analysis with Readings, 4th ed.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

Sharma, S. *Applied Multivariate Techniques.* New York: Wiley, 1996.

Multivariate Function

A FUNCTION of more than one variable.

see also MULTIVARIATE ANALYSIS, UNIVARIATE FUNCTION

Multivariate Theorem

see GAUSSIAN JOINT VARIABLE THEOREM

Müntz Space

A Müntz space is a technically defined SPACE

$$M(\Lambda) \equiv \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

which arises in the study of function approximations.

Müntz's Theorem

Müntz's theorem is a generalization of the WEIERSTRAß APPROXIMATION THEOREM, which states that any continuous function on a closed and bounded interval can be uniformly approximated by POLYNOMIALS involving constants and any INFINITE SEQUENCE of POWERS whose RECIPROCALS diverge.

In technical language, Müntz's theorem states that the MÜNTZ SPACE $M(\Lambda)$ is dense in $C[0,1]$ IFF

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

see also WEIERSTRAß APPROXIMATION THEOREM

Mutant Knot

Given an original KNOT K, the three knots produced by MUTATION together with K itself are called mutant knots. Mutant knots are often difficult to distinguish. For instant, mutants have the same HOMFLY POLYNOMIALS and HYPERBOLIC KNOT volume. Many but not all mutants also have the same GENUS (KNOT).

Mutation

Consider a KNOT as being formed from two TANGLES. The following three operations are called mutations.

1. Cut the knot open along four points on each of the four strings coming out of T_2, flipping T_2 over, and gluing the strings back together.
2. Cut the knot open along four points on each of the four strings coming out of T_2, flipping T_2 to the right, and gluing the strings back together.
3. Cut the knot, rotate it by 180°, and reglue. This is equivalent to performing (1), then (2).

Mutations applied to an alternating KNOT projection always yield an ALTERNATING KNOT. The mutation of a KNOT is always another KNOT (a opposed to a LINK).

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, p. 49, 1994.

Mutual Energy

Let Ω be a SPACE with MEASURE $\mu \geq 0$, and let $\Phi(P,Q)$ be a real function on the PRODUCT SPACE $\Omega \times \Omega$. When

$$\begin{aligned}(\mu, nu) &= \iint \Phi(P,Q)\, d\mu(Q)\, d\nu(P) \\ &= \int \Phi(P,\mu)\, d\nu(P)\end{aligned}$$

exists for measures $\mu, \nu \geq 0$, (μ,ν) is called the mutual energy. (μ,μ) is then called the ENERGY.

see also ENERGY

References

Iyanaga, S. and Kawada, Y. (Eds.). "General Potential." §335.B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1038, 1980.

Mutually Exclusive

Two events E_1 and E_2 are mutually exclusive if $E_1 \cap E_2 \equiv \varnothing$. n events $E_1, E_2, \ldots, E_n$ are mutually exclusive if $E_i \cap E_j \equiv \varnothing$ for $i \neq j$.

Mutually Singular

Let M be a SIGMA ALGEBRA M, and let λ_1 and λ_2 be MEASURES on M. If there EXISTS a pair of disjoint SETS A and B such that λ_1 is CONCENTRATED on A and λ_2 is CONCENTRATED on B, then λ_1 and λ_2 are said to be mutually singular, written $\lambda_1 \perp \lambda_2$.

see also ABSOLUTELY CONTINUOUS, CONCENTRATED, SIGMA ALGEBRA

References

Rudin, W. *Functional Analysis.* New York: McGraw-Hill, p. 121, 1991.

Myriad

The Greek word for 10,000.

Myriagon

A 10,000-sided POLYGON.

Mystic Pentagram

see PENTAGRAM

N

ℕ

The SET of NATURAL NUMBERS (the POSITIVE INTEGERS $\mathbb{Z}^+$ 1, 2, 3, ...; Sloane's A000027), denoted $\mathbb{N}$, also called the WHOLE NUMBERS. Like whole numbers, there is no general agreement on whether 0 should be included in the list of natural numbers.

Due to lack of standard terminology, the following terms are recommended in preference to "COUNTING NUMBER," "natural number," and "WHOLE NUMBER."

Set	Name	Symbol
..., −2, −1, 0, 1, 2, ...	integers	$\mathbb{Z}$
1, 2, 3, 4, ...	positive integers	$\mathbb{Z}^+$
0, 1, 2, 3, 4 ...	nonnegative integers	$\mathbb{Z}^*$
−1, −2, −3, −4, ...	negative integers	$\mathbb{Z}^-$

see also $\mathbb{C}$, CARDINAL NUMBER, COUNTING NUMBER, $\mathbb{I}$, INTEGER, $\mathbb{Q}$, $\mathbb{R}$, WHOLE NUMBER, $\mathbb{Z}$, $\mathbb{Z}^+$

References
Sloane, N. J. A. Sequence A000027/M0472 in "An On-Line Version of the Encyclopedia of Integer Sequences."

N-Cluster

A LATTICE POINT configuration with no three points COLLINEAR and no four CONCYCLIC. An example is the 6-cluster (0, 0), (132, −720), (546, −272), (960, −720), (1155, 540), (546, 1120). Call the RADIUS of the smallest CIRCLE centered at one of the points of an N-cluster which contains all the points in the N-cluster the EXTENT. Noll and Bell (1989) found 91 nonequivalent prime 6-clusters of EXTENT less than 20937, but found no 7-clusters.

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 187, 1994.
Noll, L. C. and Bell, D. I. "n-clusters for $1 < n < 7$." *Math. Comput.* **53**, 439–444, 1989.

n-Cube

see HYPERCUBE, POLYCUBE

n-in-a-Row

see TIC-TAC-TOE

n-minex

n-minex is defined as 10^{-n}.

see also n-PLEX

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 16, 1996.

n-Omino

see POLYOMINO

n-plex

n-plex is defined as 10^n.

see also GOOGOLPLEX, n-MINEX

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 16, 1996.

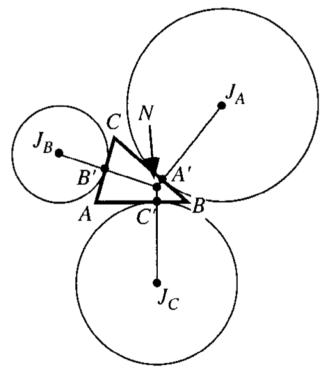
n-Sphere

see HYPERSPHERE

Nabla

see DEL, LAPLACIAN

Nagel Point



Let A' be the point at which the A-EXCIRCLE meets the side BC of a TRIANGLE ΔABC, and define B' and C' similarly. Then the lines AA', BB', and CC' CONCUR in the NAGEL POINT.

The Nagel point can also be constructed by letting A'' be the point half way around the PERIMETER of ΔABC starting at A, and B'' and C'' similarly defined. Then the lines AA'', BB'', and CC'' concur in the Nagel point. It is therefore sometimes known as the BISECTED PERIMETER POINT (Bennett *et al.* 1988, Chen *et al.* 1992, Kimberling 1994).

The Nagel point has TRIANGLE CENTER FUNCTION

$$\alpha = \frac{b+c-a}{a}.$$

It is the ISOTOMIC CONJUGATE POINT of the GERGONNE POINT.

see also EXCENTER, EXCENTRAL TRIANGLE, EXCIRCLE, MITTENPUNKT, TRISECTED PERIMETER POINT

References
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Bennett, G.; Glenn, J.; Kimberling, C.; and Cohen, J. M. "Problem E 3155 and Solution." *Amer. Math. Monthly* **95**, 874, 1988.
Chen, J.; Lo, C.-H.; and Lossers, O. P. "Problem E 3397 and Solution." *Amer. Math. Monthly* **99**, 70–71, 1992.

Eves, H. W. *A Survey of Geometry, rev. ed.* Boston, MA: Allyn and Bacon, p. 83, 1972.
Gallatly, W. *The Modern Geometry of the Triangle, 2nd ed.* London: Hodgson, p. 20, 1913.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 184 and 225–226, 1929.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. "Nagel Point." `http://www.evansville.edu/~ck6/tcenters/class/nagel.html`.

Naive Set Theory

A branch of mathematics which attempts to formalize the nature of the SET using a minimal collection of independent axioms. Unfortunately, as discovered by its earliest proponents, naive set theory quickly runs into a number of PARADOXES (such as RUSSELL'S PARADOX), so a less sweeping and more formal theory known as AXIOMATIC SET THEORY must be used.

see also AXIOMATIC SET THEORY, RUSSELL'S PARADOX, SET THEORY

Napier's Analogies

Let a SPHERICAL TRIANGLE have sides a, b, and c with A, B, and C the corresponding opposite angles. Then

$$\frac{\sin[\frac{1}{2}(A-B)]}{\sin[\frac{1}{2}(A+B)]} = \frac{\tan[\frac{1}{2}(a-b)]}{\tan(\frac{1}{2}c)} \tag{1}$$

$$\frac{\cos[\frac{1}{2}(A-B)]}{\cos[\frac{1}{2}(A+B)]} = \frac{\tan[\frac{1}{2}(a+b)]}{\tan(\frac{1}{2}c)} \tag{2}$$

$$\frac{\sin[\frac{1}{2}(a-b)]}{\sin[\frac{1}{2}(a+b)]} = \frac{\tan[\frac{1}{2}(A-B)]}{\cot(\frac{1}{2}C)} \tag{3}$$

$$\frac{\cos[\frac{1}{2}(a-b)]}{\cos[\frac{1}{2}(a+b)]} = \frac{\tan[\frac{1}{2}(A+B)]}{\cot(\frac{1}{2}C)}. \tag{4}$$

see also SPHERICAL TRIGONOMETRY

Napier's Bones

Numbered rods which can be used to perform MULTIPLICATION. This process is also called RABDOLOGY.

see also GENAILLE RODS

References

Gardner, M. "Napier's Bones." Ch. 7 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, 1986.
Pappas, T. "Napier's Bones." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 64–65, 1989.

Napier's Constant

see e

Napier's Inequality

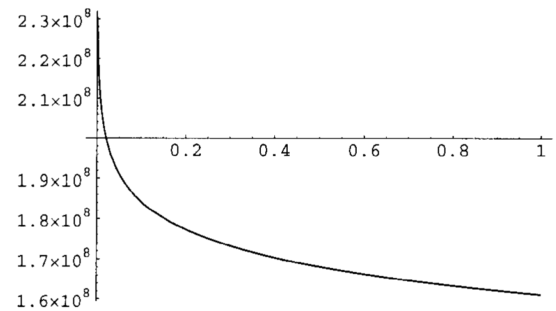
For $b > a > 0$,

$$\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

References

Nelsen, R. B. "Napier's Inequality (Two Proofs)." *College Math. J.* **24**, 165, 1993.

Napierian Logarithm



Write a number N as

$$N = 10^7(1 - 10^{-7})^L,$$

then L is the Napierian logarithm of N. This was the original definition of a LOGARITHM, and can be given in terms of the modern LOGARITHM as

$$L(N) = -\frac{\log\left(\frac{n}{10^7}\right)}{\log\left(\frac{10^7}{10^7-1}\right)}.$$

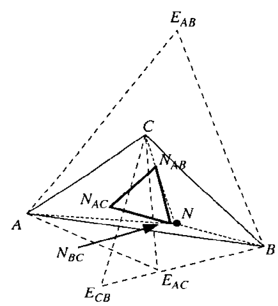
The Napierian logarithm decreases with increasing numbers and does not satisfy many of the fundamental properties of the modern LOGARITHM, e.g.,

$$\operatorname{Nlog}(xy) \neq \operatorname{Nlog} x + \operatorname{Nlog} y.$$

Napkin Ring

see SPHERICAL RING

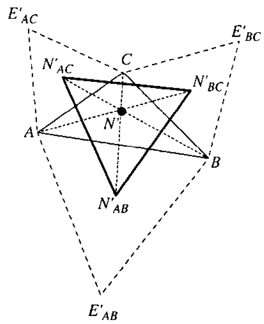
Napoleon Points



The inner Napoleon point N is the CONCURRENCE of lines drawn between VERTICES of a given TRIANGLE

ΔABC and the opposite VERTICES of the corresponding inner NAPOLEON TRIANGLE $\Delta N_{AB}N_{AC}N_{BC}$. The TRIANGLE CENTER FUNCTION of the inner Napoleon point is

$$\alpha = \csc(A - \tfrac{1}{6}\pi).$$



The outer Napoleon point N' is the CONCURRENCE of lines drawn between VERTICES of a given TRIANGLE ΔABC and the opposite VERTICES of the corresponding outer NAPOLEON TRIANGLE $\Delta N'_{AB}N'_{AC}N'_{BC}$. The TRIANGLE CENTER FUNCTION of the point is

$$\alpha = \csc(A + \tfrac{1}{6}\pi).$$

see also NAPOLEON'S THEOREM, NAPOLEON TRIANGLES

References
Casey, J. *Analytic Geometry, 2nd ed.* Dublin: Hodges, Figgis, & Co., pp. 442–444, 1893.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Napoleon's Problem

Given the center of a CIRCLE, divide the CIRCLE into four equal arcs using a COMPASS alone (a MASCHERONI CONSTRUCTION).

see also CIRCLE, COMPASS, MASCHERONI CONSTRUCTION

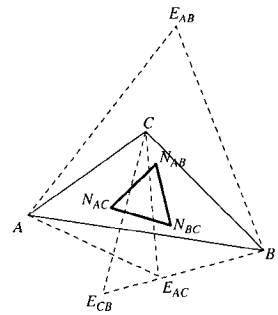
Napoleon's Theorem

If EQUILATERAL TRIANGLES are erected externally on the sides of any TRIANGLE, then the centers form an EQUILATERAL TRIANGLE (the outer NAPOLEON TRIANGLE). Furthermore, the inner NAPOLEON TRIANGLE is also EQUILATERAL and the difference between the areas of the outer and inner Napoleon triangles equals the AREA of the original TRIANGLE.

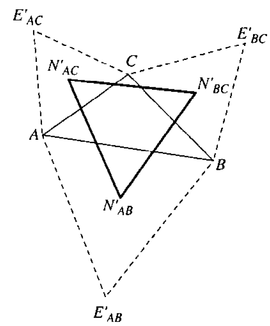
see also NAPOLEON POINTS, NAPOLEON TRIANGLES

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 60–65, 1967.
Pappas, T. "Napoleon's Theorem." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 57, 1989.
Schmidt, F. "200 Jahre französische Revolution—Problem und Satz von Napoleon." *Didaktik der Mathematik* **19**, 15–29, 1990.
Wentzel, J. E. "Converses of Napoleon's Theorem." *Amer. Math. Monthly* **99**, 339–351, 1992.

Napoleon Triangles



The inner Napoleon triangle is the TRIANGLE $\Delta N_{AB}N_{AC}N_{BC}$ formed by the centers of internally erected EQUILATERAL TRIANGLES ΔABE_{AB}, ΔACE_{AC}, and ΔBCE_{BC} on the sides of a given TRIANGLE ΔABC. It is an EQUILATERAL TRIANGLE.

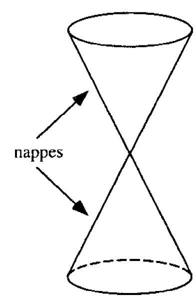


The outer Napoleon triangle is the TRIANGLE $\Delta N'_{AB}N'_{AC}N'_{BC}$ formed by the centers of externally erected EQUILATERAL TRIANGLES $\Delta ABE'_{AB}$, $\Delta ACE'_{AC}$, and $\Delta BCE'_{BC}$ on the sides of a given TRIANGLE ΔABC. It is also an EQUILATERAL TRIANGLE.

see also EQUILATERAL TRIANGLE, NAPOLEON POINTS, NAPOLEON'S THEOREM

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 60–65, 1967.

Nappe



One of the two pieces of a double CONE (i.e., two CONES placed apex to apex).

see also CONE

Narcissistic Number

An n-DIGIT number which is the SUM of the nth POWERS of its DIGITS is called an n-narcissistic number, or sometimes an ARMSTRONG NUMBER or PERFECT DIGITAL INVARIANT (Madachy 1979). The smallest example other than the trivial 1-DIGIT numbers is

$$153 = 1^3 + 5^3 + 3^3.$$

The series of smallest narcissistic numbers of n digits are 0, (none), 153, 1634, 54748, 548834, ... (Sloane's A014576). Hardy (1993) wrote, "There are just four numbers, after unity, which are the sums of the cubes of their digits: $153 = 1^3+5^3+3^3$, $370 = 3^3+7^3+0^3$, $371 = 3^3+7^3+1^3$, and $407 = 4^3+0^3+7^3$. These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to the mathematician." The following table gives the generalization of these "unappealing" numbers to other POWERS (Madachy 1979, p. 164).

n	n-Narcissistic Numbers
1	0, 1, 2, 3, 4, 5, 6, 7, 8, 9
2	none
3	153, 370, 371, 407
4	1634, 8208, 9474
5	54748, 92727, 93084
6	548834
7	1741725, 4210818, 9800817, 9926315
8	24678050, 24678051, 88593477
9	146511208, 472335975, 534494836, 912985153
10	4679307774

A total of 88 narcissistic numbers exist in base-10, as proved by D. Winter in 1985 and verified by D. Hoey. These numbers exist for only 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 16, 17, 19, 20, 21, 23, 24, 25, 27, 29, 31, 32, 33, 34, 35, 37, 38, and 39 digits. It can easily be shown that base-10 n-narcissistic numbers can exist only for $n \leq 60$, since

$$n \cdot 9^n < 10^{n-1}$$

for $n > 60$. The largest base-10 narcissistic number is the 39-narcissistic

$$115132219018736992565095597973971522401$$

A table of the largest known narcissistic numbers in various BASES is given by Pickover (1995). A tabulation of narcissistic numbers in various bases is given by (Corning).

A closely related set of numbers generalize the narcissistic number to n-DIGIT numbers which are the sums of *any* single POWER of their DIGITS. For example, 4150 is a 4-DIGIT number which is the sum of fifth POWERS of its DIGITS. Since the number of digits is not equal to the power to which they are taken for such numbers, it is *not* a narcissistic number. The smallest numbers which are sums of *any* single positive power of their digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, 370, 371, 407, 1634, 4150, 4151, 8208, 9474, ... (Sloane's A023052), with powers 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 4, 5, 5, 4, 4, ... (Sloane's A046074).

The smallest numbers which are equal to the nth powers of their digits for $n = 3, 4, \ldots$, are 153, 1634, 4150, 548834, 1741725, ... (Sloane's A003321). The n-digit numbers equal to the sum of nth powers of their digits (a finite sequence) are called ARMSTRONG NUMBERS or PLUS PERFECT NUMBERS and are given by 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, 370, 371, 407, 1634, 8208, 9474, 54748, ... (Sloane's A005188).

If the sum-of-kth-powers-of-digits operation applied iteratively to a number n eventually returns to n, the smallest number in the sequence is called a k-RECURRING DIGITAL INVARIANT.

see also ADDITIVE PERSISTENCE, DIGITAL ROOT, DIGITADITION, KAPREKAR NUMBER, MULTIPLICATIVE DIGITAL ROOT, MULTIPLICATIVE PERSISTENCE, RECURRING DIGITAL INVARIANT, VAMPIRE NUMBER

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Pickover, C. A. *Keys to Infinity.* New York: W. H. Freeman, pp. 169–170, 1995.

Rumney, M. "Digital Invariants." *Recr. Math. Mag.* No. 12, 6–8, Dec. 1962.

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Weisstein, E. W. "Narcissistic Numbers." http://www.astro.virginia.edu/~eww6n/math/notebooks/Narcissistic.dat.

Nash Equilibrium

A set of MIXED STRATEGIES for finite, noncooperative GAMES of two or more players in which no player can improve his payoff by unilaterally changing strategy.

see also FIXED POINT, GAME, MIXED STRATEGY, NASH'S THEOREM

Nash's Theorem

A theorem in GAME THEORY which guarantees the existence of a NASH EQUILIBRIUM for MIXED STRATEGIES in finite, noncooperative GAMES of two or more players.

see also MIXED STRATEGY, NASH EQUILIBRIUM

Nasik Square

see PANMAGIC SQUARE

Nasty Knot

An UNKNOT which can only be unknotted by first increasing the number of crossings.

Natural Density

see NATURAL INVARIANT

Natural Equation

A natural equation is an equation which specifies a curve independent of any choice of coordinates or parameterization. The study of natural equations began with the following problem: given two functions of one parameter, find the SPACE CURVE for which the functions are the CURVATURE and TORSION.

Euler gave an integral solution for plane curves (which always have TORSION $\tau = 0$). Call the ANGLE between the TANGENT line to the curve and the x-AXIS ϕ the TANGENTIAL ANGLE, then

$$\phi = \int \kappa(s)\,ds, \tag{1}$$

where κ is the CURVATURE. Then the equations

$$\kappa = \kappa(s) \tag{2}$$

$$\tau = 0, \tag{3}$$

where τ is the TORSION, are solved by the curve with parametric equations

$$x = \int \cos\phi\,ds \tag{4}$$

$$y = \int \sin\phi\,ds. \tag{5}$$

The equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called the natural (or INTRINSIC) equations of the space curve. An equation expressing a plane curve in terms of s and RADIUS OF CURVATURE R (or κ) is called a CESÀRO EQUATION, and an equation expressing a plane curve in terms of s and ϕ is called a WHEWELL EQUATION.

Among the special planar cases which can be solved in terms of elementary functions are the CIRCLE, LOGARITHMIC SPIRAL, CIRCLE INVOLUTE, and EPICYCLOID. Enneper showed that each of these is the projection of a HELIX on a CONIC surface of revolution along the axis of symmetry. The above cases correspond to the CYLINDER, CONE, PARABOLOID, and SPHERE.

see also CESÀRO EQUATION, INTRINSIC EQUATION, WHEWELL EQUATION

References

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Euler, L. *Comment. Acad. Petropolit.* **8**, 66–85, 1736.
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Melzak, Z. A. *Companion to Concrete Mathematics, Vol. 2.* New York: Wiley, 1976.
Struik, D. J. *Lectures on Classical Differential Geometry.* New York: Dover, pp. 26–28, 1988.

Natural Independence Phenomenon

A type of mathematical result which is considered by most logicians as more natural than the METAMATHEMATICAL incompleteness results first discovered by Gödel. Finite combinatorial examples include GOODSTEIN'S THEOREM, a finite form of RAMSEY'S THEOREM, and a finite form of KRUSKAL'S TREE THEOREM (Kirby and Paris 1982; Smorynski 1980, 1982, 1983; Gallier 1991).

see also GÖDEL'S INCOMPLETENESS THEOREM, GOODSTEIN'S THEOREM, KRUSKAL'S TREE THEOREM, RAMSEY'S THEOREM

References

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Smorynski, C. "Some Rapidly Growing Functions." *Math. Intell.* **2**, 149–154, 1980.
Smorynski, C. "The Varieties of Arboreal Experience." *Math. Intell.* **4**, 182–188, 1982.
Smorynski, C. "'Big' News from Archimedes to Friedman." *Not. Amer. Math. Soc.* **30**, 251–256, 1983.

Natural Invariant

Let $\rho(x)\,dx$ be the fraction of time a typical dynamical ORBIT spends in the interval $[x, x+dx]$, and let $\rho(x)$ be normalized such that

$$\int \rho(x)\,dx = 1$$

over the entire interval of the map. Then the fraction the time an ORBIT spends in a finite interval $[a, b]$, is given by

$$\int_a^b \rho(x)\,dx.$$

The natural invariant is also called the INVARIANT DENSITY or NATURAL DENSITY.

Natural Logarithm

The LOGARITHM having base e, where

$$e = 2.718281828\ldots, \tag{1}$$

which can be defined

$$\ln x \equiv \int_1^x \frac{dt}{t} \tag{2}$$

for $x > 0$. The natural logarithm can also be defined for COMPLEX NUMBERS as

$$\ln z \equiv \ln|z| + i\arg(z), \tag{3}$$

where $|z|$ is the MODULUS and $\arg(z)$ is the ARGUMENT. The natural logarithm is especially useful in CALCULUS because its DERIVATIVE is given by the simple equation

$$\frac{d}{dx}\ln x = \frac{1}{x}, \tag{4}$$

whereas logarithms in other bases have the more complicated DERIVATIVE

$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}. \tag{5}$$

The MERCATOR SERIES

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots \tag{6}$$

gives a TAYLOR SERIES for the natural logarithm.

CONTINUED FRACTION representations of logarithmic functions include

$$\ln(1+x) = \cfrac{x}{1+\cfrac{1^2x}{2+\cfrac{1^2x}{3+\cfrac{2^2x}{4+\cfrac{2^2x}{5+\cfrac{3^2x}{6+\cfrac{3^2x}{7+\ldots}}}}}}} \tag{7}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \cfrac{2x}{1-\cfrac{x^2}{3-\cfrac{4x^2}{5-\cfrac{9x^2}{7-\cfrac{16x^2}{9-\ldots}}}}}. \tag{8}$$

For a COMPLEX NUMBER z, the natural logarithm satisfies

$$\ln z = \ln[re^{i(\theta+2n\pi)}] = \ln r + i(\theta+2n\pi) \tag{9}$$

$$PV(\ln z) = \ln r + i\theta, \tag{10}$$

where PV is the PRINCIPAL VALUE.

Some special values of the natural logarithm are

$$\ln 1 = 0 \tag{11}$$

$$\ln 0 = -\infty \tag{12}$$

$$\ln(-1) = \pi i \tag{13}$$

$$\ln(\pm i) = \pm\tfrac{1}{2}\pi i. \tag{14}$$

An identity for the natural logarithm of 2 discovered using the PSLQ ALGORITHM is

$$\begin{aligned}(\ln 2)^2 = -\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{16^k}\Bigg[&-\frac{3}{(8k)^2} - \frac{16}{(8k+1)^2}\\ &-\frac{40}{(8k+2)^2} - \frac{8}{(8k+3)^2} - \frac{28}{(8k+4)^2} + \frac{4}{(8k+5)^2}\\ &-\frac{28}{(8k+4)^2} - \frac{4}{(8k+5)^2} + \frac{10}{(8k+5)^2} - \frac{2}{(8k+7)^2}\Bigg]\end{aligned} \tag{15}$$

(Bailey *et al.* 1995, Bailey and Plouffe).

see also e, JENSEN'S FORMULA, LG, LOGARITHM

References

Bailey, D.; Borwein, P.; and Plouffe, S. "On the Rapid Computation of Various Polylogarithmic Constants." `http://www.cecm.sfu.ca/~pborwein/PAPERS/P123.ps`.

Bailey, D. and Plouffe, S. "Recognizing Numerical Constants." `http://www.cecm.sfu.ca/organics/papers/bailey`.

Natural Measure

$\mu_i(\epsilon)$, sometimes denoted $P_i(\epsilon)$, is the probability that element i is populated, normalized such that

$$\sum_{i=1}^{N}\mu_i(\epsilon) = 1.$$

see also INFORMATION DIMENSION, q-DIMENSION

Natural Norm

Let $||\mathbf{z}||$ be a VECTOR NORM of $\mathbf{z}$ such that

$$||\mathsf{A}|| = \max_{||\mathbf{z}||=1}||\mathsf{A}\mathbf{z}||.$$

Then $||\mathsf{A}||$ is a MATRIX NORM which is said to be the natural norm INDUCED (or SUBORDINATE) to the VECTOR NORM $||\mathbf{z}||$. For any natural norm,

$$||\mathsf{I}|| = 1,$$

where I is the IDENTITY MATRIX. The natural matrix norms induced by the L_1-NORM, L_2-NORM, and L_∞-NORM are called the MAXIMUM ABSOLUTE COLUMN SUM NORM, SPECTRAL NORM, and MAXIMUM ABSOLUTE ROW SUM NORM, respectively.

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1115, 1979.

Natural Number

A POSITIVE INTEGER 1, 2, 3, ... (Sloane's A000027). The set of natural numbers is denoted $\mathbb{N}$ or $\mathbb{Z}^+$. Unfortunately, 0 is sometimes also included in the list of "natural" numbers (Bourbaki 1968, Halmos 1974), and there seems to be no general agreement about whether to include it.

Due to lack of standard terminology, the following terms are recommended in preference to "COUNTING NUMBER," "natural number," and "WHOLE NUMBER."

Set	Name	Symbol
..., −2, −1, 0, 1, 2, ...	integers	$\mathbb{Z}$
1, 2, 3, 4, ...	positive integers	$\mathbb{Z}^+$
0, 1, 2, 3, 4 ...	nonnegative integers	$\mathbb{Z}^*$
−1, −2, −3, −4, ...	negative integers	$\mathbb{Z}^-$

see also COUNTING NUMBER, INTEGER, $\mathbb{N}$, POSITIVE, $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{Z}^+$, $\mathbb{Z}^*$

References
Bourbaki, N. *Elements of Mathematics: Theory of Sets.* Paris, France: Hermann, 1968.
Courant, R. and Robbins, H. "The Natural Numbers." Ch. 1 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 1–20, 1996.
Halmos, P. R. *Naive Set Theory.* New York: Springer-Verlag, 1974.
Sloane, N. J. A. Sequence A000027/M0472 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Naught

The British word for "ZERO." It is often used to indicate 0 subscripts, so a_0 would be spoken as "a naught."

see also ZERO

Navigation Problem

A problem in the CALCULUS OF VARIATIONS. Let a vessel traveling at constant speed c navigate on a body of water having surface velocity

$$u = u(x, y)$$
$$v = v(x, y).$$

The navigation problem asks for the course which travels between two points in minimal time.

References
Sagan, H. *Introduction to the Calculus of Variations.* New York: Dover, pp. 226–228, 1992.

Near-Integer

see ALMOST INTEGER

Near Noble Number

A REAL NUMBER $0 < \nu < 1$ whose CONTINUED FRACTION is periodic, and the periodic sequence of terms is composed of a string of 1s followed by an INTEGER $n > 1$,

$$\nu = [\underbrace{\overline{1, 1, \ldots, 1}, n}_{P}]. \qquad (1)$$

This can be written in the form

$$\nu = [\underbrace{1, 1, \ldots, 1}_{P}, n, \nu^{-1}], \qquad (2)$$

which can be solved to give

$$\nu = \tfrac{1}{2}n\left(\sqrt{1 + 4\frac{nF_{P-1} + F_{P-2}}{n^2 F_P}} - 1\right), \qquad (3)$$

where F_n is a FIBONACCI NUMBER. The special case $n = 2$ gives

$$\nu = \sqrt{\frac{F_{P+2}}{F_P}} - 1. \qquad (4)$$

see also NOBLE NUMBER

References
Schroeder, M. R. *Number Theory in Science and Communication: With Applications in Cryptography, Physics, Digital Information, Computing, and Self-Similarity, 2nd enl. ed., corr. printing.* Berlin: Springer-Verlag, 1990.
Schroeder, M. "Noble and Near Noble Numbers." In *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, pp. 392–394, 1991.

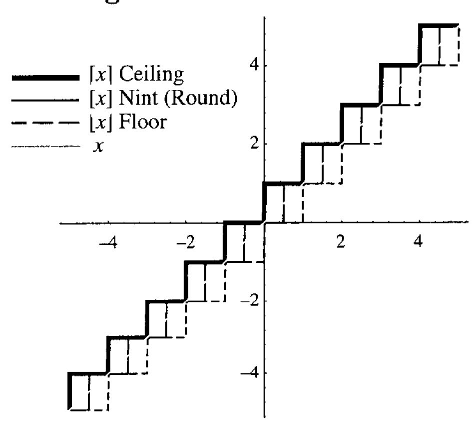
Near-Pencil

An arrangement of $n \geq 3$ points such that $n - 1$ of them are COLLINEAR.

see also GENERAL POSITION, ORDINARY LINE, PENCIL

References
Guy, R. K. "Unsolved Problems Come of Age." *Amer. Math. Monthly* **96**, 903–909, 1989.

Nearest Integer Function



The nearest integer function nint(x) of x, also called NINT or the ROUND function, is defined such that $[x]$ is the INTEGER closest to x. It is shown as the thin solid curve in the above plot. Note that while $[x]$ is used to denote the nearest integer function in this work, $[x]$ is more commonly used to denote the FLOOR FUNCTION $\lfloor x \rfloor$.

see also CEILING FUNCTION, FLOOR FUNCTION

Nearest Neighbor Problem

The problem of identifying the point from a set of points which is nearest to a given point according to some measure of distance. The nearest neighborhood problem involves identifying the locus of points lying nearer to the query point than to any other point in the set.

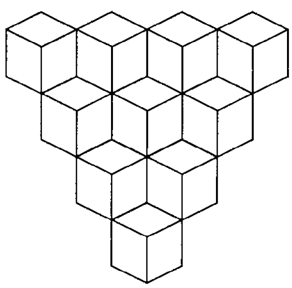
References
Martin, E. C. "Computational Geometry." http://www.mathsource.com/cgi-bin/MathSource/Enhancements/DiscreteMath/0200-181.

Necessary

A CONDITION which must hold for a result to be true, but which does not guarantee it to be true. If a CONDITION is both NECESSARY and SUFFICIENT, then the result is said to be true IFF the CONDITION holds.

see also SUFFICIENT

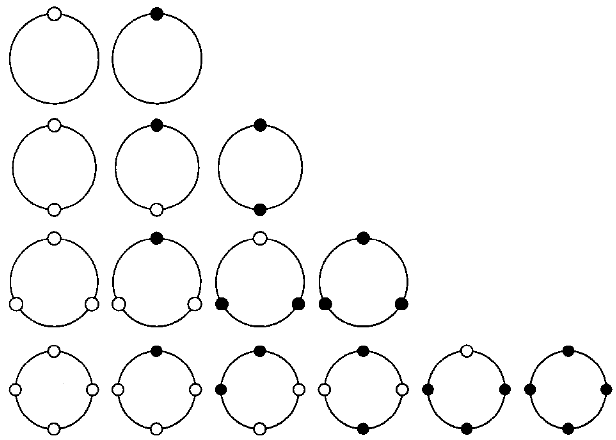
Necker Cube



An ILLUSION in which a 2-D drawing of an array of CUBES appear to simultaneously protrude and intrude into the page.

References
Fineman, M. *The Nature of Visual Illusion.* New York: Dover, pp. 25 and 118, 1996.
Jablan, S. "Impossible Figures." http://members.tripod.com/~modularity/impos.htm.
Newbold, M. "Animated Necker Cube." http://www.sover.net/~manx/necker.html.

Necklace



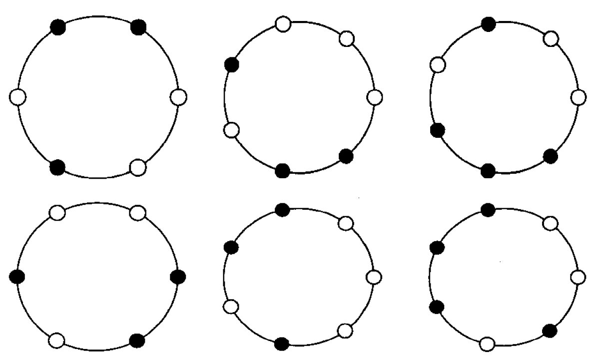
In the technical COMBINATORIAL sense, an a-ary necklace $N(n, a)$ of length n is a string of n characters, each of a possible types. Rotation is ignored, in the sense that $b_1 b_2 \ldots b_n$ is equivalent to $b_k b_{k+1} \cdots b_1 b_2 \cdots b_{k-1}$ for any k, but reversal of strings is respected. Necklaces therefore correspond to circular collections of beads in which the FIXED necklace may not be picked up out of the PLANE (so that opposite orientations are not considered equivalent).

The number of distinct FREE necklaces $N'(n, a)$ of n beads, each of a possible colors, in which opposite orientations (MIRROR IMAGES) are regarded as equivalent (so the necklace *can* be picked up out of the PLANE and flipped over) can be found as follows. Find the DIVISORS of n and label them $d_1 \equiv 1$, d_2, ..., $d_\nu(n) \equiv n$ where $\nu(n)$ is the number of DIVISORS of n. Then

$$N'(n,a) = \frac{1}{2n} \begin{cases} \sum_{i=1}^{\nu(n)} \phi(d_i) a^{n/d_i} + n a^{(n+1)/2} \\ \quad \text{for } n \text{ odd} \\ \sum_{i=1}^{\nu(n)} \phi(d_i) a^{n/d_i} + \frac{1}{2} n (1+a) a^{n/2} \\ \quad \text{for } n \text{ even,} \end{cases}$$

where $\phi(x)$ is the TOTIENT FUNCTION. For $a = 2$ and $n = p$ an ODD PRIME, this simplifies to

$$N'(p,2) = \frac{2^{p-1} - 1}{p} + 2^{(p-1)/2} + 1.$$



A table of the first few numbers of necklaces for $a = 2$ and $a = 3$ follows. Note that $N(n, 2)$ is larger than $N'(n, 2)$ for $n \geq 6$. For $n = 6$, the necklace 110100 is inequivalent to its MIRROR IMAGE 0110100, accounting for the difference of 1 between $N(6, 2)$ and $N'(6, 2)$. Similarly, the two necklaces 0010110 and 0101110 are inequivalent to their reversals, accounting for the difference of 2 between $N(7, 2)$ and $N'(7, 2)$.

n	$N(n,2)$	$N'(n,2)$	$N'(n,3)$
Sloane	000031	000029	027671
1	2	2	3
2	3	3	6
3	4	4	10
4	6	6	21
5	8	8	39
6	14	13	92
7	20	18	198
8	36	30	498
9	60	46	1219
10	108	78	3210
11	188	126	8418
12	352	224	22913
13	632	380	62415
14	1182	687	173088
15	2192	1224	481598

Ball and Coxeter (1987) consider the problem of finding the number of distinct arrangements of n people in a ring such that no person has the same two neighbors two or more times. For 8 people, there are 21 such arrangements.

see also ANTOINE'S NECKLACE, DE BRUIJN SEQUENCE, FIXED, FREE, IRREDUCIBLE POLYNOMIAL, JOSEPHUS PROBLEM, LYNDON WORD

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 49–50, 1987.

Dudeney, H. E. Problem 275 in *536 Puzzles & Curious Problems.* New York: Scribner, 1967.

Gardner, M. *Martin Gardner's New Mathematical Diversions from Scientific American.* New York: Simon and Schuster, pp. 240–246, 1966.

Gilbert, E. N. and Riordan, J. "Symmetry Types of Periodic Sequences." *Illinois J. Math.* **5**, 657–665, 1961.

Riordan, J. "The Combinatorial Significance of a Theorem of Pólya." *J. SIAM* **4**, 232–234, 1957.

Riordan, J. *An Introduction to Combinatorial Analysis.* New York: Wiley, p. 162, 1980.

Ruskey, F. "Information on Necklaces, Lyndon Words, de Bruijn Sequences." `http://sue.csc.uvic.ca/~cos/inf/neck/NecklaceInfo.html`.

Sloane, N. J. A. Sequences A000029/M0563, A000031/M0564, and A001869/M3860 in "An On-Line Version of the Encyclopedia of Integer Sequences." `http://www.research.att.com/~njas/sequences/eisonline.html`.

Sloane, N. J. A. and Plouffe, S. Extended entry for M3860 in *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.

Needle

see BUFFON-LAPLACE NEEDLE PROBLEM, BUFFON'S NEEDLE PROBLEM, KAKEYA NEEDLE PROBLEM

Negation

see NOT

Negative

A quantity less than ZERO (< 0), denoted with a MINUS SIGN, i.e., $-x$.

see also NONNEGATIVE, NONPOSITIVE, NONZERO, POSITIVE, ZERO

Negative Binomial Distribution

Also known as the PASCAL DISTRIBUTION and PÓLYA DISTRIBUTION. The probability of $r - 1$ successes and x failures in $x + r - 1$ trials, and success on the $(x + r)$th trial is

$$\begin{aligned} p\left[\binom{x+r-1}{r-1} p^{r-1}(1-p)^{[(x+r-1)-(r-1)]}\right] &= \left[\binom{x+r-1}{r-1} p^{r-1}(1-p)^x\right] p \\ &= \binom{x+r-1}{r-1} p^r (1-p)^x, \quad (1) \end{aligned}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. Let

$$P = \frac{1-p}{p} \quad (2)$$

$$Q = \frac{1}{p}. \quad (3)$$

The CHARACTERISTIC FUNCTION is given by

$$\phi(t) = (Q - Pe^{it})^{-r}, \quad (4)$$

and the MOMENT-GENERATING FUNCTION by

$$M(t) = \langle e^{tx} \rangle = \sum_{x=0}^{\infty} e^{tx} \binom{x+r-1}{r-1} p^r (1-p)^x, \quad (5)$$

but, since $\binom{N}{n} = \binom{N}{N-m}$,

$$\begin{aligned} M(t) &= p^r \sum_{x=0}^{\infty} \binom{x+r-1}{x} [(1-p)e^t]^x \\ &= p^r [1-(1-p)e^t]^{-r} \quad (6) \\ M'(t) &= p^r(-r)[1-(1-p)e^t]^{-r-1}(p-1)e^t \\ &= p^r(1-p)r[1-(1-p)e^t]^{-r-1}e^t \quad (7) \\ M''(t) &= (1-p)rp^r(1-e^t+pe^t)^{-r-2} \\ &\quad \times (-1-e^t r + e^t pr)e^t \quad (8) \\ M'''(t) &= (1-p)rp^r(1-e^t+e^t p)^{-r-3} \\ &\quad \times [1+e^t(1-p+3r-3pr) \\ &\quad + r^2 e^{2t}(1-p)^2]e^t. \quad (9) \end{aligned}$$

The MOMENTS about zero $\mu'_n = M^n(0)$ are therefore

$$\mu'_1 = \mu = \frac{r(1-p)}{p} = \frac{rq}{p} \tag{10}$$

$$\mu'_2 = \frac{r(1-p)[1-r(p-1)]}{p^2} = \frac{rq(1-rq)}{p^2} \tag{11}$$

$$\mu'_3 = \frac{(1-p)r(2-p+3r-3pr+r^2-2pr^2+p^2r^2}{p^3} \tag{12}$$

$$\mu'_4 = \frac{(-1+p)r(-6+6p-p^2-11r+15pr-4p^2r-6r^2}{p^4} + \frac{12pr^2-6p^2r^2-r^3+3pr^3-3p^2r^3+p^3r^3)}{p^4}. \tag{13}$$

(Beyer 1987, p. 487, apparently gives the MEAN incorrectly.) The MOMENTS about the mean are

$$\mu_2 = \sigma^2 = \frac{r(1-p)}{p^2} \tag{14}$$

$$\mu_3 = \frac{r(2-3p+p^2)}{p^3} = \frac{r(p-1)(p-2)}{p^3} \tag{15}$$

$$\mu_4 = \frac{r(1-p)(6-6p+p^2+3r-3pr)}{p^4}. \tag{16}$$

The MEAN, VARIANCE, SKEWNESS and KURTOSIS are then

$$\mu = \frac{r(1-p)}{p} \tag{17}$$

$$\begin{aligned}\gamma_1 &= \frac{\mu_3}{\sigma^3} = \frac{r(p-1)(p-2)}{p^3}\left[\frac{p^2}{r(1-p)}\right]^{3/2} \\ &= \frac{r(2-p)(1-p)}{p^3}\frac{p^3}{r(1-p)\sqrt{1-p}} \\ &= \frac{2-p}{\sqrt{r(1-p)}} \end{aligned} \tag{18}$$

$$\begin{aligned}\gamma_2 &= \frac{\mu_4}{\sigma^4} - 3 \\ &= \frac{-6+6p-p^2-3r+3pr}{(p-1)r}, \end{aligned} \tag{19}$$

which can also be written

$$\mu = nP \tag{20}$$

$$\mu_2 = nPQ \tag{21}$$

$$\gamma_1 = \frac{Q+P}{\sqrt{rPQ}} \tag{22}$$

$$\gamma_2 = \frac{1+6PQ}{rPQ} - 3. \tag{23}$$

The first CUMULANT is

$$\kappa_1 = nP, \tag{24}$$

and subsequent CUMULANTS are given by the recurrence relation

$$\kappa_{r+1} = PQ\frac{d\kappa_r}{dQ}. \tag{25}$$

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 533, 1987.

Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, p. 118, 1992.

Negative Binomial Series

The SERIES which arises in the BINOMIAL THEOREM for NEGATIVE integral n,

$$\begin{aligned}(x+a)^{-n} &= \sum_{k=0}^{\infty}\binom{-n}{k}x^k a^{-n-k} \\ &= \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k a^{-n-k}.\end{aligned}$$

For $a = 1$, the negative binomial series simplifies to

$$(x+1)^{-n} = 1 - nx + \tfrac{1}{2}n(n+1)x^2 - \tfrac{1}{6}n(n+1)(n+2) + \ldots.$$

see also BINOMIAL SERIES, BINOMIAL THEOREM

Negative Likelihood Ratio

The term NEGATIVE likelihood ratio is also used (especially in medicine) to test nonnested complementary hypotheses as follows,

$$\mathrm{NLR} = \frac{[\text{true negative rate}]}{[\text{false negative rate}]} = \frac{[\text{specificity}]}{1-[\text{sensitivity}]}.$$

see also LIKELIHOOD RATIO, SENSITIVITY, SPECIFICITY

Negative Integer

see $\mathbb{Z}^-$

Negative Pedal Curve

Given a curve C and O a fixed point called the PEDAL POINT, then for a point P on C, draw a LINE PERPENDICULAR to OP. The ENVELOPE of these LINES as P describes the curve C is the negative pedal of C.

see also PEDAL CURVE

References

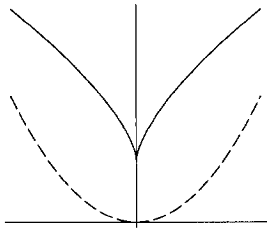
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 46–49, 1972.

Lockwood, E. H. "Negative Pedals." Ch. 19 in *A Bcok of Curves.* Cambridge, England: Cambridge University Press, pp. 156–159, 1967.

Neighborhood

The word neighborhood is a word with many different levels of meaning in mathematics. One of the most general concepts of a neighborhood of a point $x \in \mathbb{R}^n$ (also called an EPSILON-NEIGHBORHOOD or infinitesimal OPEN SET) is the set of points inside an n-BALL with center x and RADIUS $\epsilon > 0$.

Neile's Parabola



The solid curve in the above figure which is the EVOLUTE of the PARABOLA (dashed curve). In CARTESIAN COORDINATES,

$$y = \tfrac{3}{4}(2x)^{2/3} + \tfrac{1}{2}.$$

Neile's parabola is also called the SEMICUBICAL PARABOLA, and was discovered by William Neile in 1657. It was the first nontrivial ALGEBRAIC CURVE to have its ARC LENGTH computed. Wallis published the method in 1659, giving Neile the credit (MacTutor Archive).

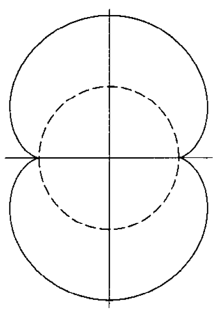
see also PARABOLA EVOLUTE

References

Lee, X. "Semicubic Parabola." `http://www.best.com/~xah/Special Plane Curves _ dir / Semicubic Parabola _ dir / semicubicParabola.html.`

MacTutor History of Mathematics Archive. "Neile's Semi-Cubical Parabola." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Neiles.html.`

Nephroid



The 2-CUSPED EPICYCLOID is called a nephroid. Since $n = 2$, $a = b/2$, and the equation for r^2 in terms of the parameter ϕ is given by EPICYCLOID equation

$$r^2 = \frac{a^2}{n^2}\left[(n^2 + 2n + 2) - 2(n+1)\cos(n\phi)\right] \tag{1}$$

with $n = 2$,

$$\begin{aligned} r^2 &= \frac{a^2}{2^2}[(2^2 + 2\cdot 2 + 2) - 2(2+1)\cos(2\phi)] \\ &= \tfrac{1}{4}a^2[10 - 6\cos(2\phi)] = \tfrac{1}{2}a^2[5 - 3\cos(2\phi)], \end{aligned} \tag{2}$$

where

$$\tan\theta = \frac{3\sin\phi - \sin(3\phi)}{3\cos\phi - \cos(3\phi)}. \tag{3}$$

This can be written

$$\left(\frac{r}{2a}\right)^{2/3} = [\sin(\tfrac{1}{2}\theta)]^{2/3} + [\cos(\tfrac{1}{2}\theta)]^{2/3}. \tag{4}$$

The parametric equations are

$$x = a[3\cos t - \cos(3t)] \tag{5}$$

$$y = a[3\sin t - \sin(3t)]. \tag{6}$$

The Cartesian equation is

$$(x^2 + y^2 - 4a^2)^3 = 108a^4y^2. \tag{7}$$

The name nephroid means "kidney shaped" and was first used for the two-cusped EPICYCLOID by Proctor in 1878 (MacTutor Archive). The nephroid has ARC LENGTH $24a$ and AREA $12\pi^2a^2$. The CATACAUSTIC for rays originating at the CUSP of a CARDIOID and reflected by it is a nephroid. Huygens showed in 1678 that the nephroid is the CATACAUSTIC of a CIRCLE when the light source is at infinity. He published this fact in *Traité de la luminère* in 1690 (MacTutor Archive).

see also ASTROID, DELTOID, FREETH'S NEPHROID

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 169–173, 1972.

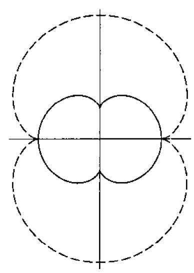
Lee, X. "Nephroid." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Nephroid_dir/nephroid.html.`

Lockwood, E. H. "The Nephroid." Ch. 7 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 62–71, 1967.

MacTutor History of Mathematics Archive. "Nephroid." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Nephroid.html.`

Yates, R. C. "Nephroid." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 152–154, 1952.

Nephroid Evolute



The EVOLUTE of the NEPHROID given by

$$x = \tfrac{1}{2}[3\cos t - \cos(3t)]$$

$$y = \tfrac{1}{2}[3\sin t - \sin(3t)]$$

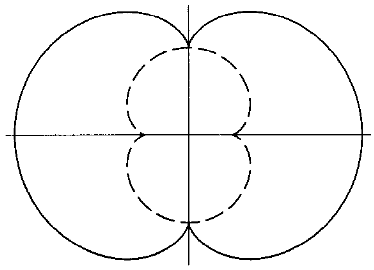
is given by

$$x = \cos^3 t$$

$$y = \tfrac{1}{4}[3\sin t + \sin(3t)],$$

which is another NEPHROID.

Nephroid Involute



The INVOLUTE of the NEPHROID given by

$$x = \tfrac{1}{2}[3\cos t - \cos(3t)]$$
$$y = \tfrac{1}{2}[3\sin t - \sin(3t)]$$

beginning at the point where the nephroid cuts the y-AXIS is given by

$$x = 4\cos^3 t$$
$$y = 3\sin t + \sin(3t),$$

another NEPHROID. If the INVOLUTE is begun instead at the CUSP, the result is CAYLEY'S SEXTIC.

Néron-Severi Group

Let V be a complete normal VARIETY, and write $G(V)$ for the group of divisors, $G_n(V)$ for the group of divisors numerically equal to 0, and $G_a(V)$ the group of divisors algebraically equal to 0. Then the finitely generated QUOTIENT GROUP $NS(V) = G(V)/G_a(V)$ is called the Néron-Severi group.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 75, 1980.

Nerve

The SIMPLICIAL COMPLEX formed from a family of objects by taking sets that have nonempty intersections.

see also DELAUNAY TRIANGULATION, SIMPLICIAL COMPLEX

Nested Hypothesis

Let S be the set of all possibilities that satisfy HYPOTHESIS H, and let S' be the set of all possibilities that satisfy HYPOTHESIS H'. Then H' is a nested hypothesis within H IFF $S' \subset S$, where $\subset$ denotes the PROPER SUBSET.

see also LOG LIKELIHOOD PROCEDURE

Nested Radical

A RADICAL of the form

$$\sqrt{n+\sqrt{n+\sqrt{n+\ldots}}}. \tag{1}$$

For this to equal a given INTEGER x, it must be true that

$$x = \sqrt{n+\sqrt{n+\sqrt{n+\ldots}}} = \sqrt{n+x}, \tag{2}$$

so

$$x^2 = n + x \tag{3}$$

and

$$n = x(x-1). \tag{4}$$

Nested radicals in the computation of PI,

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}}\cdots \tag{5}$$

and in TRIGONOMETRICAL values of COSINE and SINE for arguments of the form $\pi/2^n$, e.g.,

$$\sin\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2}} \tag{6}$$
$$\cos\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2}} \tag{7}$$
$$\sin\left(\frac{\pi}{16}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}} \tag{8}$$
$$\cos\left(\frac{\pi}{16}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}. \tag{9}$$

see also SQUARE ROOT

References

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 14–20, 1994.

Net

A generalization of a SEQUENCE used in general topology and ANALYSIS when the spaces being dealt with are not FIRST-COUNTABLE. (Sequences provide an adequate way of dealing with CONTINUITY for FIRST-COUNTABLE SPACES.) Nets are used in the study of the RIEMANN INTEGRAL.

see also FIBER BUNDLE, FIBER SPACE, FIBRATION

Net (Polyhedron)

A plane diagram in which the EDGES of a POLYHEDRON are shown. All convex POLYHEDRA have nets, but not all concave polyhedra do (the constituent POLYGONS can overlap one another when a concave POLYHEDRON is flattened out). The GREAT DODECAHEDRON and STELLA OCTANGULA are examples of a concave polyhedron which have nets.

Netto's Conjecture

The probability that two elements P_1 and P_2 of a SYMMETRIC GROUP generate the entire GROUP tends to $3/4$ as $n \to \infty$. This was proven by Dixon in 1967.

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 31, 1983.

Network

A DIRECTED GRAPH having a SOURCE, SINK, and a bound on each edge.

see also GRAPH (GRAPH THEORY), SINK (DIRECTED GRAPH), SMITH'S NETWORK THEOREM, SOURCE

Neuberg Circles

The LOCUS of the VERTEX A_1 of a TRIANGLE on a given base A_2A_3 and with a given BROCARD ANGLE ω is a CIRCLE on either side of A_2A_3. From the center N_1, the base A_2A_3 subtends the ANGLE 2ω. The RADIUS of the CIRCLE is

$$r = \tfrac{1}{2}a_1\sqrt{\cot^2\omega - 3}.$$

see also BROCARD ANGLE

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 287–290, 1929.

Neumann Algebra

see VON NEUMANN ALGEBRA

Neumann Boundary Conditions

PARTIAL DIFFERENTIAL EQUATION BOUNDARY CONDITIONS which give the normal derivative on a surface.

see also BOUNDARY CONDITIONS, CAUCHY BOUNDARY CONDITIONS

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 679, 1953.

Neumann Function

see BESSEL FUNCTION OF THE SECOND KIND

Neumann Polynomial

Polynomials which obey the RECURRENCE RELATION

$$O_{n+1}(x) = (n+1)\frac{2}{x}O_n(x) - \frac{n+1}{n-1}O_{n-1}(x) + \frac{2n}{x}\sin^2(\tfrac{1}{2}n\pi).$$

The first few are

$$O_0(x) = \frac{1}{x}$$
$$O_1(x) = \frac{1}{x^2}$$
$$O_2(x) = \frac{1}{x} + \frac{4}{x^3}.$$

see also SCHLÄFLI POLYNOMIAL

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 196, 1993.

Neumann Series (Bessel Function)

A series of the form

$$\sum_{n=0}^{\infty} a_n J_{\nu+n}(z), \tag{1}$$

where ν is a REAL and $J_{\nu+n}(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. Special cases are

$$z^\nu = 2^\nu\Gamma(\tfrac{1}{2}\nu+1)\sum_{n=0}^{\infty}\frac{(\frac{1}{2}z)^{\nu/2+n}}{n!}J_{\nu/2+n}(z), \tag{2}$$

where $\Gamma(z)$ is the GAMMA FUNCTION, and

$$\sum_{n=0}^{\infty} b_n z^{\nu+n} = \sum_{n=0}^{\infty} a_n \left(\tfrac{1}{2}z\right)^{(\nu+n)/2} J_{(\nu+n)/2}(z), \tag{3}$$

where

$$a_n \equiv \sum_{m=0}^{\lfloor n/2\rfloor}\frac{2^{\nu+n-2m}\Gamma(\frac{1}{2}\nu+\frac{1}{2}n-m+1)}{m!}b_{n-2m}, \tag{4}$$

and $\lfloor x\rfloor$ is the FLOOR FUNCTION.

see also KAPTEYN SERIES

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1966.

Neumann Series (Integral Equation)

A FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

$$\phi(x) = f(x) + \int_a^b K(x,t)\phi(t)\,dt \tag{1}$$

may be solved as follows. Take

$$\phi_0(x) \equiv f(x) \tag{2}$$

$$\phi_1(x) = f(x) + \lambda\int_a^b K(x,t)f(t)\,dt \tag{3}$$

$$\phi_2(x) = f(x) + \lambda\int_a^b K(x,t_1)f(t_1)\,dt_1 + \lambda^2\int_a^b\int_a^b K(x,t_1)K(t_1,t_2)f(t_2)\,dt_2\,dt_1 \tag{4}$$

$$\phi_n(x) = \sum_{i=0}^{n}\lambda^i u_i(x), \tag{5}$$

where

$$u_0(x) = f(x) \tag{6}$$

$$u_1(x) = \int_a^b K(x,t)f(t_1)\,dt_1 \tag{7}$$

$$u_2(x) = \int_a^b\int_a^b K(x,t_1)K(t_1,t_2)f(t_2)\,dt_2\,dt_1 \tag{8}$$

$$u_n(x) = \int_a^b\int_a^b\int_a^b K(x,t_1)K(t_1,t_2)\cdots \times K(t_{n-1},t_n)f(t_n)\,dt_n\cdots dt_1. \tag{9}$$

The Neumann series solution is then

$$\phi(x) = \lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i u_i(x). \quad (10)$$

References

Arfken, G. "Neumann Series, Separable (Degenerate) Kernels." §16.3 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 879–890, 1985.

Neusis Construction

A geometric construction, also called a VERGING CONSTRUCTION, which allows the classical GEOMETRIC CONSTRUCTION rules to be bent in order to permit sliding of a marked RULER. Using a Neusis construction, CUBE DUPLICATION and angle TRISECTION are soluble. Conway and Guy (1996) give Neusis constructions for the 7-, 9-, and 13-gons which are based on angle TRISECTION.

see also CUBE DUPLICATION, GEOMETRIC CONSTRUCTION, MASCHERONI CONSTRUCTION, RULER, TRISECTION

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 194–200, 1996.

Neville's Algorithm

An interpolation ALGORITHM which proceeds by first fitting a POLYNOMIAL P_k of degree 0 through the points (x_k, y_k) for $k = 0, \ldots, n$, i.e., $P_k = y_k$. A second iteration is then performed in which P_{12} is fit through pairs of points, yielding P_{12}, P_{23}, ... The procedure is repeated, generating a "pyramid" of approximations until the final result is reached

$$\begin{array}{ccccccc} P_1 & & & & & & \\ & P_{12} & & & & & \\ P_2 & & P_{123} & & & & \\ & P_{23} & & P_{1234}. & & & \\ P_3 & & P_{234} & & & & \\ & P_{34} & & & & & \\ P_4 & & & & & & \end{array}$$

The final result is

$$P_{i(i+1)\cdots(i+m)} = \frac{(x - x_{i+m})P_{i(i+1)\cdots(i+m-1)}}{x_i - x_{i+m}} + \frac{(x_i - x)P_{(i+1)(i+2)\cdots(i+m)}}{x_i - x_{i+m}}.$$

see also BULIRSCH-STOER ALGORITHM

Neville Theta Function

The functions

$$\vartheta_s(u) = \frac{H(u)}{H'(0)} \quad (1)$$

$$\vartheta_d(u) = \frac{\Theta(u+K)}{\Theta(k)} \quad (2)$$

$$\vartheta_c(u) = \frac{H(u)}{H(K)} \quad (3)$$

$$\vartheta_n(u) = \frac{\Theta(u)}{\Theta(0)}, \quad (4)$$

where H and Θ are the JACOBI THETA FUNCTIONS and $K(u)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND.

see also JACOBI THETA FUNCTION, THETA FUNCTION

Newcomb's Paradox

A paradox in DECISION THEORY. Given two boxes, B1 which contains $1000 and B2 which contains either nothing or a million dollars, you may pick either B2 or both. However, at some time before the choice is made, an omniscient Being has predicted what your decision will be and filled B2 with a million dollars if he expects you to take it, or with nothing if he expects you to take both.

see also ALIAS' PARADOX

References

Gardner, M. *The Unexpected Hanging and Other Mathematical Diversions.* Chicago, IL: Chicago University Press, 1991.

Gardner, M. "Newcomb's Paradox." Ch. 13 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, 1986.

Nozick, R. "Reflections on Newcomb's Paradox." Ch. 14 in Gardner, M. *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, 1986.

Newman-Conway Sequence

The sequence 1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, ... (Sloane's A004001) defined by the recurrence $P(1) = P(2) = 1$,

$$P(n) = P(P(n-1)) + P(n - P(n-1)).$$

It satisfies

$$P(2^k) = 2^{k-1}$$

and

$$P(2n) \leq 2P(n).$$

References

Bloom, D. M. "Newman-Conway Sequence." Solution to Problem 1459. *Math. Mag.* **68**, 400–401, 1995.

Sloane, N. J. A. Sequence A004001/M0276 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Newton's Backward Difference Formula

$$f_p = f_0 + p\nabla_0 + \tfrac{1}{2!}p(p+1)\nabla_0^2 + \tfrac{1}{3!}p(p+1)(p+2)\nabla_0^3 + \ldots,$$

for $p \in [0,1]$, where ∇ is the BACKWARD DIFFERENCE.

see also NEWTON'S FORWARD DIFFERENCE FORMULA

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 433, 1987.

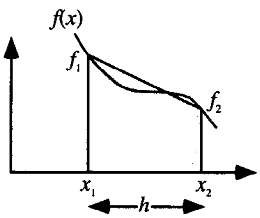
Newton-Cotes Formulas

The Newton-Cotes formulas are an extremely useful and straightforward family of NUMERICAL INTEGRATION techniques.

To integrate a function $f(x)$ over some interval $[a, b]$, divide it into n equal parts such that $f_n = f(x_n)$ and $h \equiv (b-a)/n$. Then find POLYNOMIALS which approximate the tabulated function, and integrate them to approximate the AREA under the curve. To find the fitting POLYNOMIALS, use LAGRANGE INTERPOLATING POLYNOMIALS. The resulting formulas are called Newton-Cotes formulas, or QUADRATURE FORMULAS.

Newton-Cotes formulas may be "closed" if the interval $[x_1, x_n]$ is included in the fit, "open" if the points $[x_2, x_{n-1}]$ are used, or a variation of these two. If the formula uses n points (closed or open), the COEFFICIENTS of terms sum to $n-1$.

If the function $f(x)$ is given explicitly instead of simply being tabulated at the values x_i, the best numerical method of integration is called GAUSSIAN QUADRATURE. By picking the intervals at which to sample the function, this procedure produces more accurate approximations (but is significantly more complicated to implement).



The 2-point closed Newton-Cotes formula is called the TRAPEZOIDAL RULE because it approximates the area under a curve by a TRAPEZOID with horizontal base and sloped top (connecting the endpoints x_1 and x_2). If the first point is x_1, then the other endpoint will be located at

$$x_2 = x_1 + h, \tag{1}$$

and the LAGRANGE INTERPOLATING POLYNOMIAL through the points (x_1, f_1) and (x_2, f_2) is

$$\begin{aligned} P_2(x) &= \frac{x-x_2}{x_1-x_2} f_1 + \frac{x-x_1}{x_2-x_1} f_2 \\ &= \frac{x-x_1-h}{-h} f_1 + \frac{x-x_1}{h} f_2 \\ &= \frac{x}{h}(f_2-f_1) + \left(f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2\right). \end{aligned} \tag{2}$$

Integrating over the interval (i.e., finding the area of the trapezoid) then gives

$$\begin{aligned} \int_{x_1}^{x_2} f(x)\,dx &= \int_{x_1}^{x_1+h} P_2(x)\,dx \\ &= \frac{1}{2h}(f_2-f_1)[x^2]_{x_1}^{x_2} \\ &\quad + \left(f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2\right)[x]_{x_1}^{x_2} \\ &= \frac{1}{2h}(f_2-f_1)(x_2+x_1)(x_2-x_1) \\ &\quad + (x_2-x_1)\left(f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2\right) \\ &= \tfrac{1}{2}(f_2-f_1)(2x_1+h) + f_1 h + x_1(f_1-f_2) \\ &= x_1(f_2-f_1) + \tfrac{1}{2}h(f_2-f_1) + hf_1 - x_1(f_2-f_1) \\ &= \tfrac{1}{2}h(f_1+f_2) - \tfrac{1}{2}h^3 f''(\xi). \end{aligned} \tag{3}$$

This is the trapezoidal rule, with the final term giving the amount of error (which, since $x_1 \le \xi \le x_2$, is no worse than the maximum value of $f''(\xi)$ in this range).

The 3-point rule is known as SIMPSON'S RULE. The ABSCISSAS are

$$x_2 = x_1 + h \tag{4}$$

$$x_3 = x_1 + 2h \tag{5}$$

and the LAGRANGE INTERPOLATING POLYNOMIAL is

$$\begin{aligned} P_3(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f_1 \\ &\quad + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f_3 \\ &= \frac{x^2 - x(x_2+x_3) + x_2x_3}{h(2h)} f_1 \\ &\quad + \frac{x^2 - x(x_1+x_3) + x_1x_3}{h(-h)} f_2 + \frac{x^2 - x(x_1+x_2) + x_1x_2}{2h(h)} f_3 \\ &= \frac{1}{h^2}\{x^2(\tfrac{1}{2}f_1 - f_2 - \tfrac{1}{2}f_3) \\ &\quad + x[-\tfrac{1}{2}(2x_1+3h)f_1 + (2x_1+2h)f_2 - \tfrac{1}{2}(2x_1+h)] \\ &\quad + [\tfrac{1}{2}(x_1+h)(x_1+2h)f_1 - x_1(x_1+2h)f_2 + \tfrac{1}{2}x_1(x_1+h)f_3]\}. \end{aligned} \tag{6}$$

Integrating and simplifying gives

$$\begin{aligned} \int_{x_1}^{x_3} f(x)\,dx &= \int_{x_1}^{x_1+2h} P_3(x)\,dx \\ &= \tfrac{1}{3}h(f_1+4f_2+f_3) - \tfrac{1}{90}h^5 f^{(4)}(\xi). \end{aligned} \tag{7}$$

The 4-point closed rule is SIMPSON'S 3/8 RULE,

$$\int_{x_1}^{x_4} f(x)\,dx = \tfrac{3}{8}h(f_1+3f_2+3f_3+f_4) - \tfrac{3}{80}h^5 f^{(4)}(\xi). \tag{8}$$

The 5-point closed rule is BODE'S RULE,

$$\int_{x_1}^{x_5} f(x)\,dx = \tfrac{2}{45}h(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5) - \tfrac{8}{945}h^7 f^{(6)}(\xi) \quad (9)$$

(Abramowitz and Stegun 1972, p. 886). Higher order rules include the 6-point

$$\int_{x_1}^{x_6} f(x)\,dx = \tfrac{5}{288}h(19f_1 + 75f_2 + 50f_3 + 50f_4 + 75f_5 + 19f_6) - \tfrac{275}{12096}h^7 f^{(6)}(\xi), \quad (10)$$

7-point

$$\int_{x_1}^{x_7} f(x)\,dx = \tfrac{1}{140}h(41f_1 + 216f_2 + 27f_3 + 272f_4 + 27f_5 + 216f_6 + 41f_7) - \tfrac{9}{1400}h^9 f^{(8)}(\xi), \quad (11)$$

8-point

$$\int_{x_1}^{x_8} f(x)\,dx = \tfrac{7}{17280}h(751f_1 + 3577f_2 + 1323f_2 + 2989f_3 + 2989f_5 + 1323f_6 + 3577f_7 + 751f_8) - \tfrac{8183}{518400}h^9 f^{(8)}(\xi), \quad (12)$$

9-point

$$\int_{x_1}^{x_9} f(x)\,dx = \tfrac{4}{14175}h(989f_1 + 5888f_2 - 928f_3 + 10496f_4 - 4540f_5 + 10496f_6 - 928f_7 + 5888f_8 + 989f_9) - \tfrac{2368}{467775}h^{11} f^{(10)}(\xi), \quad (13)$$

10-point

$$\int_{x_1}^{x_{10}} f(x)\,dx = \tfrac{9}{89600}h[2857(f_1 + f_{10}) + 15741(f_2 + f_9) + 1080(f_3 + f_8 + 19344(f_4 + f_7) + 5788(f_5 + f_6)] - \tfrac{173}{14620}h^{11} f^{(10)}(\xi), \quad (14)$$

and 11-point

$$\int_{x_1}^{x_{11}} f(x)\,dx = \tfrac{5}{299376}h[16067(f_1 + f_{11}) + 106300(f_2 + f_{10}) - 48525(f_3 + f_9) + 272400(f_4 + f_8) - 260550(f_5 + f_7) + 427368f_6] - \tfrac{1346350}{326918592}h^{13} f^{(12)}(\xi) \quad (15)$$

rules.

Closed "extended" rules use multiple copies of lower order closed rules to build up higher order rules. By appropriately tailoring this process, rules with particularly nice properties can be constructed. For n tabulated points, using the TRAPEZOIDAL RULE $(n-1)$ times and adding the results gives

$$\int_{x_1}^{x_n} f(x)\,dx = \left(\int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \ldots + \int_{x_{n-1}}^{x_n}\right) f(x)\,dx$$
$$= \tfrac{1}{2}h[(f_1 + f_2) + (f_2 + f_3) + \ldots + (f_{n-2} + f_{n-1}) + (f_{n-1}+f_n)] = h(\tfrac{1}{2}f_1 + f_2 + f_3 + \ldots + f_{n-2} + f_{n-1} + \tfrac{1}{2}f_n) - \tfrac{1}{12}nh^3 f''(\xi). \quad (16)$$

Using a series of refinements on the extended TRAPEZOIDAL RULE gives the method known as ROMBERG INTEGRATION. A 3-point extended rule for ODD n is

$$\int_{x_1}^{x_n} f(x)\,dx = h[(\tfrac{1}{3}f_1 + \tfrac{4}{3}f_2 + \tfrac{1}{3}f_3) + (\tfrac{1}{3}f_3 + \tfrac{4}{3}f_4 + \tfrac{1}{3}f_5) + \ldots + (\tfrac{1}{3}f_{n-4} + \tfrac{4}{3}f_{n-3} + \tfrac{1}{3}f_{n-2}) + (\tfrac{1}{3}f_{n-2} + \tfrac{4}{3}f_{n-1} + \tfrac{1}{3}f_n)]$$
$$= \tfrac{1}{3}h(f_1 + 4f_2 + 2f_3 + 4f_4 + 2f_5 + \ldots + 4f_{n-1} + f_n) - \frac{n-1}{2}\tfrac{1}{90}h^5 f^{(4)}(\xi). \quad (17)$$

Applying SIMPSON'S 3/8 RULE, then SIMPSON'S RULE (3-point) twice, and adding gives

$$\left[\int_{x_1}^{x_4} + \int_{x_4}^{x_6} + \int_{x_6}^{x_8}\right] f(x)\,dx$$
$$= h[(\tfrac{3}{8}f_1 + \tfrac{9}{8}f_2 + \tfrac{9}{8}f_3 + \tfrac{3}{8}f_4) + (\tfrac{1}{3}f_4 + \tfrac{4}{3}f_5 + \tfrac{1}{3}f_6) + (\tfrac{1}{3}f_6 + \tfrac{4}{3}f_7 + \tfrac{1}{3}f_8)]$$
$$= h[\tfrac{3}{8}f_1 + \tfrac{9}{8}f_2 + \tfrac{9}{8}f_3 + (\tfrac{3}{8} + \tfrac{1}{3})f_4 + \tfrac{4}{3}f_5 + (\tfrac{1}{3} + \tfrac{1}{3})f_6 + \tfrac{4}{3}f_7 + \tfrac{1}{3}f_8]$$
$$= h(\tfrac{3}{8}f_1 + \tfrac{9}{8}f_2 + \tfrac{9}{8}f_3 + \tfrac{17}{24}f_4 + \tfrac{4}{3}f_5 + \tfrac{2}{3}f_6 + \tfrac{4}{3}f_7 + \tfrac{1}{3}f_8). \quad (18)$$

Taking the next Simpson's 3/8 step then gives

$$\int_{x_8}^{x_{11}} f(x)\,dx = h(\tfrac{3}{8}f_8 + \tfrac{9}{8}f_9 + \tfrac{9}{8}f_{10} + \tfrac{3}{8}f_{11}). \quad (19)$$

Combining with the previous result gives

$$\int_{x_1}^{x_{11}} f(x)\,dx = h[\tfrac{3}{8}f_1 + \tfrac{9}{8}f_2 + \tfrac{9}{8}f_3 + \tfrac{17}{24}f_4 + \tfrac{4}{3}f_5 + \tfrac{2}{3}f_6 + \tfrac{4}{3}f_7 + (\tfrac{1}{3} + \tfrac{3}{8})f_8 + \tfrac{9}{8}f_9 + \tfrac{9}{8}f_{10} + \tfrac{3}{8}f_{11}]$$
$$= h(\tfrac{3}{8}f_1 + \tfrac{9}{8}f_2 + \tfrac{9}{8}f_3 + \tfrac{17}{24}f_4 + \tfrac{4}{3}f_5 + \tfrac{2}{3}f_6 + \tfrac{4}{3}f_7 + \tfrac{17}{24}f_8 + \tfrac{9}{8}f_9 + \tfrac{9}{8}f_{10} + \tfrac{3}{8}f_{11}), \quad (20)$$

where terms up to f_{10} have now been completely determined. Continuing gives

$$h(\tfrac{3}{8}f_1 + \tfrac{9}{8}f_2 + \tfrac{9}{8}f_3 + \tfrac{17}{24}f_4 + \tfrac{4}{3}f_5 + \tfrac{2}{3}f_6 + \ldots + \tfrac{2}{3}f_{n-5} + \tfrac{4}{3}f_{n-4} + \tfrac{17}{24}f_{n-3} + \tfrac{9}{8}f_{n-2} + \tfrac{9}{8}f_{n-1} + \tfrac{3}{8}f_n). \quad (21)$$

Now average with the 3-point result

$$h(\tfrac{1}{3}f_1 + \tfrac{4}{3}f_2 + \tfrac{2}{3}f_3 + \tfrac{4}{3}f_4 + \tfrac{2}{3}f_5 + \tfrac{4}{3}f_{n-1} + \tfrac{1}{3}f_n) \quad (22)$$

to obtain

$$h[\tfrac{17}{48}f_1 + \tfrac{59}{48}f_2 + \tfrac{43}{48}f_4 + \tfrac{49}{48}f_4 + (f_5 + f_6 + \ldots + f_{n-5} + f_{n-4}) + \tfrac{49}{48}f_{n-3} + \tfrac{43}{38}f_{n-2} + \tfrac{59}{48}f_{n-1} + \tfrac{17}{48}f_n] + \mathcal{O}(n^{-4}). \quad (23)$$

Note that all the middle terms now have unity COEFFICIENTS. Similarly, combining a 4-point with the (2+4)-point rule gives

$$h(\tfrac{5}{12}f_1 + \tfrac{13}{12}f_2 + f_3 + f_4 + \ldots + f_{n-3} + f_{n-2} + \tfrac{13}{12}f_{n-1} + \tfrac{5}{12}) + \mathcal{O}(n^{-3}). \quad (24)$$

Other Newton-Cotes rules occasionally encountered include DURAND'S RULE

$$\int_{x_1}^{x_n} f(x)\,dx = h(\tfrac{2}{5}f_1 + \tfrac{11}{10}f_2 + f_3 + \ldots + f_{n-2} + \tfrac{11}{10}f_{n-1} + \tfrac{2}{5}f_n) \quad (25)$$

(Beyer 1987), HARDY'S RULE

$$\int_{x_0-3h}^{x_0+3h} f(x)\,dx = \tfrac{1}{100}h(28f_{-3} + 162f_{-2} + 22f_0 + 162f_2 + 28f_3) + \tfrac{9}{1400}h^7[2f^{(4)}(\xi_2) - h^2 f^{(8)}(\xi_1)], \quad (26)$$

and WEDDLE'S RULE

$$\int_{x_1}^{x_{6n}} f(x)\,dx = \tfrac{3}{10}h(f_1 + 5f_2 + f_3 + 6f_4 + 5f_5 + f_6 + \ldots + 5f_{6n-1} + f_{6n}) \quad (27)$$

(Beyer 1987).

The open Newton-Cotes rules use points outside the integration interval, yielding the 1-point

$$\int_{x_0}^{x_2} f(x)\,dx = 2hf_1, \quad (28)$$

2-point

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_1-h}^{x_1+2h} P_2(x)\,dx = \frac{1}{2h}(f_2 - f_1)[x^2]_{x_1-h}^{x_1+2h} + \left(f_1 + \frac{x_1}{h}f_1 - \frac{x_1}{h}f_2\right)[x]_{x_1-h}^{x_1+2h} = \tfrac{3}{2}h(f_1 + f_2) + \tfrac{1}{4}h^3 f''(\xi), \quad (29)$$

3-point

$$\int_{x_0}^{x_4} f(x)\,dx = \tfrac{4}{3}h(2f_1 - f_2 + 2f_3) + \tfrac{28}{90}h^5 f^{(4)}(\xi), \quad (30)$$

4-point

$$\int_{x_0}^{x_5} f(x)\,dx = \tfrac{5}{24}h(11f_1 + f_2 + f_3 + 11f_4) + \tfrac{95}{144}h^5 f^{(4)}(\xi), \quad (31)$$

5-point

$$\int_{x_0}^{x_6} f(x)\,dx = \tfrac{6}{20}h(11f_1 - 14f_2 + 26f_3 - 14f_4 + 11f_5) - \tfrac{41}{140}h^7 f^{(6)}(\xi), \quad (32)$$

6-point

$$\int_{x_0}^{x_7} f(x)\,dx = \tfrac{7}{1440}h(611f_1 - 453f_2 + 562f_3 + 562f_4 - 453f_5 + 611f_6) - \tfrac{5257}{8640}h^7 f^{(6)}(\xi), \quad (33)$$

and 7-point

$$\int_{x_0}^{x_8} f(x)\,dx = \tfrac{8}{945}h(460f_1 - 954f_2 + 2196f_3 - 2459f_4 + 2196f_5 - 954f_6 + 460f_7) - \tfrac{3956}{14175}h^9 f^{(8)}(\xi) \quad (34)$$

rules.

A 2-point open extended formula is

$$\int_{x_1}^{x_n} f(x)\,dx = h[(\tfrac{1}{2}f_1 + f_2 + \ldots + f_{n-1} + \tfrac{1}{2}f_n) + \tfrac{1}{24}(-f_0 + f_2 + f_{n-1} + f_{n+1})] + \frac{11(n+1)}{720}h^5 f^{(4)}(\xi). \quad (35)$$

Single interval extrapolative rules estimate the integral in an interval based on the points around it. An example of such a rule is

$$hf_1 + \mathcal{O}(h^2 f') \quad (36)$$

$$\tfrac{1}{2}h(3f_1 - f_2) + \mathcal{O}(h^3 f'') \quad (37)$$

$$\tfrac{1}{12}h(23f_1 - 16f_2 + 5f_3) + \mathcal{O}(h^4 f^{(3)}) \quad (38)$$

$$\tfrac{1}{24}h(55f_1 - 59f_2 + 37f_3 - 9f_4) + \mathcal{O}(h^5 f^{(4)}). \quad (39)$$

see also BODE'S RULE, DIFFERENCE EQUATION, DURAND'S RULE, FINITE DIFFERENCE, GAUSSIAN QUADRATURE, HARDY'S RULE, LAGRANGE INTERPOLATING POLYNOMIAL, NUMERICAL INTEGRATION, SIMPSON'S RULE, SIMPSON'S 3/8 RULE, TRAPEZOIDAL RULE, WEDDLE'S RULE

References

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Newton's Diverging Parabolas

Curves with CARTESIAN equation

$$ay^2 = x(x^2 - 2bx + c)$$

with $a > 0$. The above equation represents the third class of Newton's classification of CUBIC CURVES, which Newton divided into five species depending on the ROOTS of the cubic in x on the right-hand side of the equation. Newton described these cases as having the following characteristics:

1. "All the ROOTS are REAL and unequal. Then the Figure is a diverging Parabola of the Form of a Bell, with an Oval at its Vertex.
2. Two of the ROOTS are equal. A PARABOLA will be formed, either Nodated by touching an Oval, or Punctate, by having the Oval infinitely small.
3. The three ROOTS are equal. This is the NEILIAN PARABOLA, commonly called SEMI-CUBICAL.
4. Only one REAL ROOT. If two of the ROOTS are impossible, there will be a Pure PARABOLA of a Bell-like Form"

(MacTutor Archive).

References

MacTutor History of Mathematics Archive. "Newton's Diverging Parabolas." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Newtons.html`.

Newton's Divided Difference Interpolation Formula

Let

$$\pi_n(x) \equiv \prod_{i=1}^{n}(x - x_n), \tag{1}$$

then

$$f(x) = f_0 + \sum_{k=1}^{n} x_{k-1}(x)[x_0, x_1, \ldots, x_k] + R_n, \tag{2}$$

where $[x_1, \ldots]$ is a DIVIDED DIFFERENCE, and the remainder is

$$R_n(x) = \pi_n(x)[x_0, \ldots, x_n, x] = \pi_n(x)\frac{f^{(n+1)}(\xi)}{(n+1)} \tag{3}$$

for $x_0 < \xi < x_n$.

see also DIVIDED DIFFERENCE, FINITE DIFFERENCE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 880, 1972.

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp.43–44 and 62–63, 1956.

Newton's Forward Difference Formula

A FINITE DIFFERENCE identity giving an interpolated value between tabulated points $\{f_p\}$ in terms of the first value f_0 and the POWERS of the FORWARD DIFFERENCE Δ. For $a \in [0, 1]$, the formula states

$$f_a = f_0 + a\Delta + \tfrac{1}{2!}a(a-1)\Delta^2 + \tfrac{1}{3!}a(a-1)(a-2)\Delta^3 + \ldots.$$

When written in the form

$$f(x+a) = \sum_{n=0}^{\infty} \frac{(a)_n \Delta^n f(x)}{n!}$$

with $(a)_n$ the POCHHAMMER SYMBOL, the formula looks suspiciously like a finite analog of a TAYLOR SERIES expansion. This correspondence was one of the motivating forces for the development of UMBRAL CALCULUS.

The DERIVATIVE of Newton's forward difference formula gives MARKOFF'S FORMULAS.

see also FINITE DIFFERENCE, MARKOFF'S FORMULAS, NEWTON'S BACKWARD DIFFERENCE FORMULA, NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 880, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 432, 1987.

Newton's Formulas

Let a TRIANGLE have side lengths a, b, and c with opposite angles A, B, and C. Then

$$\frac{b+c}{a} = \frac{\cos[\frac{1}{2}(B-C)]}{\sin(\frac{1}{2}A)}$$

$$\frac{c+a}{b} = \frac{\cos[\frac{1}{2}(C-A)]}{\sin(\frac{1}{2}B)}$$

$$\frac{a+b}{c} = \frac{\cos[\frac{1}{2}(A-B)]}{\sin(\frac{1}{2}C)}.$$

see also MOLLWEIDE'S FORMULAS, TRIANGLE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 146, 1987.

Newton's Identities

see also NEWTON'S RELATIONS

Newton's Iteration

An algorithm for the SQUARE ROOT of a number r quadratically as $\lim_{n\to\infty} x_n$,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{r}{x_n}\right),$$

where $x_0 = 1$. The first few approximants to $\sqrt{n}$ are given by

$$1, \tfrac{1}{2}(1+n), \frac{1+6n+n^2}{4(n+1)}, \frac{1+26n+70n^2+28n^3+n^4}{8(1+n)(1+6n+n^2)}, \ldots.$$

For $\sqrt{2}$, this gives the convergents as 1, 3/2, 17/12, 577/408, 665857/470832, ...

see also SQUARE ROOT

Newton's Method

A ROOT-finding ALGORITHM which uses the first few terms of the TAYLOR SERIES in the vicinity of a suspected ROOT to zero in on the root. The TAYLOR SERIES of a function $f(x)$ about the point $x+\epsilon$ is given by

$$f(x+\epsilon) = f(x) + f'(x)\epsilon + \tfrac{1}{2}f''(x)\epsilon^2 + \ldots. \tag{1}$$

Keeping terms only to first order,

$$f(x+\epsilon) \approx f(x) + f'(x)\epsilon. \tag{2}$$

This expression can be used to estimate the amount of offset ϵ needed to land closer to the root starting from an initial guess x_0. Setting $f(x_0+\epsilon) = 0$ and solving (2) for ϵ gives

$$\epsilon_0 = -\frac{f(x_0)}{f'(x_0)}, \tag{3}$$

which is the first-order adjustment to the ROOT's position. By letting $x_1 = x_0 + \epsilon_0$, calculating a new ϵ_1, and so on, the process can be repeated until it converges to a root.

Unfortunately, this procedure can be unstable near a horizontal ASYMPTOTE or a LOCAL MINIMUM. However, with a good initial choice of the ROOT's position, the algorithm can by applied iteratively to obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

for $n = 1, 2, 3, \ldots$.

The error ϵ_{n+1} after the $(n+1)$st iteration is given by

$$\begin{aligned}\epsilon_{n+1} &= \epsilon_n + (x_{n+1} - x_n)\\ &= \epsilon_n - \frac{f(x_n)}{f'(x_n)}.\end{aligned} \tag{5}$$

But

$$\begin{aligned}f(x_n) &= f(x) + f'(x)\epsilon_n + \tfrac{1}{2}f''(x){\epsilon_n}^2 + \ldots\\ &= f'(x)\epsilon_n + \tfrac{1}{2}f''(x){\epsilon_n}^2 + \ldots \quad (6)\\ f'(x_n) &= f'(x) + f''(x)\epsilon_n + \ldots, \quad (7)\end{aligned}$$

so

$$\begin{aligned}\frac{f(x_n)}{f'(x_x)} &= \frac{f'(x)\epsilon_n + \frac{1}{2}f''(x){\epsilon_n}^2 + \ldots}{f'(x) + f''(x)\epsilon_n + \ldots}\\ &\approx \frac{f'(x)\epsilon + \frac{1}{2}f''(x){\epsilon_n}^2}{f'(x) + f''(x)\epsilon_n} = \epsilon_n + \frac{f''(x)}{2f'(x)}{\epsilon_n}^2, \quad (8)\end{aligned}$$

and (5) becomes

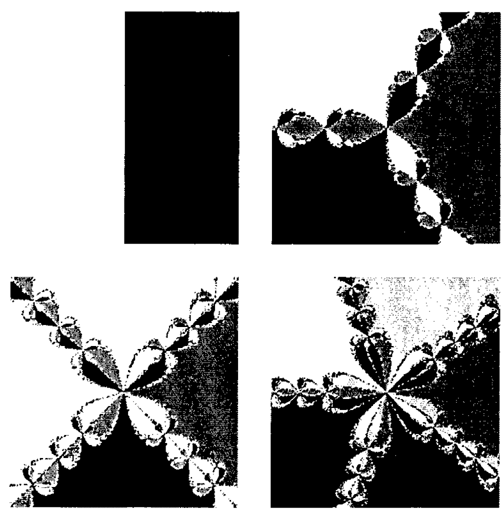
$$\epsilon_{n+1} = \epsilon_n - \left[\epsilon_n + \frac{f''(x)}{2f'(x)}{\epsilon_n}^2\right] = -\frac{f''(x)}{2f'(x)}{\epsilon_n}^2. \tag{9}$$

Therefore, when the method converges, it does so quadratically.

A FRACTAL is obtained by applying Newton's method to finding a ROOT of $z^n - 1 = 0$ (Mandelbrot 1983, Gleick 1988, Peitgen and Saupe 1988, Press *et al.* 1992, Dickau 1997). Iterating for a starting point z_0 gives

$$z_{i+1} = z_i - \frac{{z_i}^n - 1}{n{z_i}^{n-1}}. \tag{10}$$

Since this is an nth order POLYNOMIAL, there are n ROOTS to which the algorithm can converge.



Coloring the BASIN OF ATTRACTION (the set of initial points z_0 which converge to the same ROOT) for each ROOT a different color then gives the above plots, corresponding to $n = 2$, 3, 4, and 5.

see also HALLEY'S IRRATIONAL FORMULA, HALLEY'S METHOD, HOUSEHOLDER'S METHOD, LAGUERRE'S METHOD

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Ortega, J. M. and Rheinboldt, W. C. *Iterative Solution of Nonlinear Equations in Several Variables.* New York: Academic Press, 1970.

Peitgen, H.-O. and Saupe, D. *The Science of Fractal Images.* New York: Springer-Verlag, 1988.

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Ralston, A. and Rabinowitz, P. §8.4 in *A First Course in Numerical Analysis, 2nd ed.* New York: McGraw-Hill, 1978.

Newton Number

see KISSING NUMBER

Newton's Parallelogram

Approximates the possible values of y in terms of x if

$$\sum_{i,j=0}^{n} a_{ij} x^i y^i = 0.$$

Newton-Raphson Fractal

see NEWTON'S METHOD

Newton-Raphson Method

see NEWTON'S METHOD

Newton's Relations

Let s_i be the sum of the products of distinct ROOTS r_j of the POLYNOMIAL equation of degree n

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0, \qquad (1)$$

where the roots are taken i at a time (i.e., s_i is defined as the ELEMENTARY SYMMETRIC FUNCTION $\Pi_i(r_1, \ldots, r_n)$) s_i is defined for $i = 1, \ldots, n$. For example, the first few values of s_i are

$$s_1 = r_1 + r_2 + r_3 + r_4 + \ldots \qquad (2)$$

$$s_2 = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + \ldots \qquad (3)$$

$$s_3 = r_1 r_2 r_3 + r_1 r_2 r_4 + r_2 r_3 r_4 + \ldots, \qquad (4)$$

and so on. Then

$$s_i = (-1)^i \frac{a_{n-i}}{a_n}. \qquad (5)$$

This can be seen for a second DEGREE POLYNOMIAL by multiplying out,

$$a_2 x^2 + a_1 x + a_0 = a_2 (x - r_1)(x - r_2) = a_2 [x^2 - (r_1 + r_2) x + r_1 r_2], \qquad (6)$$

so

$$s_1 = \sum_{i=1}^{2} r_i = r_1 + r_2 = -\frac{a_1}{a_2} \qquad (7)$$

$$s_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{2} r_i r_j = r_1 r_2 = \frac{a_0}{a_2}, \qquad (8)$$

and for a third DEGREE POLYNOMIAL,

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = a_3 (x - r_1)(x - r_2)(x - r_3) = a_3 [x^3 - (r_1 + r_2 + r_3) x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3) x - r_1 r_2 r_3], \qquad (9)$$

so

$$s_1 = \sum_{i=1}^{3} r_i = -\frac{a_2}{a_3} \qquad (10)$$

$$s_2 = \sum_{\substack{i,j \\ i \neq j}}^{3} r_i r_j = r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{a_1}{a_3} \qquad (11)$$

$$s_3 = \sum_{\substack{i,j,k \\ i \neq j \neq k}}^{3} r_i r_j r_k = r_1 r_2 r_3 = -\frac{a_0}{a_3}. \qquad (12)$$

see also ELEMENTARY SYMMETRIC FUNCTION

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 1–2, 1959.

Newton's Theorem

If each of two nonparallel transversals with nonminimal directions meets a given curve in finite points only, then the ratio of products of the distances from the two sets of intersections to the intersection of the lines is independent of the position of the latter point.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 189, 1959.

Newtonian Form

see NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

Next Prime

The next prime function $NP(n)$ gives the smallest PRIME larger than n. The function can be given explicitly as

$$NP(n) = p_{1+\pi(n)},$$

where p_i is the ith PRIME and $\pi(n)$ is the PRIME COUNTING FUNCTION. For $n = 1, 2, \ldots$ the values are 2, 3, 5, 5, 7, 7, 11, 11, 11, 11, 13, 13, 17, 17, 17, 17, 19, ... (Sloane's A007918).

see also FORTUNATE PRIME, PRIME COUNTING FUNCTION, PRIME NUMBER

References
Sloane, N. J. A. Sequence A007918 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Nexus Number

A FIGURATE NUMBER built up of the nexus of cells less than n steps away from a given cell. In k-D, the $(n+1)$th nexus number is given by

$$N_{n+1}(k) = \sum_{i=0}^{k} \binom{k}{i} n^i,$$

where $\binom{n}{n}$ is a BINOMIAL COEFFICIENT. The first few k-dimensional nexus numbers are given in the table below.

k	N_{n+1}	name
0	1	unit
1	$1+2n$	odd number
2	$1+3n+3n^2$	hex number
3	$1+4n+6n^2+4n^3$	rhombic dodecahedral number

see also HEX NUMBER, ODD NUMBER, RHOMBIC DODECAHEDRAL NUMBER

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 53–54, 1996.

Neyman-Pearson Lemma

If there exists a critical region C of size α and a NONNEGATIVE constant k such that

$$\frac{\prod_{i=1}^n f(x_i|\theta_1)}{\prod_{i=1}^n f(x_i|\theta_0)} \geq k$$

for points in C and

$$\frac{\prod_{i=1}^n f(x_i|\theta_1)}{\prod_{i=1}^n f(x_i|\theta_0)} \leq k$$

for points not in C, then C is a best critical region of size α.

References
Hoel, P. G.; Port, S. C.; and Stone, C. J. "Testing Hypotheses." Ch. 3 in *Introduction to Statistical Theory.* New York: Houghton Mifflin, pp. 56–67, 1971.

Nicholson's Formula

Let $J_\nu(z)$ be a BESSEL FUNCTION OF THE FIRST KIND, $Y_\nu(z)$ a BESSEL FUNCTION OF THE SECOND KIND, and $K_\nu(z)$ a MODIFIED BESSEL FUNCTION OF THE FIRST KIND. Also let $\Re[z] > 0$. Then

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cos(2\nu t)\,dt.$$

see also DIXON-FERRAR FORMULA, WATSON'S FORMULA

References
Gradshteyn, I. S. and Ryzhik, I. M. Eqn. 6.664.4 in *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 727, 1979.
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1476, 1980.

Nicomachus's Theorem

The nth CUBIC NUMBER n^3 is a sum of n consecutive ODD NUMBERS, for example

$$1^3 = 1$$
$$2^3 = 3 + 5$$
$$3^3 = 7 + 9 + 11$$
$$4^3 = 13 + 15 + 17 + 19,$$

etc. This identity follows from

$$\sum_{i=1}^{n} [n(n-1) - 1 + 2i] = n^3.$$

It also follows from this fact that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

see also ODD NUMBER THEOREM

Nicomedes' Conchoid

see CONCHOID OF NICOMEDES

Nielsen-Ramanujan Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

N. Nielsen (1909) and Ramanujan (Berndt 1985) considered the integrals

$$a_k = \int_1^2 \frac{(\ln x)^k}{x-1}\,dx. \tag{1}$$

They found the values for $k = 1$ and 2. The general constants for $k > 3$ were found by V. Adamchik (Finch)

$$a_p = p!\zeta(p+1) - \frac{p(\ln 2)^{p+1}}{p+1} - p!\sum_{k=0}^{p-1} \frac{\mathrm{Li}_{p+1-k}(\frac{1}{2})(\ln 2)^k}{k!}, \tag{2}$$

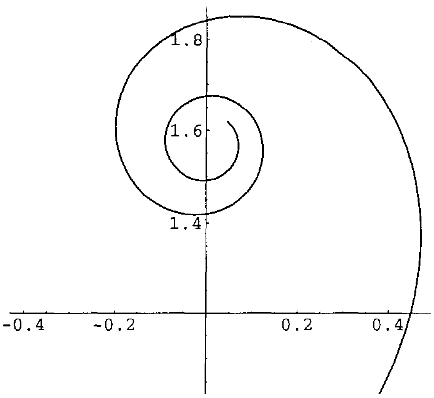
where $\zeta(z)$ is the RIEMANN ZETA FUNCTION and $\mathrm{Li}_n(x)$ is the POLYLOGARITHM. The first few values are

$$a_1 = \tfrac{1}{2}\zeta(2) = \tfrac{1}{12}\pi^2 \tag{3}$$
$$a_2 = \tfrac{1}{4}\zeta(3) \tag{4}$$
$$a_3 = \tfrac{1}{15}\pi^4 + \tfrac{1}{4}\pi^2(\ln 2)^2 - \tfrac{1}{4}(\ln 2)^4 - 6\,\mathrm{Li}_4(\tfrac{1}{2}) - \tfrac{21}{4}\ln 2\zeta(3) \tag{5}$$
$$a_4 = \tfrac{2}{3}\pi^2(\ln 2)^3 - \tfrac{4}{5}(\ln 2)^5 - 24\ln 2\,\mathrm{Li}_4(\tfrac{1}{2}) - 24\,\mathrm{Li}_5(\tfrac{1}{2}) - \tfrac{21}{2}(\ln 2)^2\zeta(3) + 24\zeta(5). \tag{6}$$

see also POLYLOGARITHM, RIEMANN ZETA FUNCTION

References
Berndt, B. C. *Ramanujan's Notebooks, Part I.* New York: Springer-Verlag, 1985.
Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/nielram/nielram.html.
Flajolet, P. and Salvy, B. "Euler Sums and Contour Integral Representation." Submitted to *Experim. Math* 1997. http://pauillac.inria.fr/algo/flajolet/Publications/publist.html.

Nielsen's Spiral



The SPIRAL with parametric equations

$$x(t) = a\,\mathrm{ci}(t) \tag{1}$$
$$y(t) = a\,\mathrm{si}(t), \tag{2}$$

where $\mathrm{ci}(t)$ is the COSINE INTEGRAL and $\mathrm{si}(t)$ is the SINE INTEGRAL. The CESÀRO EQUATION is

$$\kappa = \frac{e^{s/a}}{a}. \tag{3}$$

see also CORNU SPIRAL, COSINE INTEGRAL, SINE INTEGRAL

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 119, 1993.

Nil Geometry

The GEOMETRY of the LIE GROUP consisting of REAL MATRICES of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix},$$

i.e., the HEISENBERG GROUP.

see also HEISENBERG GROUP, LIE GROUP, THURSTON'S GEOMETRIZATION CONJECTURE

Nilmanifold

Let N be a NILPOTENT, connected, SIMPLY CONNECTED LIE GROUP, and let D be a discrete SUBGROUP of N with compact right QUOTIENT SPACE. Then N/D is called a nilmanifold.

Nilpotent Element

An element B of a RING is nilpotent if there exists a POSITIVE INTEGER k for which $B^k = 0$.

see also ENGEL'S THEOREM

Nilpotent Group

A GROUP G for which the chain of groups

$$I = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_n$$

with Z_{k+1}/Z_k (equal to the CENTER of G/Z_k) terminates finitely with $G = Z_n$ is called a nilpotent group.

see also CENTER (GROUP), NILPOTENT LIE GROUP

Nilpotent Lie Group

A LIE GROUP which has a simply connected covering group HOMEOMORPHIC to $\mathbb{R}^n$. The prototype is any connected closed subgroup of upper triangular COMPLEX matrices with 1s on the diagonal. The HEISENBERG GROUP is such a group.

References
Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Nilpotent Matrix

A SQUARE MATRIX whose EIGENVALUES are all 0. A related definition is a SQUARE MATRIX M such that M^n is $\mathsf{0}$ for some POSITIVE integral POWER.

see also EIGENVALUE, SQUARE MATRIX

Nim

A game, also called TACTIX, which is played by the following rules. Given one or more piles (NIM-HEAPS), players alternate by taking all or some of the counters in a single heap. The player taking the last counter or stack of counters is the winner. Nim-like games are also called TAKE-AWAY GAMES and DISJUNCTIVE GAMES.

If optimal strategies are used, the winner can be determined from any intermediate position by its associated NIM-VALUE.

see also MISÈRE FORM, NIM-VALUE, WYTHOFF'S GAME

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 36–38, 1987.
Bogomolny, A. "The Game of Nim." `http://www.cut-the-knot.com/bottom_nim.html`.
Bouton, C. L. "Nim, A Game with a Complete Mathematical Theory." *Ann. Math. Princeton* **3**, 35–39, 1901–1902.
Gardner, M. "Nim and Hackenbush." Ch. 14 in *Wheels, Life, and other Mathematical Amusements.* New York: W. H. Freeman, 1983.
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Oxford University Press, pp. 117–120, 1990.
Kraitchik, M. "Nim." §3.12.2 in *Mathematical Recreations.* New York: W. W. Norton, pp. 86–88, 1942.

Nim-Heap

A pile of counters in a game of NIM.

Nim-Sum

see NIM-VALUE

Nim-Value

Every position of every IMPARTIAL GAME has a nim-value, making it equivalent to a NIM-HEAP. To find the nim-value (also called the SPRAGUE-GRUNDY NUMBER), take the MEX of the nim-values of the possible moves. The nim-value can also be found by writing the number of counters in each heap in binary, adding without carrying, and replacing the digits with their values mod 2. If the nim-value is 0, the position is SAFE; otherwise, it is UNSAFE. With two heaps, safe positions are (x, x) where $x \in [1, 7]$. With three heaps, (1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), and (3, 4, 7).

see also GRUNDY'S GAME, IMPARTIAL GAME, MEX, NIM, SAFE, UNSAFE

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 36–38, 1987.
Grundy, P. M. "Mathematics and Games." *Eureka* **2**, 6–8, 1939.
Sprague, R. "Über mathematische Kampfspiele." *Tôhoku J. Math.* **41**, 438–444, 1936.

Nine-Point Center

The center F (or N) of the NINE-POINT CIRCLE. It has TRIANGLE CENTER FUNCTION

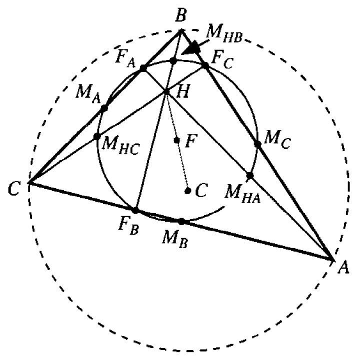
$$\begin{aligned}\alpha &= \cos(B - C) = \cos A + 2\cos B\cos C \\ &= bc[a^2b^2 + a^2c^2 + (b^2 - c^2)^2],\end{aligned}$$

and is the MIDPOINT of the line between the CIRCUMCENTER C and ORTHOCENTER H. It lies on the EULER LINE.

see also EULER LINE, NINE-POINT CIRCLE, NINE-POINT CONIC

References
Carr, G. S. *Formulas and Theorems in Pure Mathematics, 2nd ed.* New York: Chelsea, p. 624, 1970.
Dixon, R. *Mathographics.* New York: Dover, pp. 57–58, 1991.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. "Nine-Point Center." `http://www.evansville.edu/~ck6/tcenters/class/npcenter.html`.

Nine-Point Circle



The CIRCLE, also called EULER'S CIRCLE and the FEUERBACH CIRCLE, which passes through the feet of the PERPENDICULAR F_A, F_B, and F_C dropped from the VERTICES of any TRIANGLE ΔABC on the sides opposite them. Euler showed in 1765 that it also passes through the MIDPOINTS M_A, M_B, M_C of the sides of ΔABC.

By FEUERBACH'S THEOREM, the nine-point circle also passes through the MIDPOINTS M_{HA}, M_{HB}, M_{HC} of the segments which join the VERTICES and the ORTHOCENTER H. These three triples of points make nine in all, giving the circle its name. The center F of the nine-point circle is called the NINE-POINT CENTER.

The RADIUS of the nine-point circle is $R/2$, where R is the CIRCUMRADIUS. The center of KIEPERT'S HYPERBOLA lies on the nine-point circle. The nine-point circle bisects any line from the ORTHOCENTER to a point on the CIRCUMCIRCLE. The nine-point circle of the INCENTER and EXCENTERS of a TRIANGLE is the CIRCUMCIRCLE.

The sum of the powers of the VERTICES with regard to the nine-point circle is

$$\tfrac{1}{4}({a_1}^2 + {a_2}^2 + {a_3}^2).$$

Also,

$$\overline{FA_1}^2 + \overline{FA_2}^2 + \overline{FA_3}^2 + \overline{FH}^2 = 3R^2,$$

where F is the NINE-POINT CENTER, A_i are the VERTICES, H is the ORTHOCENTER, and R is the CIRCUMRADIUS. All triangles inscribed in a given CIRCLE and having the same ORTHOCENTER have the same nine-point circle.

see also COMPLETE QUADRILATERAL, EIGHT-POINT CIRCLE THEOREM, FEUERBACH'S THEOREM, FONTENÉ THEOREMS, GRIFFITHS' THEOREM, NINE-POINT CENTER, NINE-POINT CONIC, ORTHOCENTRIC SYSTEM

References

Altshiller-Court, N. *College Geometry: A Second Course in Plane Geometry for Colleges and Normal Schools, 2nd ed., rev. enl.* New York: Barnes and Noble, pp. 93–97, 1952.
Brand, L. "The Eight-Point Circle and the Nine-Point Circle." *Amer. Math. Monthly* **51**, 84–85, 1944.
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* New York: Random House, pp. 20–22, 1967.
Dörrie, H. "The Feuerbach Circle." §28 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 142–144, 1965.
Gardner, M. *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage Books, p. 59, 1977.
Guggenbuhl, L. "Karl Wilhelm Feuerbach, Mathematician." Appendix to *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., pp. 89–100, 1995.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 165 and 195–212, 1929.
Lange, J. *Geschichte des Feuerbach'schen Kreises.* Berlin, 1894.
Mackay, J. S. "History of the Nine-Point Circle." *Proc. Edinburgh Math. Soc.* **11**, 19–61, 1892.
Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 119–120, 1990.
Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., pp. 1–4, 1995.

Nine-Point Conic

A CONIC SECTION on which the MIDPOINTS of the sides of any COMPLETE QUADRANGLE lie. The three diagonal points also lie on this conic.

see also COMPLETE QUADRANGLE, CONIC SECTION, NINE-POINT CIRCLE

Nint

see NEAREST INTEGER FUNCTION

Nint Zeta Function

Let

$$S_N(s) = \sum_{n=1}^{\infty} [(n^{1/N})]^{-s}, \tag{1}$$

where $[x]$ denotes NINT, the INTEGER closest to x. For $s > 3$,

$$S_2(s) = 2\zeta(s-1) \tag{2}$$
$$S_3(s) = 3\zeta(s-2) + 4^{-s}\zeta(s) \tag{3}$$
$$S_4(s) = 4\zeta(s-3) + \zeta(s-1). \tag{4}$$

$S_N(n)$ is a POLYNOMIAL in π whose COEFFICIENTS are ALGEBRAIC NUMBERS whenever $n - N$ is ODD. The first few values are given explicitly by

$$S_3(4) = \frac{\pi^2}{2} + \frac{\pi^4}{23046} \tag{5}$$

$$S_5(6) = \frac{5\pi^2}{6} + \frac{\pi^4}{36} + \frac{\pi^6}{4^{12}} \times \left(\frac{1}{945} - \frac{170912 + 49928\sqrt{2}}{25}\sqrt{1 - \sqrt{\frac{1}{2}}}\right) \tag{6}$$

$$S_6(7) = \pi^2 + \frac{\pi^4}{18} + \frac{\pi^6}{2520} + \frac{246013 + 353664\sqrt{2}}{45}\frac{\pi^7}{2^{27}}. \tag{7}$$

References

Borwein, J. M.; Hsu, L. C.; Mabry, R.; Neu, K.; Roppert, J.; Tyler, D. B.; and de Weger, B. M. M. "Nearest Integer Zeta-Functions." *Amer. Math. Monthly* **101**, 579–580, 1994.

Nirenberg's Conjecture

If the GAUSS MAP of a complete minimal surface omits a NEIGHBORHOOD of the SPHERE, then the surface is a PLANE. This was proven by Osserman (1959). Xavier (1981) subsequently generalized the result as follows. If the GAUSS MAP of a complete MINIMAL SURFACE omits ≥ 7 points, then the surface is a PLANE.

see also GAUSS MAP, MINIMAL SURFACE, NEIGHBORHOOD

References

do Carmo, M. P. *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 42, 1986.
Osserman, R. "Proof of a Conjecture of Nirenberg." *Comm. Pure Appl. Math.* **12**, 229–232, 1959.
Xavier, F. "The Gauss Map of a Complete Nonflat Minimal Surface Cannot Omit 7 Points on the Sphere." *Ann. Math.* **113**, 211–214, 1981.

Niven's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Given a POSITIVE INTEGER $m > 1$, let its PRIME FACTORIZATION be written

$$m = {p_1}^{a_1}{p_2}^{a_2}{p_3}^{a_3}\cdots{p_k}^{a_k}. \tag{1}$$

Define the functions h and H by $h(1) = 1$, $H(1) = 1$, and

$$h(m) = \min(a_1, a_2, \ldots, a_k) \tag{2}$$
$$H(m) = \max(a_1, a_2, \ldots, a_k). \tag{3}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} h(m) = 1 \tag{4}$$

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} h(m) - n}{\sqrt{n}} = \frac{\zeta(\frac{3}{2})}{\zeta(3)}, \tag{5}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION (Niven 1969). Niven (1969) also proved that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} H(m) = C, \tag{6}$$

where

$$C = 1 + \left\{ \sum_{j=2}^{\infty} \left[1 - \frac{1}{\zeta(j)} \right] \right\} = 1.705221\ldots \tag{7}$$

(Sloane's A033150).

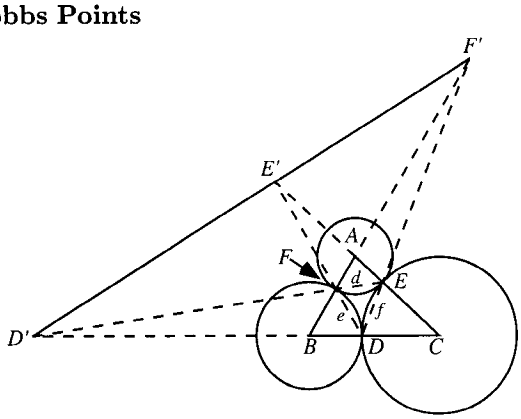
The CONTINUED FRACTION of Niven's constant is 1, 1, 2, 2, 1, 1, 4, 1, 1, 3, 4, 4, 8, 4, 1, ... (Sloane's A033151). The positions at which the digits 1, 2, ... first occur in the CONTINUED FRACTION are 1, 3, 10, 7, 47, 41, 34, 13, 140, 252, 20, ... (Sloane's A033152). The sequence of largest terms in the CONTINUED FRACTION is 1, 2, 4, 8, 11, 14, 29, 372, 559, ... (Sloane's A0033153), which occur at positions 1, 3, 7, 13, 20, 35, 51, 68, 96, ... (Sloane's A033154).

References
Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/niven/niven.html.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 41, 1983.
Niven, I. "Averages of Exponents in Factoring Integers." *Proc. Amer. Math. Soc.* **22**, 356–360, 1969.
Plouffe, S. "The Niven Constant." http://www.lacim.uqam.ca/piDATA/niven.txt.

Niven Number

see HARSHAD NUMBER

Nobbs Points



Given a TRIANGLE ΔABC, construct the CONTACT TRIANGLE ΔDEF. Then the Nobbs points are the three points D', E', and F' from which ΔABC and ΔDEF are PERSPECTIVE, as illustrated above. The Nobbs points are COLLINEAR and fall along the GERGONNE LINE.

see also COLLINEAR, CONTACT TRIANGLE, EVANS POINT, FLETCHER POINT, GERGONNE LINE, PERSPECTIVE TRIANGLES

References
Oldknow, A. "The Euler-Gergonne-Soddy Triangle of a Triangle." *Amer. Math. Monthly* **103**, 319–329, 1996.

Noble Number

A noble number is defined as an IRRATIONAL NUMBER which has a CONTINUED FRACTION which becomes an infinite sequence of 1s at some point,

$$\nu \equiv [a_1, a_2, \ldots, a_n, \bar{1}].$$

The prototype is the GOLDEN RATIO ϕ whose CONTINUED FRACTION is composed *entirely* of 1s, $[\bar{1}]$. Any noble number can written as

$$\nu = \frac{A_n + \phi A_{n-1}}{B_n + \phi B_{n+1}},$$

where A_k and B_k are the NUMERATOR and DENOMINATOR of the kth CONVERGENT of $[a_1, a_2, \ldots, a_n]$. The noble numbers are a SUBFIELD of $\mathbb{Q}(\sqrt{5})$.

see also NEAR NOBLE NUMBER

References
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, p. 236, 1979.
Schroeder, M. "Noble and Near Noble Numbers." In *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, pp. 392–394, 1991.

Node (Algebraic Curve)

see ORDINARY DOUBLE POINT

Node (Fixed Point)

A FIXED POINT for which the STABILITY MATRIX has both EIGENVALUES of the same sign (i.e., both are POSITIVE or both are NEGATIVE). If $\lambda_1 < \lambda_2 < 0$, then the node is called STABLE; if $\lambda_1 > \lambda_2 > 0$, then the node is called an UNSTABLE NODE.

see also STABLE NODE, UNSTABLE NODE

Node (Graph)

Synonym for the VERTICES of a GRAPH, i.e., the points connected by EDGES.

see also ACNODE, CRUNODE, TACNODE

Noether's Fundamental Theorem

If two curves ϕ and ψ of MULTIPLICITIES $r_i \neq 0$ and $s_i \neq 0$ have only ordinary points or ordinary singular points and CUSPS in common, then every curve which has at least MULTIPLICITY

$$r_i + s_i - 1$$

at every point (distinct or infinitely near) can be written

$$f \equiv \phi\psi' + \psi\phi' = 0,$$

where the curves ϕ' and ψ' have MULTIPLICITIES at least $r_i - 1$ and $s_i - 1$.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 29–30, 1959.

Noether-Lasker Theorem

Let M be a finitely generated MODULE over a commutative NOETHERIAN RING R. Then there exists a finite set $\{N_i | 1 \leq i \leq l\}$ of submodules of M such that

1. $\cap_{i=1}^{l} N_i = 0$ and $\cap_{i \neq i_0} N_i$ is not contained in N_{i_0} for all $1 \leq i_0 \leq l$.
2. Each quotient M/N_i is primary for some prime P_i.
3. The P_i are all distinct for $1 \leq i \leq l$.
4. Uniqueness of the primary component N_i is equivalent to the statement that P_i does not contain P_j for any $j \neq i$.

Noether's Transformation Theorem

Any irreducible curve may be carried by a factorable CREMONA TRANSFORMATION into one with none but ordinary singular points.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 207, 1959.

Noetherian Module

A MODULE M is Noetherian if every submodule is finitely generated.

see also NOETHERIAN RING

Noetherian Ring

An abstract commutative RING satisfying the abstract chain condition.

see also LOCAL RING, NOETHER-LASKER THEOREM

Noise

An error which is superimposed on top of a true signal. Noise may be random or systematic. Noise can be greatly reduced by transmitting signals digitally instead of in analog form because each piece of information is allowed only discrete values which are spaced farther apart than the contribution due to noise.

CODING THEORY studies how to encode information efficiently, and ERROR-CORRECTING CODES devise methods for transmitting and reconstructing information in the presence of noise.

see also ERROR

References
Davenport, W. B. and Root, W. L. *An Introduction to the Theory of Random Signals and Noise.* New York: IEEE Press, 1987.
McDonough, R. N. and Whalen, A. D. *Detection of Signals in Noise, 2nd ed.* Orlando, FL: Academic Press, 1995.
Pierce, J. R. *Symbols, Signals and Noise: The Nature and Process of Communication.* New York: Harper & Row, 1961.
Vainshtein, L. A. and Zubakov, V. D. *Extraction of Signals from Noise.* New York: Dover, 1970.
van der Ziel, A. *Noise: Sources, Characterization, Measurement.* New York: Prentice-Hall, 1954.
van der Ziel, A. *Noise in Measurement.* New York: Wiley, 1976.
Wax, N. *Selected Papers on Noise and Stochastic Processes.* New York: Dover, 1954.

Noise Sphere

A mapping of RANDOM NUMBER TRIPLES to points in SPHERICAL COORDINATES,

$$\theta = 2\pi X_n$$
$$\phi = \pi X_{n+1}$$
$$r = \sqrt{X_{n+2}}.$$

The graphical result can yield unexpected structure which indicates correlations between triples and therefore that the numbers are not truly RANDOM.

References
Pickover, C. A. *Computers and the Imagination.* New York: St. Martin's Press, 1991.
Pickover, C. A. "Computers, Randomness, Mind, and Infinity." Ch. 31 in *Keys to Infinity.* New York: W. H. Freeman, pp. 233–247, 1995.
Richards, T. "Graphical Representation of Pseudorandom Sequences." *Computers and Graphics* **13**, 261–262, 1989.

Nolid

An assemblage of faces forming a POLYHEDRON of zero VOLUME (Holden 1991, p. 124).

see also ACOPTIC POLYHEDRON

References
Holden, A. *Shapes, Space, and Symmetry.* New York: Dover, 1991.

Nome

Given a THETA FUNCTION, the nome is defined as

$$q(m) \equiv e^{\pi\tau i} = e^{-\pi K(1-m)/K(m)} \equiv e^{-\pi K'(m)/K(m)}, \quad (1)$$

where $K(k)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND, and m is the PARAMETER.

$$\vartheta_i(z,q) \equiv \vartheta(z|\tau) \quad (2)$$

$$\vartheta_i \equiv \vartheta(0,q). \quad (3)$$

Solving the nome for the PARAMETER m gives

$$m(q) = \frac{\vartheta_2{}^4(0,q)}{\vartheta_3{}^4(0,q)}, \tag{4}$$

where $\vartheta_i(z,q)$ is a THETA FUNCTION.

see also AMPLITUDE, CHARACTERISTIC (ELLIPTIC INTEGRAL), ELLIPTIC INTEGRAL, MODULAR ANGLE, MODULUS (ELLIPTIC INTEGRAL), PARAMETER

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 591, 1972.

Nomogram

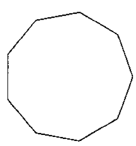
A graphical plot which can be used for solving certain types of equations.

References

Iyanaga, S. and Kawada, Y. (Eds.). "Nomograms." §282 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 891–893, 1980.

Menzel, D. (Ed.). *Fundamental Formulas of Physics, Vol. 1.* New York: Dover, p. 141, 1960.

Nonagon



The unconstructible regular POLYGON with nine sides and SCHLÄFLI SYMBOL $\{9\}$. It is sometimes called an ENNEAGON.

Although the regular nonagon is not a CONSTRUCTIBLE POLYGON, Dixon (1991) gives several close approximations. While the ANGLE subtended by a side is $360°/9 = 40°$, Dixon gives constructions containing angles of $\tan^{-1}(5/6) \approx 39.8805571°$ and $2\tan^{-1}((\sqrt{3}-1)/2) \approx 40.207818°$.

Madachy (1979) illustrates how to construct a nonagon by folding and knotting a strip of paper.

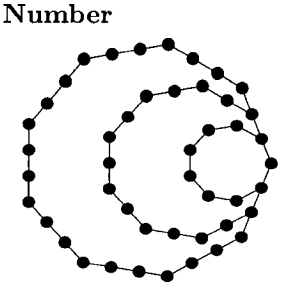
see also NONAGRAM, TRIGONOMETRY VALUES—$\pi/9$

References

Dixon, R. *Mathographics.* New York: Dover, pp. 40–44, 1991.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 60–61, 1979.

Nonagonal Number

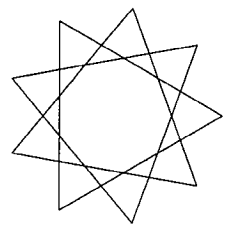


A FIGURATE NUMBER of the form $n(7n-5)/2$, also called an ENNEAGONAL NUMBER. The first few are 1, 9, 24, 46, 75, 111, 154, 204, ... (Sloane's A001106).

References

Sloane, N. J. A. Sequence A001106/M4604 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Nonagram



A STAR POLYGON composed of three EQUILATERAL TRIANGLES rotated at angles 0°, 40°, and 80°. It has been called the STAR OF GOLIATH by analogy with the STAR OF DAVID (HEXAGRAM).

see also HEXAGRAM, NONAGON, TRIGONOMETRY VALUES—$\pi/9$

Nonassociative Algebra

An ALGEBRA which does not satisfy

$$a(bc) = (ab)c$$

is called a nonassociative algebra. Bott and Milnor (1958) proved that the only nonassociative DIVISION ALGEBRAS are for $n = 1$, 2, 4, and 8. Each gives rise to an ALGEBRA with particularly useful physical applications (which, however, is not itself necessarily nonassociative), and these four cases correspond to REAL NUMBERS, COMPLEX NUMBERS, QUATERNIONS, and CAYLEY NUMBERS, respectively.

see also ALGEBRA, CAYLEY NUMBER, COMPLEX NUMBER, DIVISION ALGEBRA, QUATERNION, REAL NUMBER

References

Bott, R. and Milnor, J. "On the Parallelizability of the Spheres." *Bull. Amer. Math. Soc.* **64**, 87–89, 1958.

Nonassociative Product

The number of nonassociative n-products with k elements preceding the rightmost left parameter is

$$\begin{aligned} F(n,k) &= F(n-1,k) + F(n-1,k-1) \\ &= \binom{n+k-2}{k} - \binom{n+k-1}{k-1}, \end{aligned}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. The number of n-products in a nonassociative algebra is

$$F(n) = \sum_{j=0}^{n-2} F(n,j) = \frac{(2n-2)!}{n!(n-1)!}.$$

References
Niven, I. M. *Mathematics of Choice: Or, How to Count Without Counting.* Washington, DC: Math. Assoc. Amer., pp. 140–152, 1965.

Nonaveraging Sequence

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

An infinite sequence of POSITIVE INTEGERS

$$1 \leq a_1 < a_2 < a_3 < \ldots$$

is a nonaveraging sequence if it contains no three terms which are in an ARITHMETIC PROGRESSION, so that

$$a_i + a_j \neq 2a_k$$

for all distinct a_i, a_j, a_k. Wróblewski (1984) showed that

$$S(A) \equiv \sup_{\substack{\text{all nonaveraging}\\ \text{sequences}}} \sum_{k=1}^{\infty} \frac{1}{a_k} > 3.00849.$$

References
Behrend, F. "On Sets of Integers which Contain no Three Terms in an Arithmetic Progression." *Proc. Nat. Acad. Sci. USA* **32**, 331–332, 1946.
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/erdos/erdos.html`.
Gerver, J. L. "The Sum of the Reciprocals of a Set of Integers with No Arithmetic Progression of k Terms." *Proc. Amer. Math. Soc.* **62**, 211–214, 1977.
Gerver, J. L. and Ramsey, L. "Sets of Integers with no Long Arithmetic Progressions Generated by the Greedy Algorithm." *Math. Comput.* **33**, 1353–1360, 1979.
Guy, R. K. "Nonaveraging Sets. Nondividing Sets." §C16 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 131–132, 1994.
Wróblewski, J. "A Nonaveraging Set of Integers with a Large Sum of Reciprocals." *Math. Comput.* **43**, 261–262, 1984.

Noncentral Distribution

see CHI-SQUARED DISTRIBUTION, F-DISTRIBUTION, STUDENT'S t-DISTRIBUTION

Noncommutative Group

A group whose elements do not commute. The simplest noncommutative GROUP is the DIHEDRAL GROUP D_3 of ORDER six.

see also COMMUTATIVE, FINITE GROUP—D_3

Nonconformal Mapping

Let γ be a path in $\mathbb{C}$, $w = f(z)$, and θ and ϕ be the tangents to the curves γ and $f(\gamma)$ at z_0 and w_0. If there is an N such that

$$f^{(N)}(z_0) \neq 0 \quad (1)$$

$$f^{(n)}(z_0) = 0 \quad (2)$$

for all $n < N$ (or, equivalently, if $f'(z)$ has a zero of order $N-1$), then

$$f(z) = f(z_0) + \frac{f^{(N)}(z_0)}{N!}(z-z_0)^N + \frac{f^{(N+1)}(z_0)}{(N+1)!}(z-z_0)^{N+1} + \ldots \quad (3)$$

$$f(z) - f(z_0) = (z-z_0)^N \left[\frac{f(N)(z_0)}{N!} + \frac{f^{(N+1)}(z_0)}{(N+1)!}(z-z_0) + \ldots\right], \quad (4)$$

so the ARGUMENT is

$$\arg[f(z) - f(z_0)] = N \arg(z-z_0) + \arg\left[\frac{f(N)(z_0)}{N!} + \frac{f^{(N+1)}(z_0)}{(N+1)!}(z-z_0) + \ldots\right]. \quad (5)$$

As $z \to z_0$, $\arg(z-z_0) \to \theta$ and $|\arg[f(z)-f(z_0)]| \to \phi$,

$$\phi = N\theta + \arg\left[\frac{f(N)(z_0)}{N!}\right] = N\theta + \arg[f(N)(z_0)]. \quad (6)$$

see also CONFORMAL TRANSFORMATION

Nonconstructive Proof

A PROOF which indirectly shows a mathematical object exists without providing a specific example or algorithm for producing an example.

see also PROOF

References
Courant, R. and Robbins, H. "The Indirect Method of Proof." §2.4.4 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 86–87, 1996.

Noncototient

A POSITIVE value of n for which $x - \phi(x) = n$ has no solution, where $\phi(x)$ is the TOTIENT FUNCTION. The first few are 10, 26, 34, 50, 52, ... (Sloane's A005278).

see also NONTOTIENT, TOTIENT FUNCTION

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 91, 1994.
Sloane, N. J. A. Sequence A005278/M4688 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Noncylindrical Ruled Surface

A RULED SURFACE parameterization $\mathbf{x}(u,v) = \mathbf{b}(u) + v\mathbf{g}(u)$ is called noncylindrical if $\mathbf{g} \times \mathbf{g}'$ is nowhere $\mathbf{0}$. A noncylindrical ruled surface always has a parameterization of the form

$$\mathbf{x}(u,v) = \boldsymbol{\sigma}(u) + v\boldsymbol{\delta}(u),$$

where $|\boldsymbol{\delta}| = 1$ and $\boldsymbol{\sigma}' \cdot \boldsymbol{\delta}' = 0$, where $\boldsymbol{\sigma}$ is called the STRICTION CURVE of $\mathbf{x}$ and $\boldsymbol{\delta}$ the DIRECTOR CURVE.

see also DISTRIBUTION PARAMETER, RULED SURFACE, STRICTION CURVE

References

Gray, A. "Noncylindrical Ruled Surfaces." §17.3 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 345–349, 1993.

Nondecreasing Function

A function $f(x)$ is said to be nondecreasing on an INTERVAL I if $f(b) \geq f(a)$ for all $b > a$, where $a, b \in I$. Conversely, a function $f(x)$ is said to be nonincreasing on an INTERVAL I if $f(b) \leq f(a)$ for all $b > a$ with $a, b \in I$.

see also DECREASING FUNCTION, NONINCREASING FUNCTION

Nondividing Set

A SET in which no element divides the SUM of any other.

References

Guy, R. K. "Nonaveraging Sets. Nondividing Sets." §C16 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 131–132, 1994.

Nonessential Singularity

see REGULAR SINGULAR POINT

Non-Euclidean Geometry

In 3 dimensions, there are three classes of constant curvature GEOMETRIES. All are based on the first four of EUCLID'S POSTULATES, but each uses its own version of the PARALLEL POSTULATE. The "flat" geometry of everyday intuition is called EUCLIDEAN GEOMETRY (or PARABOLIC GEOMETRY), and the non-Euclidean geometries are called HYPERBOLIC GEOMETRY (or LOBACHEVSKY-BOLYAI-GAUSS GEOMETRY) and ELLIPTIC GEOMETRY (or RIEMANNIAN GEOMETRY). It was not until 1868 that Beltrami proved that non-Euclidean geometries were as logically consistent as EUCLIDEAN GEOMETRY.

see also ABSOLUTE GEOMETRY, ELLIPTIC GEOMETRY, EUCLID'S POSTULATES, EUCLIDEAN GEOMETRY, HYPERBOLIC GEOMETRY, PARALLEL POSTULATE

References

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Carslaw, H. S. *The Elements of Non-Euclidean Plane Geometry and Trigonometry.* London: Longmans, 1916.

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Iyanaga, S. and Kawada, Y. (Eds.). "Non-Euclidean Geometry." §283 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 893–896, 1980.

Martin, G. E. *The Foundations of Geometry and the Non-Euclidean Plane.* New York: Springer-Verlag, 1975.

Pappas, T. "A Non-Euclidean World." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 90–92, 1989.

Ramsay, A. and Richtmeyer, R. D. *Introduction to Hyperbolic Geometry.* New York: Springer-Verlag, 1995.

Sommerville, D. Y. *The Elements of Non-Euclidean Geometry.* London: Bell, 1914.

Sommerville, D. Y. *Bibliography of Non-Euclidean Geometry, 2nd ed.* New York: Chelsea, 1960.

Sved, M. *Journey into Geometries.* Washington, DC: Math. Assoc. Amer., 1991.

Trudeau, R. J. *The Non-Euclidean Revolution.* Boston, MA: Birkhäuser, 1987.

Nonillion

In the American system, 10^{30}.

see also LARGE NUMBER

Nonincreasing Function

A function $f(x)$ is said to be nonincreasing on an INTERVAL I if $f(b) \leq f(a)$ for all $b > a$, where $a, b \in I$. Conversely, a function $f(x)$ is said to be nondecreasing on an INTERVAL I if $f(b) \geq f(a)$ for all $b > a$ with $a, b \in I$.

see also INCREASING FUNCTION, NONDECREASING FUNCTION

Nonlinear Least Squares Fitting

Given a function $f(x)$ of a variable x tabulated at m values $y_1 = f(x_1)$, ..., $y_m = f(x_m)$, assume the function is of known analytic form depending on n parameters $f(x; \lambda_1, \ldots, \lambda_n)$, and consider the overdetermined set of m equations

$$y_1 = f(x_1; \lambda_1, \lambda_2, \ldots, \lambda_n) \tag{1}$$

$$y_m = f(x_m; \lambda_1, \lambda_2, \ldots, \lambda_n). \tag{2}$$

We desire to solve these equations to obtain the values λ_1, ..., λ_n which best satisfy this system of equations. Pick an initial guess for the λ_i and then define

$$d\beta_i = y_i - f(x_i; \lambda_1, \ldots, \lambda_n). \tag{3}$$

Now obtain a linearized estimate for the changes $d\lambda_i$ needed to reduce $d\beta_i$ to 0,

$$d\beta_i = \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda_j}\, d\lambda_j \bigg|_{x_j, \lambda} \tag{4}$$

for $i = 1, \ldots, n$. This can be written in component form as

$$d\beta_i = A_{ij}\, d\lambda_i, \tag{5}$$

where A is the $m \times n$ MATRIX

$$A_{ij} = \begin{bmatrix} \left.\frac{\partial f}{\partial \lambda_1}\right|_{x_1,\lambda} & \left.\frac{\partial f}{\partial \lambda_n}\right|_{x_1,\lambda} & \cdots \\ \left.\frac{\partial f}{\partial \lambda_2}\right|_{x_2,\lambda} & \left.\frac{\partial f}{\partial \lambda_2}\right|_{x_2,\lambda} & \cdots \\ \vdots & \vdots & \ddots \\ \left.\frac{\partial f}{\partial \lambda_1}\right|_{x_m,\lambda} & \left.\frac{\partial f}{\partial \lambda_n}\right|_{x_m,\lambda} & \cdots \end{bmatrix}. \tag{6}$$

In more concise MATRIX form,

$$d\boldsymbol{\beta} = \mathsf{A}\, d\lambda, \tag{7}$$

where $d\boldsymbol{\beta}$ and $d\lambda$ are m-VECTORS. Applying the MATRIX TRANSPOSE of A to both sides gives

$$\mathsf{A}^{\mathrm{T}}\, d\boldsymbol{\beta} = (\mathsf{A}^{\mathrm{T}}\mathsf{A})\, d\lambda. \tag{8}$$

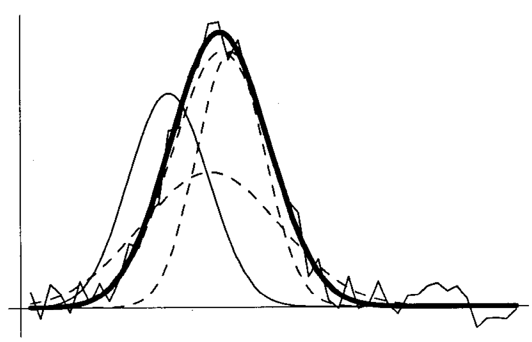
Defining

$$\mathsf{a} \equiv \mathsf{A}^{\mathrm{T}}\mathsf{A} \tag{9}$$
$$\mathbf{b} \equiv \mathsf{A}^{\mathrm{T}}\, d\boldsymbol{\beta} \tag{10}$$

in terms of the known quantities A and $d\boldsymbol{\beta}$ then gives the MATRIX EQUATION

$$\mathsf{a} d\lambda = \mathbf{b}, \tag{11}$$

which can be solved for $d\lambda$ using standard matrix techniques such as GAUSSIAN ELIMINATION. This offset is then applied to λ and a new $d\beta$ is calculated. By iteratively applying this procedure until the elements of $d\lambda$ become smaller than some prescribed limit, a solution is obtained. Note that the procedure may not converge very well for some functions and also that convergence is often greatly improved by picking initial values close to the best-fit value. The sum of square residuals is given by $R^2 = d\boldsymbol{\beta} \cdot d\boldsymbol{\beta}$ after the final iteration.



An example of a nonlinear least squares fit to a noisy GAUSSIAN FUNCTION

$$f(A, x_0, \sigma; x) = Ae^{-(x-x_0)^2/(2\sigma^2)} \tag{12}$$

is shown above, where the thin solid curve is the initial guess, the dotted curves are intermediate iterations, and the heavy solid curve is the fit to which the solution converges. The actual parameters are $(A, x_0, \sigma) = (1, 20, 5)$, the initial guess was (0.8, 15, 4), and the converged values are (1.03105, 20.1369, 4.86022), with $R^2 = 0.148461$. The PARTIAL DERIVATIVES used to construct the matrix A are

$$\frac{\partial f}{\partial A} = e^{-(x-x_0)^2/(2\sigma^2)} \tag{13}$$
$$\frac{\partial f}{\partial x_0} = \frac{A(x-x_0)}{\sigma^2} e^{-(x-x_0)^2/(2\sigma^2)} \tag{14}$$
$$\frac{\partial f}{\partial \sigma} = \frac{A(x-x_0)^2}{\sigma^3} e^{-(x-x_0)^2/(2\sigma^2)}. \tag{15}$$

The technique could obviously be generalized to multiple Gaussians, to include slopes, etc., although the convergence properties generally worsen as the number of free parameters is increased.

An analogous technique can be used to solve an overdetermined set of equations. This problem might, for example, arise when solving for the best-fit EULER ANGLES corresponding to a noisy ROTATION MATRIX, in which case there are three unknown angles, but nine correlated matrix elements. In such a case, write the n *different* functions as $f_i(\lambda_1, \ldots, \lambda_n)$ for $i = 1, \ldots, n$, call their actual values y_i, and define

$$\mathsf{A} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial \lambda_1}\right|_{\lambda_i} & \left.\frac{\partial f_1}{\partial \lambda_2}\right|_{\lambda_i} & \cdots & \left.\frac{\partial f_1}{\partial \lambda_n}\right|_{\lambda_i} \\ \vdots & \vdots & \ddots & \vdots \\ \left.\frac{\partial f_m}{\partial \lambda_1}\right|_{\lambda_i} & \left.\frac{\partial f_m}{\partial \lambda_2}\right|_{\lambda_i} & \cdots & \left.\frac{\partial f_m}{\partial \lambda_n}\right|_{\lambda_i} \end{bmatrix}, \tag{16}$$

and

$$d\boldsymbol{\beta} = \mathbf{y} - f_i(\lambda_1, \ldots, \lambda_n), \tag{17}$$

where λ_i are the numerical values obtained after the ith iteration. Again, set up the equations as

$$\mathsf{A}\, d\lambda = d\boldsymbol{\beta}, \tag{18}$$

and proceed exactly as before.

see also LEAST SQUARES FITTING, LINEAR REGRESSION, MOORE-PENROSE GENERALIZED MATRIX INVERSE

Nonnegative

A quantity which is either 0 (ZERO) or POSITIVE, i.e., ≥ 0.

see also NEGATIVE, NONNEGATIVE INTEGER, NONPOSITIVE, NONZERO, POSITIVE, ZERO

Nonnegative Integer

see $\mathbb{Z}^*$

Nonnegative Partial Sum

The number of sequences with NONNEGATIVE partial sums which can be formed from n 1s and n -1s (Bailey 1996, Buraldi 1992) is given by the CATALAN NUMBERS. Bailey (1996) gives the number of NONNEGATIVE partial sums of n 1s and k -1s $a_1, a_2, \ldots, a_{n+k}$, so that

$$a_1 + a_2 + \ldots + a_i \geq 0 \quad (1)$$

for all $1 \leq i \leq n+k$. The closed form expression is

$$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 1 \quad (2)$$

for $n \geq 0$,

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = n \quad (3)$$

for $n \geq 1$, and

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{(n+1-k)(n+2)(n+3)\cdots(n+k)}{k!}, \quad (4)$$

for $n \geq k \geq 2$. Setting $k = n$ then recovers the CATALAN NUMBERS

$$C_n = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = \frac{1}{n+1}\binom{2n}{n}. \quad (5)$$

see also CATALAN NUMBER

References

Bailey, D. F. "Counting Arrangements of 1's and -1's." *Math. Mag.* **69**, 128–131, 1996.

Buraldi, R. A. *Introductory Combinatorics, 2nd ed.* New York: Elsevier, 1992.

Nonorientable Surface

A surface such as the MÖBIUS STRIP on which there exists a closed path such that the directrix is reversed when moved around this path. The EULER CHARACTERISTIC of a nonorientable surface is ≤ 0. The real PROJECTIVE PLANE is also a nonorientable surface, as are the BOY SURFACE, CROSS-CAP, and ROMAN SURFACE, all of which are homeomorphic to the REAL PROJECTIVE PLANE (Pinkall 1986). There is a general method for constructing nonorientable surfaces which proceeds as follows (Banchoff 1984, Pinkall 1986). Choose three HOMOGENEOUS POLYNOMIALS of POSITIVE EVEN degree and consider the MAP

$$\mathbf{f} = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)) : \mathbb{R}^3 \to \mathbb{R}^3. \quad (1)$$

Then restricting x, y, and z to the surface of a sphere by writing

$$x = \cos\theta \sin\phi \quad (2)$$

$$y = \sin\theta \sin\phi \quad (3)$$

$$z = \cos\phi \quad (4)$$

and restricting θ to $[0, 2\pi)$ and ϕ to $[0, \pi/2]$ defines a map of the REAL PROJECTIVE PLANE to $\mathbb{R}^3$.

In 3-D, there is no unbounded nonorientable surface which does not intersect itself (Kuiper 1961, Pinkall 1986).

see also BOY SURFACE, CROSS-CAP, MÖBIUS STRIP, ORIENTABLE SURFACE, PROJECTIVE PLANE, ROMAN SURFACE

References

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Gray, A. "Nonorientable Surfaces." Ch. 12 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 229–249, 1993.

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Nonpositive

A quantity which is either 0 (ZERO) or NEGATIVE, i.e., ≤ 0.

see also NEGATIVE, NONNEGATIVE, NONZERO, POSITIVE, ZERO

Nonsquarefree

see SQUAREFUL

Nonstandard Analysis

Nonstandard analysis is a branch of mathematical LOGIC which weakens the axioms of usual ANALYSIS to include only the first-order ones. It also introduces HYPERREAL NUMBERS to allow for the existence of "genuine INFINITESIMALS," numbers which are less than 1/2, 1/3, 1/4, 1/5, ..., but greater than 0. Abraham Robinson developed nonstandard analysis in the 1960s. The theory has since been investigated for its own sake and has been applied in areas such as BANACH SPACES, differential equations, probability theory, microeconomic theory, and mathematical physics (Apps).

see also AX-KOCHEN ISOMORPHISM THEOREM, LOGIC, MODEL THEORY

References

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Anderson, R. "Nonstandard Analysis with Applications to Economics." In *Handbook of Mathematical Economics, Vol. 4.* New York: Elsevier, 1991.

Apps, P. "What is Nonstandard Analysis?" `http://www.math.wisc.edu/~apps/nonstandard.html`.

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Lindstrøom, T. "An Invitation to Nonstandard Analysis." In *Nonstandard Analysis and Its Applications* (Ed. N. Cutland). New York: Cambridge University Press, 1988.
Robinson, A. *Non-Standard Analysis.* Princeton, NJ: Princeton University Press, 1996.
Stewart, I. "Non-Standard Analysis." In *From Here to Infinity: A Guide to Today's Mathematics.* Oxford, England: Oxford University Press, pp. 80–81, 1996.

Nontotient

A POSITIVE EVEN value of n for which $\phi(x) = n$, where $\phi(x)$ is the TOTIENT FUNCTION, has no solution. The first few are 14, 26, 34, 38, 50, ... (Sloane's A005277).

see also NONCOTOTIENT, TOTIENT FUNCTION

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 91, 1994.
Sloane, N. J. A. Sequence A005277/M4927 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Nonwandering

A point x in a MANIFOLD M is said to be nonwandering if, for every open NEIGHBORHOOD U of x, it is true that $\phi^{-n}U \cup U \neq \emptyset$ for a MAP ϕ for some $n > 0$. In other words, every point close to x has some iterate under ϕ which is also close to x. The set of all nonwandering points is denoted $\Omega(\phi)$, which is known as the nonwandering set of ϕ.

see also ANOSOV DIFFEOMORPHISM, AXIOM A DIFFEOMORPHISM, SMALE HORSESHOE MAP

Nonzero

A quantity which does not equal ZERO is said to be nonzero. A REAL nonzero number must be either POSITIVE or NEGATIVE, and a COMPLEX nonzero number can have either REAL or IMAGINARY PART nonzero.

see also NEGATIVE, NONNEGATIVE, NONPOSITIVE, POSITIVE, ZERO

Nordstrand's Weird Surface

An attractive CUBIC SURFACE defined by Nordstrand. It is given by the implicit equation

$$\begin{aligned}25[x^3(y+z)+y^3(x+z)+z^3(x+y)]+50(x^2y^2+x^2z^2\\+y^2z^2)-125(x^2yz+y^2xz+z^2xy)+60xyz\\-4(xy+xz+yz)=0.\end{aligned}$$

References

Nordstrand, T. "Weird Cube." `http://www.uib.no/people/nfytn/weirdtxt.htm`.

Norm

Given a n-D VECTOR

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

a VECTOR NORM $||\mathbf{x}||$ is a NONNEGATIVE number satisfying

1. $||x|| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $||\mathbf{x}|| = 0$ IFF $\mathbf{x} = \mathbf{0}$,
2. $||k\mathbf{x}|| = |k|\,||\mathbf{x}||$ for any SCALAR k,
3. $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

The most common norm is the vector L_2-NORM, defined by

$$||\mathbf{x}||_2 = |\mathbf{x}| = \sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2}.$$

Given a SQUARE MATRIX A, a MATRIX NORM $||\mathsf{A}||$ is a NONNEGATIVE number associated with A having the properties

1. $||\mathsf{A}|| > 0$ when $\mathsf{A} \neq 0$ and $||\mathsf{A}|| = 0$ IFF $\mathsf{A} = 0$,
2. $||k\mathsf{A}|| = |k|\,||\mathsf{A}||$ for any SCALAR k,
3. $||\mathsf{A}+\mathsf{B}|| \leq ||\mathsf{A}|| + ||\mathsf{B}||$,
4. $||\mathsf{AB}|| \leq ||\mathsf{A}||\,||\mathsf{B}||$.

see also BOMBIERI NORM, COMPATIBLE, EUCLIDEAN NORM, HILBERT-SCHMIDT NORM, INDUCED NORM, L_1-NORM, L_2-NORM, L_∞-NORM, MATRIX NORM, MAXIMUM ABSOLUTE COLUMN SUM NORM, MAXIMUM ABSOLUTE ROW SUM NORM, NATURAL NORM, NORMALIZED VECTOR, NORMED SPACE, PARALLELOGRAM LAW, POLYNOMIAL NORM, SPECTRAL NORM, SUBORDINATE NORM, VECTOR NORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1114–1125, 1979.

Norm Theorem

If a PRIME number divides a norm but not the bases of the norm, it is itself a norm.

Normal

see NORMAL CURVE, NORMAL DISTRIBUTION, NORMAL DISTRIBUTION FUNCTION, NORMAL EQUATION, NORMAL FORM, NORMAL GROUP, NORMAL MAGIC SQUARE, NORMAL MATRIX, NORMAL NUMBER, NORMAL PLANE, NORMAL SUBGROUP, NORMAL VECTOR

Normal (Algebraically)

see GALOISIAN

Normal Curvature

Let $\mathbf{u_p}$ be a unit TANGENT VECTOR of a REGULAR SURFACE $M \subset \mathbb{R}^3$. Then the normal curvature of M in the direction $\mathbf{u_p}$ is

$$\kappa(\mathbf{u_p}) = S(\mathbf{u_p}) \cdot \mathbf{u_p}, \tag{1}$$

where S is the SHAPE OPERATOR. Let $M \subset \mathbb{R}^3$ be a REGULAR SURFACE, $\mathbf{p} \in M$, $\mathbf{x}$ be an injective REGULAR PATCH of M with $\mathbf{p} = \mathbf{x}(u_0, v_0)$, and

$$\mathbf{v_p} = a\mathbf{x}_u(u_0, v_0) + b\mathbf{x}_v(u_0, v_0), \tag{2}$$

where $\mathbf{v_p} \in M_\mathbf{p}$. Then the normal curvature in the direction $\mathbf{v_p}$ is

$$\kappa(v\mathbf{p}) = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}, \tag{3}$$

where E, F, and G are first FUNDAMENTAL FORMS and e, f, and g second FUNDAMENTAL FORMS.

The MAXIMUM and MINIMUM values of the normal curvature on a REGULAR SURFACE at a point on the surface are called the PRINCIPAL CURVATURES κ_1 and κ_2.

see also CURVATURE, FUNDAMENTAL FORMS, GAUSSIAN CURVATURE, MEAN CURVATURE, PRINCIPAL CURVATURES, SHAPE OPERATOR, TANGENT VECTOR

References

Euler, L. "Récherches sur la coubure des surfaces." *Mém. de l'Acad. des Sciences, Berlin* **16**, 119–143, 1760.

Gray, A. "Normal Curvature." §14.2 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 270–273 and 277, 1993.

Meusnier, J. B. "Mémoire sur la courbure des surfaces." *Mém. des savans étrangers* **10** (lu 1776), 477–510, 1785.

Normal Curve

see GAUSSIAN DISTRIBUTION

Normal Developable

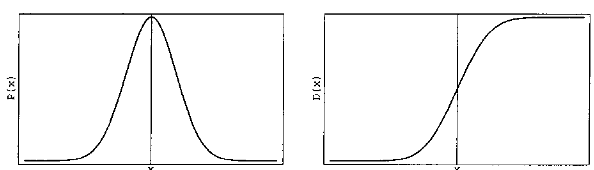
A RULED SURFACE M is a normal developable of a curve $\mathbf{y}$ if M can be parameterized by $\mathbf{x}(u,v) = \mathbf{y}(u) + v\hat{\mathbf{N}}(u)$, where $\mathbf{N}$ is the NORMAL VECTOR.

see also BINORMAL DEVELOPABLE, TANGENT DEVELOPABLE

References

Gray, A. "Developables." §17.6 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 352–354, 1993.

Normal Distribution



Another name for a GAUSSIAN DISTRIBUTION. Given a normal distribution in a VARIATE x with MEAN μ and VARIANCE σ^2,

$$P(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx,$$

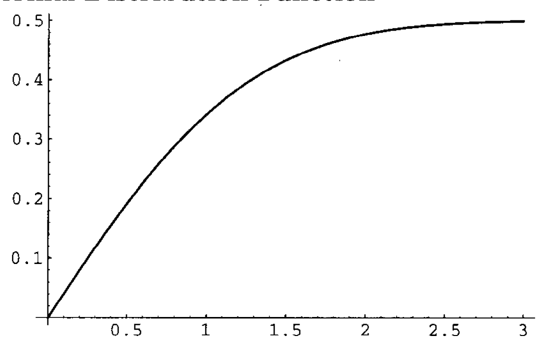
the so-called "STANDARD NORMAL DISTRIBUTION" is given by taking $\mu = 0$ and $\sigma^2 = 1$. An arbitrary normal distribution can be converted to a STANDARD NORMAL DISTRIBUTION by changing variables to $z \equiv (x - \mu)/\sigma$, so $dz = dx/\sigma$, yielding

$$P(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz.$$

The FISHER-BEHRENS PROBLEM is the determination of a test for the equality of MEANS for two normal distributions with different VARIANCES.

see also FISHER-BEHRENS PROBLEM, GAUSSIAN DISTRIBUTION, HALF-NORMAL DISTRIBUTION, KOLMOGOROV-SMIRNOV TEST, NORMAL DISTRIBUTION FUNCTION, STANDARD NORMAL DISTRIBUTION, TETRACHORIC FUNCTION

Normal Distribution Function



A normalized form of the cumulative GAUSSIAN DISTRIBUTION function giving the probability that a variate assumes a value in the range $[0, x]$,

$$\Phi(x) \equiv Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\,dt. \tag{1}$$

It is related to the PROBABILITY INTEGRAL

$$\alpha(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-t^2/2}\,dt \tag{2}$$

by

$$\Phi(x) = \tfrac{1}{2}\alpha(x). \tag{3}$$

Let $u \equiv t/\sqrt{2}$ so $du = dt/\sqrt{2}$. Then

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-u^2}\,du = \tfrac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{4}$$

Here, ERF is a function sometimes called the error function. The probability that a normal variate assumes a value in the range $[x_1, x_2]$ is therefore given by

$$\Phi(x_1, x_2) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x_2}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{x_1}{\sqrt{2}}\right)\right]. \qquad (5)$$

Neither $\Phi(z)$ nor ERF can be expressed in terms of finite additions, subtractions, multiplications, and root extractions, and so must be either computed numerically or otherwise approximated.

Note that a function different from $\Phi(x)$ is sometimes defined as "the" normal distribution function

$$\Phi'(x) \equiv \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = \tfrac{1}{2} + \Phi(x) \qquad (6)$$

(Beyer 1987, p. 551), although this function is less widely encountered than the usual $\Phi(x)$.

The value of a for which $P(x)$ falls within the interval $[-a, a]$ with a given probability P is a related quantity called the CONFIDENCE INTERVAL.

For small values $x \ll 1$, a good approximation to $\Phi(x)$ is obtained from the MACLAURIN SERIES for ERF,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}(2x - \tfrac{1}{3}x^3 + \tfrac{1}{20}x^5 - \tfrac{1}{168}x^7 + \ldots). \qquad (7)$$

For large values $x \gg 1$, a good approximation is obtained from the asymptotic series for ERF,

$$\Phi(x) = \frac{1}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}}(x^{-1} - x^{-3} + 3x^{-5} - 15x^{-7} + 105x^{-9} + \ldots). \qquad (8)$$

The value of $\Phi(x)$ for intermediate x can be computed using the CONTINUED FRACTION identity

$$\int_0^x e^{-u^2}\,du = \frac{\sqrt{\pi}}{2} - \cfrac{\frac{1}{2}e^{-x^2}}{x + \cfrac{1}{2x + \cfrac{2}{x + \cfrac{3}{2x + \cfrac{4}{x + \ldots}}}}}. \qquad (9)$$

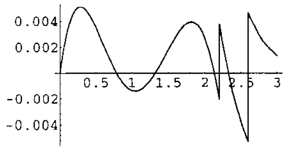
A simple approximation of $\Phi(x)$ which is good to two decimal places is given by

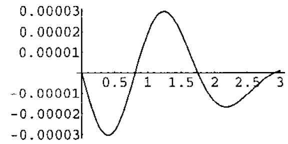
$$\Phi_1(x) \approx \begin{cases} 0.1x(4.4 - x) & \text{for } 0 \leq x \leq 2.2 \\ 0.49 & \text{for } 2.2 < x < 2.6 \\ 0.50 & \text{for } x \geq 2.6. \end{cases} \qquad (10)$$

Abramowitz and Stegun (1972) and Johnson and Kotz (1970) give other functional approximations. An approximation due to Bagby (1995) is

$$\Phi_2(x) = \tfrac{1}{2}\{1 - \tfrac{1}{30}[7e^{-x^2/2} + 16e^{-x^2(2-\sqrt{2})} + (7 + \tfrac{1}{4}\pi x^2)e^{-x^2}]\}^{1/2}. \qquad (11)$$

The plots below show the differences between Φ and the two approximations.





The first QUARTILE of a standard NORMAL DISTRIBUTION occurs when

$$\int_0^t \Phi(z)\,dz = \tfrac{1}{4}. \qquad (12)$$

The solution is $t = 0.6745\ldots$. The value of t giving $\frac{1}{4}$ is known as the PROBABLE ERROR of a normally distributed variate.

see also CONFIDENCE INTERVAL, ERF, ERFC, FISHER-BEHRENS PROBLEM, GAUSSIAN DISTRIBUTION, GAUSSIAN INTEGRAL, HH FUNCTION, NORMAL DISTRIBUTION, PROBABILITY INTEGRAL, TETRACHORIC FUNCTION

References

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Johnson, N.; Kotz, S.; and Balakrishnan, N. *Continuous Univariate Distributions, Vol. 1, 2nd ed.* Boston, MA: Houghton Mifflin, 1994.

Normal Equation

Given an overdetermined MATRIX EQUATION

$$\mathsf{Ax} = \mathbf{b},$$

the normal equation is that which minimizes the sum of the square differences between left and right sides

$$\mathsf{A}^{\mathrm{T}}\mathsf{Ax} = \mathsf{A}^{\mathrm{T}}\mathbf{b}.$$

see also LEAST SQUARES FITTING, MOORE-PENROSE GENERALIZED MATRIX INVERSE, NONLINEAR LEAST SQUARES FITTING

Normal Form

A way of representing objects so that, although each may have many different names, every possible name corresponds to exactly one object.

see also CANONICAL FORM

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, p. 7, 1996.

Normal Function

A SQUARE INTEGRABLE function ϕ is said to be normal if

$$\int \phi^2\,dt = 1$$

However, the NORMAL DISTRIBUTION FUNCTION is also sometimes called "the normal function."

see also NORMAL DISTRIBUTION FUNCTION, SQUARE INTEGRABLE

References
Sansone, G. *Orthogonal Functions, rev. English ed.* New York: Dover, p. 6, 1991.

Normal Group

see NORMAL SUBGROUP

Normal Magic Square

see MAGIC SQUARE

Normal Matrix

A normal matrix A is a MATRIX for which

$$[\mathsf{A}, \mathsf{A}^\dagger] = 0,$$

where $[a, b]$ is the COMMUTATOR and † denotes the ADJOINT OPERATOR.

Normal Number

An IRRATIONAL NUMBER for which any FINITE pattern of numbers occurs with the expected limiting frequency in the expansion in any base. It is not known if π or e are normal. Tests of $\sqrt{n}$ for $n = 2$, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15 indicate that these SQUARE ROOTS may be normal. The only numbers known to be normal are artificially constructed ones such as the CHAMPERNOWNE CONSTANT and the COPELAND-ERDŐS CONSTANT.

see also CHAMPERNOWNE CONSTANT, COPELAND-ERDŐS CONSTANT, e, PI

Normal Order

$f(n)$ has the normal order $F(n)$ if $f(n)$ is approximately $F(n)$ for ALMOST ALL values of n. More precisely, if

$$(1-\epsilon)F(n) < f(n) < (1+\epsilon)F(n)$$

for every positive ϵ and ALMOST ALL values of n, then the normal order of $f(n)$ is $F(n)$.

see also ALMOST ALL

References
Hardy, G. H. and Weight, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Oxford University Press, p. 356, 1979.

Normal Plane

The PLANE spanned by $\mathbf{N}$ and $\mathbf{B}$ (the NORMAL VECTOR and BINORMAL VECTOR).

see also BINORMAL VECTOR, NORMAL VECTOR, PLANE

Normal to a Plane

see NORMAL VECTOR

Normal Section

Let $M \subset \mathbb{R}^3$ be a REGULAR SURFACE and $\mathbf{u_p}$ a unit TANGENT VECTOR to M, and let $\Pi(\mathbf{u_p}, \mathbf{N}(\mathbf{p}))$ be the PLANE determined by $\mathbf{u_p}$ and the normal to the surface $\mathbf{N}(\mathbf{p})$. Then the normal section of M is defined as the intersection of $\Pi(\mathbf{u_p}, \mathbf{N}(\mathbf{p}))$ and M.

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 271, 1993.

Normal Subgroup

Let H be a SUBGROUP of a GROUP G. Then H is a normal subgroup of G, written $H \triangleleft G$, if

$$xHx^{-1} = H$$

for every element x in H. Normal subgroups are also known as INVARIANT SUBGROUPS.

see also GROUP, SUBGROUP

Normal Vector

The normal to a PLANE specified by

$$f(x,y,z) = ax + by + cz + d = 0 \tag{1}$$

is given by

$$\mathbf{N} = \nabla f = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{2}$$

The normal vector at a point (x_0, y_0) on a surface $z = f(x,y)$ is

$$\mathbf{N} = \begin{bmatrix} f_x(x_0,y_0) \\ f_y(x_0,y_0) \\ -1 \end{bmatrix}. \tag{3}$$

In the PLANE, the unit normal vector is defined by

$$\hat{\mathbf{N}} \equiv \frac{d\hat{\mathbf{T}}}{d\phi}, \tag{4}$$

where $\hat{\mathbf{T}}$ is the unit TANGENT VECTOR and ϕ is the polar angle. Given a unit TANGENT VECTOR

$$\hat{\mathbf{T}} \equiv u_1\hat{\mathbf{x}} + u_2\hat{\mathbf{y}} \tag{5}$$

with $\sqrt{{u_1}^2 + {u_2}^2} = 1$, the normal is

$$\hat{\mathbf{N}} = -u_2\hat{\mathbf{x}} + u_1\hat{\mathbf{y}}. \tag{6}$$

For a function given parametrically by $(f(t), g(t))$, the normal vector relative to the point $(f(t), g(t))$ is therefore given by

$$x(t) = -\frac{g'}{\sqrt{f'^2 + g'^2}} \tag{7}$$

$$y(t) = \frac{f'}{\sqrt{f'^2 + g'^2}}. \tag{8}$$

To actually place the vector normal to the curve, it must be displaced by $(f(t), g(t))$.

In 3-D SPACE, the unit normal is

$$\hat{\mathbf{N}} \equiv \frac{\frac{d\hat{\mathbf{T}}}{ds}}{\left|\frac{d\hat{\mathbf{T}}}{ds}\right|} = \frac{\frac{d\hat{\mathbf{T}}}{dt}}{\left|\frac{d\hat{\mathbf{T}}}{dt}\right|} = \frac{1}{\kappa}\frac{d\hat{\mathbf{T}}}{ds}, \tag{9}$$

where κ is the CURVATURE. Given a 3-D surface $F(x, y, z) = 0$,

$$\hat{\mathbf{n}} = \frac{F_x + F_y + F_z}{\sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2}}. \tag{10}$$

If the surface is defined parametrically in the form

$$x = x(\phi, \psi) \tag{11}$$

$$y = y(\phi, \psi) \tag{12}$$

$$z = z(\phi, \psi), \tag{13}$$

define the VECTORS

$$\mathbf{a} \equiv \begin{bmatrix} x_\phi \\ y_\phi \\ z_\phi \end{bmatrix} \tag{14}$$

$$\mathbf{b} \equiv \begin{bmatrix} x_\psi \\ y_\psi \\ z_\psi \end{bmatrix}. \tag{15}$$

Then the unit normal vector is

$$\hat{\mathbf{N}} = \frac{\mathbf{a} \times \mathbf{b}}{\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}\cdot\mathbf{b}|^2}}. \tag{16}$$

Let g be the discriminant of the METRIC TENSOR. Then

$$\mathbf{N} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\sqrt{g}} = \epsilon_{ij}\mathbf{r}^j. \tag{17}$$

see also BINORMAL VECTOR, CURVATURE, FRENET FORMULAS, TANGENT VECTOR

References

Gray, A. "Tangent and Normal Lines to Plane Curves." §5.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 85–90, 1993.

Normalized Vector

The normalized vector of $\mathbf{X}$ is a VECTOR in the same direction but with NORM (length) 1. It is denoted $\hat{\mathbf{X}}$ and given by

$$\hat{\mathbf{X}} \equiv \frac{\mathbf{X}}{|\mathbf{X}|},$$

where $|\mathbf{X}|$ is the NORM of $\mathbf{X}$. It is also called a UNIT VECTOR.

see also UNIT VECTOR

Normalizer

A set of elements g of a GROUP such that

$$g^{-1}Hg = H,$$

is said to be the normalizer $N_G(H)$ with respect to a subset of group elements H.

see also CENTRALIZER, TIGHTLY EMBEDDED

Normed Space

A VECTOR SPACE possessing a NORM.

Nosarzewska's Inequality

Given a convex PLANE region with AREA A and PERIMETER p,

$$A - \tfrac{1}{2}p < N \leq A + \tfrac{1}{2}p + 1,$$

where N is the number of enclosed LATTICE POINTS (Nosarzewska 1948). This improves on JARNICK'S INEQUALITY

$$|N - A| < p.$$

see also JARNICK'S INEQUALITY, LATTICE POINT

References

Nosarzewska, M. "Évaluation de la différence entre l'aire d'une région plane convexe et le nombre des points aux coordonnées entières couverts par elle." *Colloq. Math.* **1**, 305–311, 1948.

Not

An operation in LOGIC which converts TRUE to FALSE and FALSE to TRUE. NOT A is denoted $!A$ or $\neg A$.

A	$\neg A$
F	T
T	F

see also AND, OR, TRUTH TABLE, XOR

Notation

A NOTATION is a set of well-defined rules for representing quantities and operations with symbols.

see also ARROW NOTATION, CHAINED ARROW NOTATION, CIRCLE NOTATION, CLEBSCH-ARONHOLD NOTATION, CONWAY'S KNOT NOTATION, DOWKER NOTATION, DOWN ARROW NOTATION, PETROV NOTATION, SCIENTIFIC NOTATION, STEINHAUS-MOSER NOTATION

References

Cajori, F. *A History of Mathematical Notations, Vols. 1–2.* New York: Dover, 1993.

Miller, J. "Earliest Uses of Various Mathematical Symbols." `http://members.aol.com/jeff570/mathsym.html`.

Miller, J. "Earliest Uses of Some of the Words of Mathematics." `http://members.aol.com/jeff570/mathword.html`.

Nöther

see NOETHER'S FUNDAMENTAL THEOREM, NOETHER-LASKER THEOREM, NOETHER'S TRANSFORMATION THEOREM, NOETHERIAN MODULE, NOETHERIAN RING

Novemdecillion

In the American system, 10^{60}.

see also LARGE NUMBER

NP-Complete Problem

A problem which is both NP (solvable in nondeterministic POLYNOMIAL time) and NP-HARD (can be translated into any other NP-PROBLEM). Examples of NP-hard problems include the HAMILTONIAN CYCLE and TRAVELING SALESMAN PROBLEMS.

In a landmark paper, Karp (1972) showed that 21 intractable combinatorial computational problems are all NP-complete.

see also HAMILTONIAN CYCLE, NP-HARD PROBLEM, NP-PROBLEM, P-PROBLEM, TRAVELING SALESMAN PROBLEM

References

Karp, R. M. "Reducibility Among Combinatorial Problems." In *Complexity of Computer Computations,* (Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972). New York: Plenum, pp. 85–103, 1972.

NP-Hard Problem

A problem is NP-hard if an ALGORITHM for solving it can be translated into one for solving any other NP-PROBLEM (nondeterministic POLYNOMIAL time) problem. NP-hard therefore means "at least as hard as any NP-PROBLEM," although it might, in fact, be harder.

see also COMPLEXITY THEORY, HITTING SET, NP-COMPLETE PROBLEM, NP-PROBLEM, P-PROBLEM, SATISFIABILITY PROBLEM

NP-Problem

A problem is assigned to the NP (nondeterministic POLYNOMIAL time) class if it is solvable in polynomial time by a nondeterministic TURING MACHINE. (A nondeterministic TURING MACHINE is a "parallel" TURING MACHINE which can take many computational paths simultaneously, with the restriction that the parallel Turing machines cannot communicate.) A P-PROBLEM (whose solution time is bounded by a polynomial) is always also NP. If a solution to an NP problem is known, it can be reduced to a single P (POLYNOMIAL time) verification.

LINEAR PROGRAMMING, long known to be NP and thought *not* to be P, was shown to be P by L. Khachian in 1979. It is not known if all apparently NP problems are actually P.

A problem is said to be NP-HARD if an ALGORITHM for solving it can be translated into one for solving any other NP-problem problem. It is much easier to show that a problem is NP than to show that it is NP-HARD. A problem which is both NP and NP-HARD is called an NP-COMPLETE PROBLEM.

see also COMPLEXITY THEORY, NP-COMPLETE PROBLEM, NP-HARD PROBLEM, P-PROBLEM, TURING MACHINE

References

Borwein, J. M. and Borwein, P. B. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, 1987.

Greenlaw, R.; Hoover, H. J.; and Ruzzo, W. L. *Limits to Parallel Computation: P-Completeness Theory.* Oxford, England: Oxford University Press, 1995.

NSW Number

The numbers

$$S_{2m+1} = \frac{(1+\sqrt{2}\,)^{2m+1} + (1-\sqrt{2}\,)^{2m+1}}{2}$$

for positive integer m. The first few terms are 1, 7, 41, 239, 1393, ... (Sloane's A002315). The indices giving PRIME NSW numbers are 3, 5, 7, 19, 29, 47, 59, 163, 257, 421, 937, 947, 1493, 1901, ... (Sloane's A005850).

References

Ribenboim, P. "The NSW Primes." §5.9 in *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 367–369, 1996.

Sloane, N. J. A. Sequences A002315/M4423 and A005850/M2426 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Nu Function

$$\nu(x) \equiv \int_0^\infty \frac{x^t\,dt}{\Gamma(t+1)}$$

$$\nu(x,\alpha) \equiv \int_0^\infty \frac{x^{\alpha+t}\,dt}{\Gamma(\alpha+t+1)},$$

where $\Gamma(z)$ is the GAMMA FUNCTION. See Gradshteyn and Ryzhik (1980, p. 1079).

see also LAMBDA FUNCTION, MU FUNCTION

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, 1979.

Null Function

A null function $\delta^0(x)$ satisfies

$$\int_a^b \delta^0(x)\,dx = 0 \tag{1}$$

for all a, b, so

$$\int_{-\infty}^\infty |\delta^0(x)|\,dx = 0. \tag{2}$$

Like a DELTA FUNCTION, they satisfy

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases} \tag{3}$$

see also DELTA FUNCTION, LERCH'S THEOREM

Null Graph
A GRAPH containing only VERTICES and no EDGES.

Null Hypothesis
A hypothesis which is tested for possible rejection under the assumption that it is true (usually that observations are the result of chance). The concept was introduced by R. A. Fisher.

Null Tetrad

$$g_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

It can be expressed as

$$g_{ab} = l_a n_b + l_b n_a - m_a \bar{m}_b - m_b \bar{m}_a.$$

see also TETRAD

References
d'Inverno, R. *Introducing Einstein's Relativity.* Oxford, England: Oxford University Press, pp. 248–249, 1992.

Nullspace
Also called the KERNEL. If T is a linear transformation of $\mathbb{R}^n$, then Null(T) is the set of all VECTORS $\mathbf{X}$ such that $\mathrm{T}(X) = 0$, i.e.,

$$\mathrm{Null}(T) \equiv \{\mathbf{X} : T(\mathbf{X}) = 0\}.$$

Nullstellansatz
see HILBERT'S NULLSTELLANSATZ

Number
The word "number" is a general term which refers to a member of a given (possibly ordered) SET. The meaning of "number" is often clear from context (i.e., does it refer to a COMPLEX NUMBER, INTEGER, REAL NUMBER, etc.?). Wherever possible in this work, the word "number" is used to refer to quantities which are INTEGERS, and "CONSTANT" is reserved for nonintegral numbers which have a fixed value. Because terms such as REAL NUMBER, BERNOULLI NUMBER, and IRRATIONAL NUMBER are commonly used to refer to nonintegral quantities, however, it is not possible to be entirely consistent in nomenclature.

see also ABUNDANT NUMBER, ACKERMANN NUMBER, ALGEBRAIC NUMBER, ALMOST PERFECT NUMBER, AMENABLE NUMBER, AMICABLE NUMBERS, ANTIMORPHIC NUMBER, APOCALYPSE NUMBER, APOCALYPTIC NUMBER, ARMSTRONG NUMBER, ARRANGEMENT NUMBER, BELL NUMBER, BERNOULLI NUMBER, BERTELSEN'S NUMBER, BETROTHED NUMBERS, BETTI NUMBER, BEZOUT NUMBERS, BINOMIAL NUMBER, BRAUER NUMBER, BROWN NUMBERS, CARDINAL NUMBER, CARMICHAEL NUMBER, CATALAN NUMBER, CAYLEY NUMBER, CENTERED CUBE NUMBER, CENTERED SQUARE NUMBER, CHAITIN'S NUMBER, CHERN NUMBER, CHOICE NUMBER, CHRISTOFFEL NUMBER, CLIQUE NUMBER, COLUMBIAN NUMBER, COMPLEX NUMBER, COMPUTABLE NUMBER, CONDITION NUMBER, CONGRUENT NUMBERS, CONSTRUCTIBLE NUMBER, COTES NUMBER, CROSSING NUMBER (GRAPH), CROSSING NUMBER (LINK), CUBIC NUMBER, CULLEN NUMBER, CUNNINGHAM NUMBER, CYCLIC NUMBER, CYCLOMATIC NUMBER, D-NUMBER, DE MOIVRE NUMBER, DEFICIENT NUMBER, DELANNOY NUMBER, DEMLO NUMBER, DIAGONAL RAMSEY NUMBER, e-PERFECT NUMBER, EBAN NUMBER, EDDINGTON NUMBER, EDGE NUMBER, ENNEAGONAL NUMBER, ENTRINGER NUMBER, ERDŐS NUMBER, EUCLID NUMBER, EULER'S IDONEAL NUMBER, EULER NUMBER, EULERIAN NUMBER, EULER ZIGZAG NUMBER, EVEN NUMBER, FACTORIAL NUMBER, FERMAT NUMBER, FIBONACCI NUMBER, FIGURATE NUMBER, G-NUMBER, GENOCCHI NUMBER, GIUGA NUMBER, GNOMIC NUMBER, GONAL NUMBER, GRAHAM'S NUMBER, GREGORY NUMBER, HAILSTONE NUMBER, HANSEN NUMBER, HAPPY NUMBER, HARMONIC DIVISOR NUMBER, HARMONIC NUMBER, HARSHAD NUMBER, HEEGNER NUMBER, HEESCH NUMBER, HELLY NUMBER, HEPTAGONAL NUMBER, HETEROGENEOUS NUMBERS, HEX NUMBER, HEX PYRAMIDAL NUMBER, HEXAGONAL NUMBER, HOMOGENEOUS NUMBERS, HURWITZ NUMBER, HYPERCOMPLEX NUMBER, HYPERPERFECT NUMBER, i, IDONEAL NUMBER, IMAGINARY NUMBER, INDEPENDENCE NUMBER, INFINARY MULTIPERFECT NUMBER, INFINARY PERFECT NUMBER, IRRATIONAL NUMBER, IRREDUCIBLE SEMIPERFECT NUMBER, IRREDUNDANT RAMSEY NUMBER, j, KAPREKAR NUMBER, KEITH NUMBER, KISSING NUMBER, KNÖDEL NUMBERS, LAGRANGE NUMBER (DIOPHANTINE EQUATION), LAGRANGE NUMBER (RATIONAL APPROXIMATION), LARGE NUMBER, LEAST DEFICIENT NUMBER, LEHMER NUMBER, LEVIATHAN NUMBER, LIOUVILLE NUMBER, LOGARITHMIC NUMBER, LUCAS NUMBER, LUCKY NUMBER, MACMAHON'S PRIME NUMBER OF MEASUREMENT, MARKOV NUMBER, MCNUGGET NUMBER, MÉNAGE NUMBER, MERSENNE NUMBER, MOTZKIN NUMBER, MULTIPLICATIVE PERFECT NUMBER, MULTIPLY PERFECT NUMBER, NARCISSISTIC NUMBER, NATURAL NUMBER, NEAR NOBLE NUMBER, NEXUS NUMBER, NIVEN NUMBER, NOBLE NUMBER, NONAGONAL NUMBER, NORMAL NUMBER, NSW NUMBER, NUMBER GUESSING, OBLONG NUMBER, OCTAGONAL NUMBER, OCTAHEDRAL NUMBER, ODD NUMBER, ORE NUMBER, ORDINAL NUMBER, PENTAGONAL NUMBER, PENTATOPE NUMBER, PERFECT DIGITAL INVARIANT, PERFECT NUMBER, PERSISTENT NUMBER, PLUPERFECT NUMBER, PLUS PERFECT NUMBER, PLUTARCH NUMBERS, POLYGONAL NUMBER,

PONTRYAGIN NUMBER, POULET NUMBER, POWERFUL NUMBER, PRACTICAL NUMBER, PRIMARY, PRIMITIVE ABUNDANT NUMBER, PRIMITIVE PSEUDOPERFECT NUMBER, PRIMITIVE SEMIPERFECT NUMBER, PSEUDOPERFECT NUMBER, PSEUDORANDOM NUMBER, PSEUDOSQUARE, PYRAMIDAL NUMBER, Q-NUMBER, QUASIPERFECT NUMBER, RAMSEY NUMBER, RATIONAL NUMBER, REAL NUMBER, RENCONTRES NUMBER, RECURRING DIGITAL INVARIANT, REPFIGIT NUMBER, RHOMBIC DODECAHEDRAL NUMBER, RIESEL NUMBER, ROTATION NUMBER, RSA NUMBER, SARRUS NUMBER, SCHRÖDER NUMBER, SCHUR NUMBER, SECANT NUMBER, SEGMENTED NUMBER, SELF-DESCRIPTIVE NUMBER, SELF NUMBER, SEMIPERFECT NUMBER, SIERPIŃSKI NUMBER OF THE FIRST KIND, SIERPIŃSKI NUMBER OF THE SECOND KIND, SINGLY EVEN NUMBER, SKEWES NUMBER, SMALL NUMBER, SMITH NUMBER, SMOOTH NUMBER, SOCIABLE NUMBERS, SPRAGUE-GRUNDY NUMBER, SQUARE NUMBER, SQUARE PYRAMIDAL NUMBER, STAR NUMBER, STELLA OCTANGULA NUMBER, STIEFEL-WHITNEY NUMBER, STIRLING CYCLE NUMBER, STIRLING SET NUMBER, STØRMER NUMBER, SUBLIME NUMBER, SUITABLE NUMBER, SUM-PRODUCT NUMBER, SUPER-3 NUMBER, SUPER CATALAN NUMBER, SUPERABUNDANT NUMBER, SUPERPERFECT NUMBER, SUPER-POULET NUMBER, TANGENT NUMBER, TAXICAB NUMBER, TETRAHEDRAL NUMBER, TRANSCENDENTAL NUMBER, TRANSFINITE NUMBER, TRIANGULAR NUMBER, TRIBONACCI NUMBER, TRIMORPHIC NUMBER, TRUNCATED OCTAHEDRAL NUMBER, TRUNCATED TETRAHEDRAL NUMBER, TWIST NUMBER, U-NUMBER, ULAM NUMBER, UNDULATING NUMBER, UNHAPPY NUMBER, UNITARY MULTIPERFECT NUMBER, UNITARY PERFECT NUMBER, UNTOUCHABLE NUMBER, VAMPIRE NUMBER, VAN DER WAERDEN NUMBER, VR NUMBER, WEIRD NUMBER, WHOLE NUMBER, WOODALL NUMBER, Z-NUMBER, ZAG NUMBER, ZEISEL NUMBER, ZIG NUMBER

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Wells, D. W. *The Penguin Dictionary of Curious and Interesting Numbers.* Harmondsworth, England: Penguin Books, 1986.

Number Axis

see REAL LINE

Number Field

If r is an ALGEBRAIC NUMBER of degree n, then the totality of all expressions that can be constructed from r by repeated additions, subtractions, multiplications, and divisions is called a number field (or an ALGEBRAIC NUMBER FIELD) generated by r, and is denoted $F[r]$. Formally, a number field is a finite extension $\mathbb{Q}(\alpha)$ of the FIELD $\mathbb{Q}$ of RATIONAL NUMBERS.

The numbers of a number field which are ROOTS of a POLYNOMIAL

$$z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0$$

with integral coefficients and leading coefficient 1 are called the ALGEBRAIC INTEGERS of that field.

see also ALGEBRAIC FUNCTION FIELD, ALGEBRAIC INTEGER, ALGEBRAIC NUMBER, FIELD, FINITE FIELD, $\mathbb{Q}$, QUADRATIC FIELD

References

Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 127, 1996.

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 151–152, 1993.

Number Field Sieve Factorization Method

An extremely fast factorization method developed by Pollard which was used to factor the RSA-130 NUMBER. This method is the most powerful known for factoring general numbers, and has complexity

$$\mathcal{O}\{\exp[c(\log n)^{1/3}(\log\log n)^{2/3}]\},$$

reducing the *exponent* over the CONTINUED FRACTION FACTORIZATION ALGORITHM and QUADRATIC SIEVE FACTORIZATION METHOD. There are three values of c relevant to different flavors of the method (Pomerance 1996). For the "special" case of the algorithm applied to numbers near a large POWER,

$$c = \left(\tfrac{32}{9}\right)^{1/3} = 1.523\ldots,$$

for the "general" case applicable to any ODD POSITIVE number which is not a POWER,

$$c = \left(\tfrac{64}{9}\right)^{1/3} = 1.923\ldots,$$

and for a version using many POLYNOMIALS (Coppersmith 1993),

$$c = \tfrac{1}{3}(92 + 26\sqrt{13})^{1/3} = 1.902\ldots.$$

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Number Group

see FIELD

Number Guessing

By asking a small number of innocent-sounding questions about an unknown number, it is possible to reconstruct the number with absolute certainty (assuming that the questions are answered correctly). Ball and Coxeter (1987) give a number of sets of questions which can be used.

One of the simplest algorithms uses only three questions to determine an unknown number n:

1. Triple n and announce if the result $n' = 3n$ is EVEN or ODD.
2. If you were told that n' is EVEN, ask the person to reveal the number n'' which is half of n'. If you were told that n' is ODD, ask the person to reveal the number n'' which is half of $n' + 1$.
3. Ask the person to reveal the number of times k which 9 divides evenly into $n''' = 3n''$.

The original number n is then given by $2k$ if n' was EVEN, or $2k + 1$ if n' was ODD. For $n = 2m$ even, $n' = 6m$, $n'' = 3m$, $n''' = 9m$, $k = m$, so $2k = 2m = n$. For $n = 2m + 1$ odd, $n' = 6m + 3$, $n'' = 3m + 2$, $n''' = 9m + 6$, $k = m$, so $2k + 1 = 2m + 1 = n$.

Another method asks:

1. Multiply the number n by 5.
2. Add 6 to the product.
3. Multiply the sum by 4.
4. Add 9 to the product.
5. Multiply the sum by 5 and reveal the result n'.

The original number is then given by $n = (n' - 165)/100$, since the above steps give $n' = 5(4(5n+6)+9) = 100n + 165$.

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Number Pyramid

A set of numbers obeying a pattern like the following.

$$\begin{aligned} 91 \cdot 37 &= 3367 \\ 9901 \cdot 3367 &= 33336667 \\ 999001 \cdot 333667 &= 333333666667 \\ 99990001 \cdot 33336667 &= 3333333366666667 \end{aligned}$$

$$\begin{aligned} 4^2 &= 16 \\ 34^2 &= 1156 \\ 334^2 &= 111556 \end{aligned}$$

$$\begin{aligned} 7^2 &= 49 \\ 67^2 &= 4489 \\ 667^2 &= 444889. \end{aligned}$$

see also AUTOMORPHIC NUMBER

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Number System

see BASE (NUMBER)

Number Theoretic Transform

Simplemindedly, a number theoretic transform is a generalization of a FAST FOURIER TRANSFORM obtained by replacing $e^{-2\pi i k/N}$ with an nth PRIMITIVE ROOT OF UNITY. This effectively means doing a transform over the QUOTIENT RING $\mathbb{Z}/p\mathbb{Z}$ instead of the COMPLEX NUMBERS $\mathbb{C}$. The theory is rather elegant and uses the language of FINITE FIELDS and NUMBER THEORY.

see also FAST FOURIER TRANSFORM, FINITE FIELD

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Number Theory

A vast and fascinating field of mathematics consisting of the study of the properties of whole numbers. PRIMES and PRIME FACTORIZATION are especially important in number theory, as are a number of functions such as the DIVISOR FUNCTION, RIEMANN ZETA FUNCTION, and TOTIENT FUNCTION. Excellent introductions to number theory may be found in Ore (1988) and Beiler (1966). The classic history on the subject (now slightly dated) is that of Dickson (1952).

see also ARITHMETIC, CONGRUENCE, DIOPHANTINE EQUATION, DIVISOR FUNCTION, GÖDEL'S INCOMPLETENESS THEOREM, PEANO'S AXIOMS, PRIME COUNTING FUNCTION, PRIME FACTORIZATION, PRIME NUMBER, QUADRATIC RECIPROCITY THEOREM, RIEMANN ZETA FUNCTION, TOTIENT FUNCTION

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Number Triangle

see BELL TRIANGLE, CLARK'S TRIANGLE, EULER'S TRIANGLE, LEIBNIZ HARMONIC TRIANGLE, PASCAL'S TRIANGLE, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, TRINOMIAL TRIANGLE

Number Wall

see QUOTIENT-DIFFERENCE TABLE

Numerator

The number p in a FRACTION p/q.

see also DENOMINATOR, FRACTION, RATIONAL NUMBER

Numeric Function

A FUNCTION $f : A \to B$ such that B is a SET of numbers.

Numerical Derivative

While it is usually much easier to compute a DERIVATIVE instead of an INTEGRAL (which is a little strange, considering that "more" functions have integrals than derivatives), there are still many applications where derivatives need to be computed numerically. The simplest approach simply uses the definition of the DERIVATIVE

$$f'(x) \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

for some small numerical value of $h \ll 1$.

see also NUMERICAL INTEGRATION

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Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Numerical Derivatives." §5.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 180–184, 1992.

Numerical Integration

The approximate computation of an INTEGRAL. The numerical computation of an INTEGRAL is sometimes called QUADRATURE. There are a wide range of methods available for numerical integration. A good source for such techniques is Press *et al.* (1992).

The most straightforward numerical integration technique uses the NEWTON-COTES FORMULAS (also called QUADRATURE FORMULAS), which approximate a function tabulated at a sequent of regularly spaced INTERVALS by various degree POLYNOMIALS. If the endpoints are tabulated, then the 2- and 3-point formulas are called the TRAPEZOIDAL RULE and SIMPSON'S RULE, respectively. The 5-point formula is called BODE'S RULE. A generalization of the TRAPEZOIDAL RULE is ROMBERG INTEGRATION, which can yield accurate results for many fewer function evaluations.

If the functions are known analytically instead of being tabulated at equally spaced intervals, the best numerical method of integration is called GAUSSIAN QUADRATURE. By picking the abscissas at which to evaluate the function, Gaussian quadrature produces the most accurate approximations possible. However, given the speed of modern computers, the additional complication of the GAUSSIAN QUADRATURE formalism often makes it less desirable than simply brute-force calculating twice as many points on a regular grid (which also permits the already computed values of the function to be re-used). An excellent reference for GAUSSIAN QUADRATURE is Hildebrand (1956).

see also DOUBLE EXPONENTIAL INTEGRATION, FILON'S INTEGRATION FORMULA, INTEGRAL, INTEGRATION, NUMERICAL DERIVATIVE, QUADRATURE

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Numerology

The study of numbers for the supposed purpose of predicting future events or seeking connections with the occult.

see also BEAST NUMBER, NUMBER THEORY

NURBS Curve

A nonuniform rational B-SPLINE curve defined by

$$\mathbf{C}(t) = \frac{\sum_{i=0}^n N_{i,p}(t) w_i \mathbf{P}_i}{\sum_{i=0}^n N_{i,p}(t) w_i},$$

where p is the order, $N_{i,p}$ are the B-SPLINE basis functions, $\mathbf{P}_i$ are control points, and the weight w_i of $\mathbf{P}_i$ is the last ordinate of the homogeneous point $\mathbf{P}_i^w$. These curves are closed under perspective transformations and can represent CONIC SECTIONS exactly.

see also B-SPLINE, BÉZIER CURVE, NURBS SURFACE

References

Piegl, L. and Tiller, W. *The NURBS Book, 2nd ed* New York: Springer-Verlag, 1997.

NURBS Surface

A nonuniform rational B-SPLINE surface of degree (p, q) is defined by

$$\mathbf{S}(u,v) = \frac{\sum_{i=0}^m \sum_{j=0}^n N_{i,p}(u) N_{j,q}(v) w_{i,j} \mathbf{P}_{i,j}}{\sum_{i=0}^m \sum_{j=0}^n N_{i,p}(u) N_{j,q}(v) w_{i,j}},$$

where $N_{i,p}$ and $N_{j,q}$ are the B-SPLINE basis functions, $\mathbf{P}_{i,j}$ are control points, and the weight $w_{i,j}$ of $\mathbf{P}_{i,j}$ is the last ordinate of the homogeneous point $\mathbf{P}^w_{i,j}$.

see also B-SPLINE, BÉZIER CURVE, NURBS CURVE

Nyquist Frequency

In order to recover all FOURIER components of a periodic waveform, it is necessary to sample twice as fast as the *highest* waveform frequency ν,

$$f_{\text{Nyquist}} = 2\nu.$$

The minimum sampling frequency is called the Nyquist frequency.

see also FOURIER SERIES, FOURIER TRANSFORM, NYQUIST SAMPLING, OVERSAMPLING, SAMPLING THEOREM

Nyquist Sampling

Sampling at the NYQUIST FREQUENCY.

O

Obelus

The symbol $\div$ used to indicate DIVISION. In typography, an obelus has a more general definition as any symbol, such as the dagger (†), used to indicate a footnote.

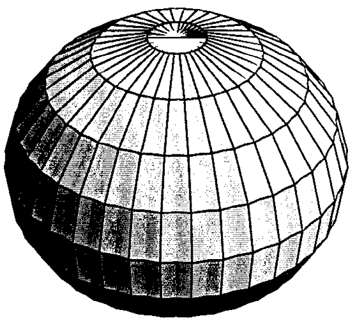
see also DIVISION, SOLIDUS

Object

A mathematical structure (e.g., a GROUP, VECTOR SPACE, or DIFFERENTIABLE MANIFOLD) in a CATEGORY.

see also MORPHISM

Oblate Spheroid



A "squashed" SPHEROID for which the equatorial radius a is greater than the polar radius c, so $a > c$. To first approximation, the shape assumed by a rotating fluid (including the Earth, which is "fluid" over astronomical time scales) is an oblate spheroid. The oblate spheroid can be specified parametrically by the usual SPHEROID equations (for a SPHEROID with z-AXIS as the symmetry axis),

$$x = a\sin v\cos u \tag{1}$$

$$y = a\sin v\sin u \tag{2}$$

$$z = c\cos v, \tag{3}$$

with $a > c$, $u \in [0, 2\pi)$, and $v \in [0, \pi]$. Its Cartesian equation is

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1. \tag{4}$$

The ELLIPTICITY of an oblate spheroid is defined by

$$e \equiv \sqrt{\frac{a^2-c^2}{a^2}}, \tag{5}$$

so that

$$1 - e^2 = \frac{c^2}{a^2}. \tag{6}$$

Then the radial distance from the rotation axis is given by

$$r(\delta) = a\left(1 + \frac{e^2}{1-e^2}\sin^2\delta\right)^{-1/2} \tag{7}$$

as a function of the LATITUDE δ.

The SURFACE AREA and VOLUME of an oblate spheroid are

$$S = 2\pi a^2 + \pi\frac{c^2}{e}\ln\left(\frac{1+e}{1-e}\right) \tag{8}$$

$$V = \tfrac{4}{3}\pi a^2 c. \tag{9}$$

An oblate spheroid with its origin at a FOCUS has equation

$$r = \frac{a(1-e^2)}{1+e\cos\phi}. \tag{10}$$

Define k and expand up to POWERS of e^6,

$$\begin{aligned} k &\equiv e^2(1-e^2)^{-1} = e^2(1+e^2-2e^4+6e^6+\ldots) \\ &= e^2+e^4-2e^6+\ldots \end{aligned} \tag{11}$$

$$k^2 = e^4 + e^6 + \ldots \tag{12}$$

$$k^3 = e^6 + \ldots. \tag{13}$$

Expanding r in POWERS of ELLIPTICITY to e^6 therefore yields

$$\begin{aligned} \frac{r}{a} &= 1 - \tfrac{1}{2}(e^2+e^4-2e^4+6e^6)\sin^2\delta \\ &\quad + \tfrac{3}{4}(e^4+e^6)\sin^4\delta - \tfrac{15}{8}e^6\sin^6\delta + \ldots. \end{aligned} \tag{14}$$

In terms of LEGENDRE POLYNOMIALS,

$$\begin{aligned} \frac{r}{a} &= (1 - \tfrac{1}{6}e^2 - \tfrac{11}{20}e^4 - \tfrac{103}{1680}e^6) \\ &\quad + (-\tfrac{1}{3}e^2 - \tfrac{5}{42}e^4 - \tfrac{3}{56}e^6)P_2 \\ &\quad + (\tfrac{3}{35}e^4 + \tfrac{57}{770}e^6)P_4 - \tfrac{5}{231}e^6P_6 + \ldots. \end{aligned} \tag{15}$$

The ELLIPTICITY may also be expressed in terms of the OBLATENESS (also called FLATTENING), denoted ϵ or f.

$$\epsilon \equiv \frac{a-c}{a} \tag{16}$$

$$c = a(1-\epsilon) \tag{17}$$

$$c^2 = a^2(1-\epsilon)^2 \tag{18}$$

$$(1-\epsilon)^2 = 1 - e^2, \tag{19}$$

so

$$\epsilon = 1 - \sqrt{1-e^2} \tag{20}$$

and

$$e^2 = 1-(1-\epsilon)^2 = 1-(1-2\epsilon+\epsilon^2) = 2\epsilon-\epsilon^2 \tag{21}$$

$$r = a\left[1 + \frac{2\epsilon-\epsilon^2}{(1-\epsilon)^2}\sin^2\delta\right]^{-1/2}. \tag{22}$$

Define k and expand up to POWERS of ϵ^6

$$\begin{aligned} k &\equiv (2\epsilon-\epsilon)(1-\epsilon)^{-2} = (2\epsilon-\epsilon^2)(1+2\epsilon-6\epsilon^2+\ldots) \\ &= 2\epsilon+4\epsilon^4-12\epsilon^3-\epsilon^2-2\epsilon^3+\ldots \\ &= 2\epsilon+3\epsilon^2-14\epsilon^3+\ldots \end{aligned} \tag{23}$$

$$k^2 = 4\epsilon^2+6\epsilon^3+\ldots \tag{24}$$

$$k^3 = 8\epsilon^3+\ldots. \tag{25}$$

Expanding r in POWERS of the OBLATENESS to ϵ^3 yields

$$\begin{aligned} \frac{r}{a} &= 1-\tfrac{1}{2}(2\epsilon+3\epsilon^2-14\epsilon^3)\sin^2\delta \\ &\quad +\tfrac{3}{4}(4\epsilon^2+6\epsilon^3)\sin^4\delta+8\epsilon^3\sin^6\delta+\ldots. \end{aligned} \tag{26}$$

In terms of LEGENDRE POLYNOMIALS,

$$\begin{aligned} \frac{r}{a} &= (1-\tfrac{1}{3}\epsilon-\tfrac{2}{5}\epsilon^2-\tfrac{13}{105}\epsilon^3)+(-\tfrac{2}{3}\epsilon-\tfrac{1}{7}\epsilon^2-\tfrac{1}{21}\epsilon^3)P_2 \\ &\quad +(\tfrac{12}{35}\epsilon^2-\tfrac{96}{385}\epsilon^3)P_4-\tfrac{40}{231}\epsilon^3P_6+\ldots. \end{aligned} \tag{27}$$

To find the projection of an oblate spheroid onto a PLANE, set up a coordinate system such that the z-AXIS is towards the observer, and the x-axis is in the PLANE of the page. The equation for an oblate spheroid is

$$r(\theta) = a\left[1+\frac{2\epsilon-\epsilon^2}{(1-\epsilon)^2}\cos^2\theta\right]^{-1/2}. \tag{28}$$

Define

$$k \equiv \frac{2\epsilon-\epsilon^2}{(1-\epsilon)^2}, \tag{29}$$

and $x \equiv \sin\theta$. Then

$$r(\theta) = a[1+k(1-x^2)]^{-1/2} = a(1+k-kx^2)^{-1/2}. \tag{30}$$

Now rotate that spheroid about the x-axis by an ANGLE B so that the new symmetry axes for the spheroid are $x' \equiv x$, y', and z'. The projected height of a point in the $x=0$ PLANE on the y-axis is

$$\begin{aligned} y &= r(\theta)\cos(\theta-B) = r(\theta)(\cos\theta\cos B-\sin\theta\sin B) \\ &= r(\theta)(\sqrt{1-x^2}\cos B+x\sin B). \end{aligned} \tag{31}$$

To find the highest projected point,

$$\frac{dy}{d\theta} = \frac{a\sin(B-\theta)}{(1+k\cos^2\theta)^{1/2}}+ak\frac{\cos(B-\theta)\cos\theta\sin\theta}{(1+k\cos^2\theta)^{3/2}} = 0. \tag{32}$$

Simplifying,

$$\tan(B-\theta)(1+k\cos^2\theta)+k\cos\theta\sin\theta = 0. \tag{33}$$

But

$$\begin{aligned} \tan(B-\theta) &= \frac{\tan B-\tan\theta}{1+\tan B\tan\theta} = \frac{\tan B-\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}}{1+\tan B\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}} \\ &= \frac{\sqrt{1-\sin^2\theta}\tan B-\sin\theta}{\sqrt{1-\sin^2\theta}+\tan B\sin\theta}. \end{aligned} \tag{34}$$

Plugging (34) into (33),

$$\frac{\sqrt{1-x^2}\tan B-x}{\sqrt{1-x^2}+x\tan B}[1+k(1-x^2)]+kx\sqrt{1-x^2} = 0 \tag{35}$$

and performing a number of algebraic simplifications

$$\begin{aligned} &(\sqrt{1-x^2}\tan B-x)(1+k-kx^2) \\ &\quad +kx\sqrt{1-x^2}(\sqrt{1-x^2}+x\tan B) = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} &[(1+k)\sqrt{1-x^2}\tan B-kx^2\sqrt{1-x^2}\tan B \\ &\quad -x-kx+kx^3]+[kx(1-x^2)+kx^2\sqrt{1-x^2}\tan B] \end{aligned} \tag{37}$$

$$(1+k)\tan B\sqrt{1-x^2}-kx(1-x^2)-x+kx(1-x^2) = 0 \tag{38}$$

$$(1+k)\tan B\sqrt{1-x^2} = x \tag{39}$$

$$(1+k)^2\tan^2 B(1-x^2) = x^2 \tag{40}$$

$$x^2[1+(1+k)^2\tan^2 B] = (1+k)^2\tan^2 B \tag{41}$$

finally gives the expression for x in terms of B and k,

$$x^2 = \frac{\tan^2 B(1+k)^2}{1+(1+k)^2\tan^2 B}. \tag{42}$$

Combine (30) and (31) and plug in for x,

$$\begin{aligned} y &= a\frac{\sqrt{1-x^2}\cos B+x\sin B}{\sqrt{1+k-kx^2}} \\ &= a\frac{\cos B+(1+k)\frac{\sin^2 B}{\cos B}}{\sqrt{(1+k)[1+(1+k)\tan^2 B]}} \\ &= a\frac{\cos^2 B+(1+k)\sin^2 B}{\cos B\sqrt{(1+k)[1+(1+k)\tan^2 B]}}. \end{aligned} \tag{43}$$

Now re-express k in terms of a and c, using $\epsilon \equiv 1-c/a$,

$$\begin{aligned} k &\equiv \frac{(2-\epsilon)\epsilon}{(1-\epsilon)^2} = \frac{\left(1+\frac{c}{a}\right)\left(1-\frac{c}{a}\right)}{\left(\frac{c}{a}\right)^2} \\ &= \frac{1-\left(\frac{c}{a}\right)^2}{\left(\frac{c}{a}\right)^2} = \left(\frac{a}{c}\right)^2-1, \end{aligned} \tag{44}$$

so

$$1+k = \left(\frac{a}{c}\right)^2. \tag{45}$$

Plug (44) and (45) into (43) to obtain the SEMIMINOR AXIS of the projected oblate spheroid,

$$\begin{aligned} c' &= a\frac{\cos^2 B+\left(\frac{a}{c}\right)^2\sin^2 B}{\cos B\sqrt{\left(\frac{a}{c}\right)^2\left[1+\left(\frac{a}{c}\right)^2\tan^2 B\right]}} \\ &= a\frac{\cos^2 B+\left(\frac{a}{c}\right)^2\sin^2 B}{\frac{a}{c}\sqrt{\cos^2 B+\left(\frac{a}{c}\right)^2\sin^2 B}} \\ &= c\sqrt{\cos^2 B+\left(\frac{a}{c}\right)^2\sin^2 B} = \sqrt{c^2\cos^2 B+a^2\sin^2 B} \\ &= a\sqrt{(1-\epsilon)^2\cos^2 B+\sin^2 B}. \end{aligned} \tag{46}$$

We wish to find the equation for a spheroid which has been rotated about the $x \equiv x'$-axis by ANGLE B, then the z-axis by ANGLE P

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos B & \sin B \\ 0 & -\sin B & \cos B \end{bmatrix} \begin{bmatrix} \cos P & 0 & \sin P \\ 0 & 1 & 0 \\ -\sin P & 0 & \cos P \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} \cos P & 0 & \sin P \\ -\sin B \sin P & \cos B & \sin B \cos P \\ -\cos B \sin P & -\sin B & \cos B \cos P \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{47}$$

Now, in the original coordinates (x', y', z'), the spheroid is given by the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{c^2} + \frac{z'^2}{a^2} = 1, \tag{48}$$

which becomes in the new coordinates,

$$\frac{(x \cos P + y \sin P)^2}{a^2} + \frac{(-x \sin B \sin P + z \cos B + y \sin B \cos P)^2}{a^2} + \frac{(-x \cos B \sin P - z \sin B + y \cos B \cos P)^2}{c^2} = 1. \tag{49}$$

Collecting COEFFICIENTS,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 1, \tag{50}$$

where

$$A \equiv \frac{\cos^2 P + \sin^2 B \sin^2 P}{a^2} + \frac{\cos^2 B \sin^2 P}{c^2} \tag{51}$$
$$B \equiv \frac{\sin^2 P + \sin^2 B \cos^2 P}{a^2} + \frac{\cos^2 B \cos^2 P}{c^2} \tag{52}$$
$$C \equiv \frac{\cos^2 B}{a^2} + \frac{\sin^2 B}{c^2} \tag{53}$$
$$D \equiv 2 \cos P \sin P \left(\frac{1 - \sin^2 B}{a^2} - \frac{\cos^2 B}{c^2} \right)$$
$$= 2 \cos P \sin P \cos^2 B \left(\frac{1}{a^2} - \frac{1}{c^2} \right) \tag{54}$$
$$E \equiv 2 \sin B \cos B \sin P \left(\frac{1}{b^2} - \frac{1}{a^2} \right) \tag{55}$$
$$F \equiv 2 \sin B \cos B \cos P \left(\frac{1}{a^2} - \frac{1}{b^2} \right). \tag{56}$$

If we are interested in computing z, the radial distance from the symmetry axis of the spheroid (y) corresponding to a point

$$Cz^2 + (Ex + Fy)z + (Ax^2 + By^2 + Dxy - 1)$$
$$= Cz^2 + G(x, y)z + H(x, y) = 0, \tag{57}$$

where

$$G(x, y) \equiv Ex + Fy \tag{58}$$
$$H(x, y) \equiv Ax^2 + By^2 + Dxy - 1. \tag{59}$$

z can now be computed using the quadratic equation when (x, y) is given,

$$z = \frac{-G(x, y) \pm \sqrt{G^2(x, y) - 4CG(x, y)}}{2C}. \tag{60}$$

If $P = 0$, then we have $\sin P = 0$ and $\cos P = 1$, so (51) to (56) and (58) to (59) become

$$A \equiv \frac{1}{a^2} \tag{61}$$
$$B \equiv \frac{\sin^2 B}{a^2} + \frac{\cos^2 B}{b^2} \tag{62}$$
$$C \equiv \frac{\cos^2 B}{a^2} + \frac{\sin^2 B}{b^2} \tag{63}$$
$$D \equiv 0 \tag{64}$$
$$E \equiv 0 \tag{65}$$
$$F \equiv 2 \sin B \cos B \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \tag{66}$$
$$G(x, y) \equiv Fy = 2y \sin B \cos B \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \tag{67}$$
$$H(x, y) \equiv Ax^2 + By^2 - 1$$
$$= \frac{x^2}{a^2} + y^2 \left(\frac{\sin^2 B}{a^2} + \frac{\cos^2 B}{b^2} \right) - 1. \tag{68}$$

see also DARWIN-DE SITTER SPHEROID, ELLIPSOID, OBLATE SPHEROIDAL COORDINATES, PROLATE SPHEROID, SPHERE, SPHEROID

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Oblate Spheroid Geodesic

The GEODESIC on an OBLATE SPHEROID can be computed analytically for a spheroid specified parametrically by

$$x = a \sin v \cos u \tag{1}$$
$$y = a \sin v \sin u \tag{2}$$
$$z = c \cos v, \tag{3}$$

with $a > c$, although it is much more unwieldy than for a simple SPHERE. Using the first PARTIAL DERIVATIVES

$$\frac{\partial x}{\partial u} = -a \sin v \sin u \qquad \frac{\partial x}{\partial v} = a \cos v \cos u \tag{4}$$
$$\frac{\partial y}{\partial u} = a \sin v \cos u \qquad \frac{\partial y}{\partial v} = a \cos v \sin u \tag{5}$$
$$\frac{\partial z}{\partial u} = 0 \qquad \frac{\partial z}{\partial v} = -c \sin v, \tag{6}$$

and second PARTIAL DERIVATIVES

$$\frac{\partial^2 x}{\partial u^2} = -a \sin v \cos u \qquad \frac{\partial^2 x}{\partial v^2} = -a \sin v \cos u \tag{7}$$

$$\frac{\partial^2 y}{\partial u^2} = -a \sin v \sin u \qquad \frac{\partial^2 y}{\partial v^2} = -a \sin v \sin u \tag{8}$$

$$\frac{\partial^2 z}{\partial u^2} = 0 \qquad \frac{\partial^2 z}{\partial v^2} = -z \cos v, \tag{9}$$

gives the GEODESICS functions as

$$\begin{aligned} P &\equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ &= a^2(\sin^2 v \cos^2 u + \sin^2 v \sin^2 u) \\ &= a^2 \sin^2 v \end{aligned} \tag{10}$$

$$Q \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0 \tag{11}$$

$$\begin{aligned} R &\equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\ &= a^2 + (c^2 - a^2)\sin^2 v = a^2(1 - e^2 \sin^2 v). \end{aligned} \tag{12}$$

Since $Q = 0$ and P and R are explicit functions of v only, we can use the special form of the GEODESIC equation.

$$\begin{aligned} u &= \int \sqrt{\frac{R}{P^2 - c_1{}^2 P}}\, dv \\ &= \int \sqrt{\frac{a^2(1 - e^2 \sin^2 v)}{a^4 \sin^4 v - c_1{}^2 a^2 \sin^2 v}}\, dv \\ &= c_1 \int \sqrt{\frac{1 - e^2 \sin^2 v}{\left(\frac{a}{c_1}\right)^2 \sin^2 v - 1}}\, \frac{dv}{\sin v}. \end{aligned} \tag{13}$$

Integrating gives

$$u = -c_1 \frac{e^2 F\left(\phi \middle| \frac{(d^2-1)e^2}{d^2-e^2}\right) - b^2 \Pi\left(d^2 - 1, \phi \middle| \frac{(d^2-1)e^2}{d^2-e^2}\right)}{\sqrt{d^2 - e^2}}, \tag{14}$$

where

$$d \equiv \frac{a}{c_1} \tag{15}$$

$$\cos\phi \equiv \frac{d \cos v}{\sqrt{d^2 - 1}}, \tag{16}$$

$F(\phi|m)$ is an ELLIPTIC INTEGRAL OF THE FIRST KIND with PARAMETER m, and $\Pi(\phi|m, k)$ is an ELLIPTIC INTEGRAL OF THE THIRD KIND.

GEODESICS other than MERIDIANS of an OBLATE SPHEROID undulate between two parallels with latitudes equidistant from the equator. Using the WEIERSTRAß SIGMA FUNCTION and WEIERSTRAß ZETA FUNCTION, the GEODESIC on the OBLATE SPHEROID can be written as

$$x + iy = \kappa \frac{\sigma(a + u)}{\sigma(u)\sigma(a)} e^{u[\eta - \zeta(\omega + a)]} \tag{17}$$

$$x - iy = \kappa \frac{\sigma(a - u)}{\sigma(u)\sigma(a)} e^{-u[\eta - \zeta(\omega + a)]} \tag{18}$$

$$z^2 = \lambda^2 \frac{\sigma(\omega'' + u)\sigma(\omega'' - u)}{\sigma^2(u)\sigma^2(a)} \tag{19}$$

(Forsyth 1960, pp. 108–109; Halphen 1886–1891).

The equation of the GEODESIC can be put in the form

$$d\phi = \frac{\sqrt{1 - e^2 \sin^2 v}\, \sin a}{\sqrt{\sin^2 v - \sin^2 a}\, \sin v}\, dv, \tag{20}$$

where a is the smallest value of v on the curve. Furthermore, the difference in longitude between points of highest and next lowest latitude on the curve is

$$\pi - 2\frac{\sqrt{1 - e^2 \sin^2 a}}{\sin a} \int_0^\kappa \frac{\operatorname{dn} u - \operatorname{dn}^2 u}{1 + \cot^2 a \operatorname{sn}^2 u}\, du, \tag{21}$$

where the MODULUS of the ELLIPTIC FUNCTION is

$$k = \frac{e \cos a}{\sqrt{1 - e^2 \sin^2 a}} \tag{22}$$

(Forsyth 1960, p. 446).

see also ELLIPSOID GEODESIC, OBLATE SPHEROID, SPHERE GEODESIC

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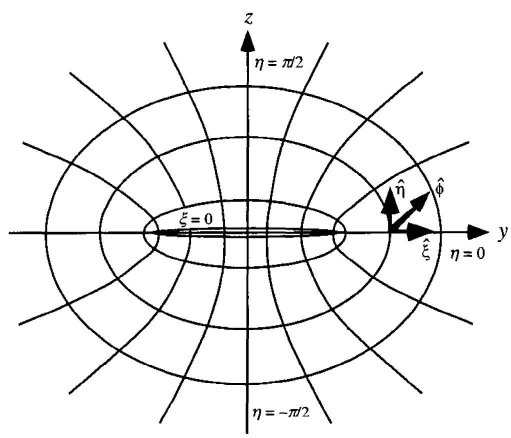
Oblate Spheroidal Coordinates



A system of CURVILINEAR COORDINATES in which two sets of coordinate surfaces are obtained by revolving the curves of the ELLIPTIC CYLINDRICAL COORDINATES about the y-AXIS which is relabeled the z-AXIS. The third set of coordinates consists of planes passing through this axis.

$$x = a \cosh\xi \cos\eta \cos\phi \tag{1}$$

$$y = a \cosh\xi \cos\eta \sin\phi \tag{2}$$

$$z = a \sinh\xi \sin\eta, \tag{3}$$

where $\xi \in [0,\infty)$, $\eta \in [-\pi/2, \pi/2]$, and $\phi \in [0, 2\pi)$. Arfken (1970) uses (u, v, φ) instead of (ξ, η, ϕ). The SCALE FACTORS are

$$h_\xi = a\sqrt{\sinh^2\xi + \sin^2\eta} \tag{4}$$
$$h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta} \tag{5}$$
$$h_\phi = a\cosh\xi\cos\eta. \tag{6}$$

The LAPLACIAN is

$$\begin{aligned}\nabla^2 f &= \frac{1}{a^3(\sinh^2\xi + \sin^2\eta)\cosh\xi\cos\eta} \\ &\quad\times\left[\frac{\partial f}{\partial\xi}\left(a\cosh\xi\cos\eta\frac{\partial f}{\partial\xi}\right)\right. \\ &\quad\left.+\frac{\partial f}{\partial\eta}\left(a\cosh\xi\cos\eta\frac{\partial f}{\partial\eta}\right)+\frac{a^2(\sinh^2\xi+\sin^2\eta)}{a\cosh\xi\cos\eta}\frac{\partial^2 f}{\partial\phi^2}\right] \\ &= \frac{1}{a^3(\sinh^2\xi+\sin^2\eta)\cosh\xi\cos\eta}\left[a\sinh\xi\cos\eta\frac{\partial f}{\partial\xi}\right. \\ &\quad a\cosh\xi\cos\eta\frac{\partial^2 f}{\partial\xi^2}+a\sinh\xi\cos\eta\frac{\partial f}{\partial\eta} \\ &\quad\left.+a\cosh\xi\cos\eta\frac{\partial^2 f}{\partial\eta^2}\right]+\frac{1}{a^2(\sinh^2\xi+\sin^2\eta)}\frac{\partial^2 f}{\partial\phi^2} \\ &= \frac{1}{a^2(\sinh^2\xi+\sin^2\eta)}\left[\frac{1}{\cosh\xi}\frac{\partial}{\partial\xi}\left(\cosh\xi\frac{\partial f}{\partial\xi}\right)\right. \\ &\quad\left.+\frac{1}{\cos\eta}\frac{\partial}{\partial\eta}\left(\cos\eta\frac{\partial f}{\partial\eta}\right)\right]+\frac{1}{a^2(\cosh^2\xi+\cos^2\eta)}\frac{\partial^2 f}{\partial\phi^2} \qquad (7) \\ &= \frac{1}{\sin^2\eta+\sinh^2\xi}\left[(\operatorname{sech}^2\xi\tan^2\eta+\sec^2\tanh^2\xi)\frac{\partial^2}{\partial\phi^2}\right. \\ &\quad\left.+\tanh\xi\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\xi^2}-\tan\eta\frac{\partial}{\eta}+\frac{\partial^2}{\eta^2}\right]. \qquad (8)\end{aligned}$$

An alternate form useful for "two-center" problems is defined by

$$\xi_1 = \sinh\xi \tag{9}$$
$$\xi_1' = \cosh\xi \tag{10}$$
$$\xi_2 = \cos\eta \tag{11}$$
$$\xi_3 = \phi, \tag{12}$$

where $\xi_1 \in [1,\infty]$, $\xi_2 \in [-1,1]$, and $\xi_3 \in [0,2\pi)$. In these coordinates,

$$y = a\xi_1'\xi_2\sin\xi_3 \tag{13}$$
$$z = a\sqrt{({\xi_1'}^2-1)(1-{\xi_2}^2)} \tag{14}$$
$$x = a\xi_1'\xi_2\cos\xi_3 \tag{15}$$

(Abramowitz and Stegun 1972). The SCALE FACTORS are

$$h_{\xi_1} = a\sqrt{\frac{{\xi_1}^2-{\xi_2}^2}{{\xi_1}^2-1}} \tag{16}$$
$$h_{\xi_2} = a\sqrt{\frac{{\xi_1}^2-{\xi_2}^2}{1-{\xi_2}^2}} \tag{17}$$
$$h_{\xi_3} = a\xi\eta, \tag{18}$$

and the LAPLACIAN is

$$\begin{aligned}\nabla^2 f = \frac{1}{a^2}&\left\{\frac{1}{{\xi_1}^2+{\xi_2}^2}\frac{\partial}{\partial\xi_1}\left[({\xi_1}^2+1)\frac{\partial f}{\partial\xi_1}\right]\right. \\ &+\frac{1}{{\xi_1}^2+{\xi_2}^2}\frac{\partial}{\partial\xi_2}\left[(1-{\xi_2}^2)\frac{\partial f}{\partial\xi_2}\right] \\ &\left.+\frac{1}{({\xi_1}^2+1)(1-{\xi_2}^2)}\frac{\partial^2 f}{\partial{\xi_3}^2}\right\}. \qquad (19)\end{aligned}$$

The HELMHOLTZ DIFFERENTIAL EQUATION is separable.

see also HELMHOLTZ DIFFERENTIAL EQUATION—OBLATE SPHEROIDAL COORDINATES, LATITUDE, LONGITUDE, PROLATE SPHEROIDAL COORDINATES, SPHERICAL COORDINATES

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Oblate Spheroidal Wave Function

The wave equation in OBLATE SPHEROIDAL COORDINATES is

$$\begin{aligned}\nabla^2\Phi + k^2\Phi &= \frac{\partial}{\partial\xi_1}\left[({\xi_1}^2+1)\frac{\partial\Phi}{\partial\xi_1}\right] \\ &\quad+\frac{\partial}{\partial\xi_2}\left[(1-{\xi_2}^2)\frac{\partial\Phi}{\partial\xi_2}\right]+\frac{{\xi_1}^2+{\xi_2}^2}{({\xi_1}^2+1)(1-{x_2}^2)}\frac{\partial^2\Phi}{\partial\phi^2} \\ &\quad+c^2({\xi_1}^2+{\xi_2}^2)\Phi = 0, \qquad (1)\end{aligned}$$

where

$$c \equiv \tfrac{1}{2}ak. \tag{2}$$

Substitute in a trial solution

$$\Phi = R_{mn}(c,\xi_1)S_{mn}(c,\xi_2)\begin{matrix}\cos\\ \sin\end{matrix}(m\phi). \tag{3}$$

The radial differential equation is

$$\begin{aligned}&\frac{d}{d\xi_2}\left[(1+{\xi_2}^2)\frac{d}{d\xi_2}S_{mn}(c,\xi_2)\right] \\ &\quad-\left(\lambda_{mn}-c^2{\xi_2}^2+\frac{m^2}{1+{\xi_2}^2}\right)R_{mn}(c,\xi_2) = 0, \qquad (4)\end{aligned}$$

and the angular differential equation is

$$\frac{d}{d\xi_2}\left[(1-{\xi_2}^2)\frac{d}{d\xi_2}S_{mn}(c,\xi_2)\right] - \left(\lambda_{mn} - c^2{\xi_2}^2 + \frac{m^2}{1-{\xi_2}^2}\right)R_{mn}(c,\xi_2) = 0 \quad (5)$$

(Abramowitz and Stegun 1972, pp. 753–755).

see also PROLATE SPHEROIDAL WAVE FUNCTION

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Spheroidal Wave Functions." Ch. 21 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 751–759, 1972.

Oblateness

see FLATTENING

Oblique Angle

An ANGLE which is not a RIGHT ANGLE.

Oblong Number

see PRONIC NUMBER

Obstruction

Obstruction theory studies the extentability of MAPS using algebraic GADGETS. While the terminology rapidly becomes technical and convoluted (as Iyanaga and Kawada note, "It is extremely difficult to discuss higher obstructions in general since they involve many complexities"), the ideas associated with obstructions are very important in modern ALGEBRAIC TOPOLOGY.

see also ALGEBRAIC TOPOLOGY, CHERN CLASS, EILENBERG-MAC LANE SPACE, STIEFEL-WHITNEY CLASS

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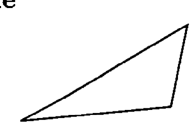
Obtuse Angle

An ANGLE greater than $\pi/2$ RADIANS (90°).

see also ACUTE ANGLE, OBTUSE TRIANGLE, RIGHT ANGLE, STRAIGHT ANGLE

Obtuse Triangle

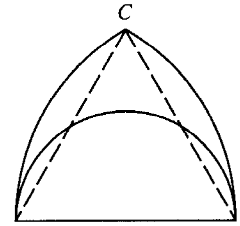


An obtuse triangle is a TRIANGLE in which one of the ANGLES is an OBTUSE ANGLE. (Obviously, only a single ANGLE in a TRIANGLE can be OBTUSE or it wouldn't be a TRIANGLE.) A triangle must be either obtuse, ACUTE, or RIGHT.

A famous problem is to find the chance that three points picked randomly in a PLANE are the VERTICES of an obtuse triangle (Eisenberg and Sullivan 1996). Unfortunately, the solution of the problem depends on the procedure used to pick the "random" points (Portnoy 1994). In fact, it is impossible to pick random variables which are uniformly distributed in the plane (Eisenberg and Sullivan 1996). Guy (1993) gives a variety of solutions to the problem. Woolhouse (1886) solved the problem by picking uniformly distributed points in the unit DISK, and obtained

$$P_2 = 1 - \left(\frac{4}{\pi^2} - \frac{1}{8}\right) = \frac{9}{8} - \frac{4}{\pi^2} = 0.719715\ldots. \quad (1)$$

The problem was generalized by Hall (1982) to n-D BALL TRIANGLE PICKING, and Buchta (1986) gave closed form evaluations for Hall's integrals.



Lewis Carroll (1893) posed and gave another solution to the problem as follows. Call the longest side of a TRIANGLE AB, and call the DIAMETER $2r$. Draw arcs from A and B of RADIUS $2r$. Because the longest side of the TRIANGLE is defined to be AB, the third VERTEX of the TRIANGLE must lie within the region $ABCA$. If the third VERTEX lies within the SEMICIRCLE, the TRIANGLE is an obtuse triangle. If the VERTEX lies *on* the SEMICIRCLE (which will happen with probability 0), the TRIANGLE is a RIGHT TRIANGLE. Otherwise, it is an ACUTE TRIANGLE. The chance of obtaining an obtuse triangle is then the ratio of the AREA of the SEMICIRCLE to that of $ABCA$. The AREA of $ABCA$ is then twice the AREA of a SECTOR minus the AREA of the TRIANGLE.

$$A_{\text{whole figure}} = 2\left(\frac{4\pi r^2}{6}\right) - \sqrt{3}\,r^2 = r^2(\tfrac{4}{3}\pi - \sqrt{3}). \quad (2)$$

Therefore,

$$P = \frac{\frac{1}{2}\pi r^2}{r^2(\frac{4}{3}\pi - \sqrt{3})} = \frac{3\pi}{8\pi - 6\sqrt{3}} = 0.63938\ldots. \quad (3)$$

Let the VERTICES of a triangle in n-D be NORMAL (GAUSSIAN) variates. The probability that a Gaussian triangle in n-D is obtuse is

$$\begin{aligned} P_n &= \frac{3\Gamma(n)}{\Gamma^2(\frac{1}{2}n)}\int_0^{1/3}\frac{x^{(n-2)/2}}{(1+x)^n}\,dx \\ &= \frac{3\Gamma(n)}{\Gamma^2(\frac{1}{2}n)2^{n-1}}\int_0^{\pi/3}\sin^{n-1}\theta\,d\theta \\ &= \frac{6\Gamma(n)\,{}_2F_1(\frac{1}{2}n, n, 1+\frac{1}{2}n; -\frac{1}{3})}{3^{n/2}n\Gamma^2(\frac{1}{2}n)}, \end{aligned} \quad (4)$$

where $\Gamma(n)$ is the GAMMA FUNCTION and ${}_2F_1(a, b; c; x)$ is the HYPERGEOMETRIC FUNCTION. For EVEN $n \equiv 2k$,

$$P_{2k} = 3 \sum_{j=k}^{2k-1} \binom{2k-1}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{2k-1-j} \tag{5}$$

(Eisenberg and Sullivan 1996). The first few cases are explicitly

$$P_2 = \tfrac{3}{4} = 0.75 \tag{6}$$

$$P_3 = 1 - \frac{3\sqrt{3}}{4\pi} = 0.586503\ldots \tag{7}$$

$$P_4 = \tfrac{15}{32} = 0.46875\ldots \tag{8}$$

$$P_5 = 1 - \frac{9\sqrt{3}}{8\pi} = 0.379755\ldots. \tag{9}$$

see also ACUTE ANGLE, ACUTE TRIANGLE, BALL TRIANGLE PICKING, OBTUSE ANGLE, RIGHT TRIANGLE, TRIANGLE

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Ochoa Curve

The ELLIPTIC CURVE

$$3Y^2 = 2X^3 + 386X^2 + 256X - 58195,$$

given in Weierstraß form as

$$y^2 = x^3 - 440067x + 106074110.$$

The complete set of solutions to this equation consists of $(x, y) = (-761, 504)$, $(-745, 4520)$, $(-557, 13356)$, $(-446, 14616)$, $(-17, 10656)$, $(91, 8172)$, $(227, 4228)$, $(247, 3528)$, $(271, 2592)$, $(455, 200)$, $(499, 3276)$, $(523, 4356)$, $(530, 4660)$, $(599, 7576)$, $(751, 14112)$, $(1003, 25956)$, $(1862, 75778)$, $(3511, 204552)$, $(5287, 381528)$, $(23527, 3607272)$, $(64507, 16382772)$, $(100102, 31670478)$, and $(1657891, 2134685628)$ (Stroeker and de Weger 1994).

References

Guy, R. K. "The Ochoa Curve." *Crux Math.* **16**, 65–69, 1990.

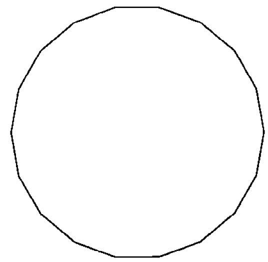
Ochoa Melida, J. "La ecuacion diofántica $b_0y^3 - b_1y^2 + b_2y - b_3 = z^2$." *Gaceta Math.* 139–141, 1978.

Stroeker, R. J. and de Weger, B. M. M. "On Elliptic Diophantine Equations that Defy Thue's Method: The Case of the Ochoa Curve." *Experiment. Math.* **3**, 209–220, 1994.

Octacontagon

An 80-sided POLYGON.

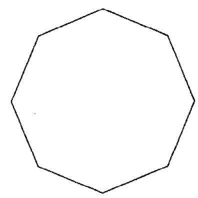
Octadecagon



An 18-sided POLYGON, sometimes also called an OCTAKAIDECAGON.

see also POLYGON, REGULAR POLYGON, TRIGONOMETRY VALUES—$\pi/18$

Octagon



The regular 8-sided POLYGON. The INRADIUS r, CIRCUMRADIUS R, and AREA A can be computed directly from the formulas for a general regular POLYGON with side length s and $n = 8$ sides,

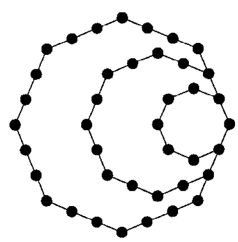
$$r = \tfrac{1}{2}s\cot\left(\frac{\pi}{8}\right) = \tfrac{1}{2}(1 + \sqrt{2})s \tag{1}$$

$$R = \tfrac{1}{2}s\csc\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{4 + 2\sqrt{2}}\,s \tag{2}$$

$$A = \tfrac{1}{4}ns^2\cot\left(\frac{\pi}{8}\right) = 2(1 + \sqrt{2})s^2. \tag{3}$$

see also OCTAHEDRON, POLYGON, REGULAR POLYGON, TRIGONOMETRY VALUES—$\pi/8$

Octagonal Number



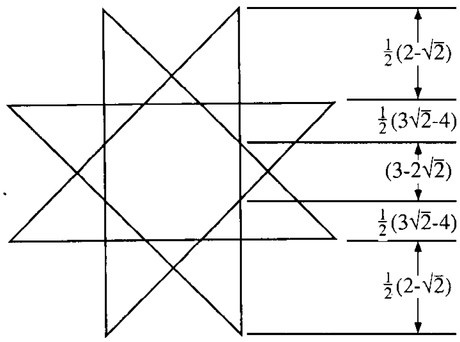
A POLYGONAL NUMBER of the form $n(3n - 2)$. The first few are 1, 8, 21, 40, 65, 96, 133, 176, ... (Sloane's A000567). The GENERATING FUNCTION for the octagonal numbers is

$$\frac{x(5x + 1)}{(1 - x)^3} = x + 8x^2 + 21x^3 + 40x^4 + \ldots.$$

References

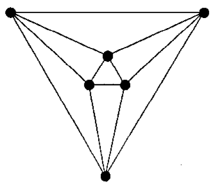
Sloane, N. J. A. Sequence A000567/M4493 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Octagram



The STAR POLYGON $\{8,3\}$.

Octahedral Graph



The POLYHEDRAL GRAPH having the topology of the OCTAHEDRON.

see also CUBICAL GRAPH, DODECAHEDRAL GRAPH, ICOSAHEDRAL GRAPH, OCTAHEDRON, TETRAHEDRAL GRAPH

Octahedral Group

The POINT GROUP of symmetries of the OCTAHEDRON, denoted O_h. It is also the symmetry group of the CUBE, CUBOCTAHEDRON, and TRUNCATED OCTAHEDRON. It has symmetry operations E, $8C_3$, $6C_4$, $6C_2$, $3C_2 = C_4^2$, i, $6S_4$, $8S_6$, $3\sigma_h$, and $6\sigma_4$ (Cotton 1990).

see also CUBE, CUBOCTAHEDRON, ICOSAHEDRAL GROUP, OCTAHEDRON, POINT GROUPS, TETRAHEDRAL GROUP, TRUNCATED OCTAHEDRON

References

Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 47–49, 1990.

Lomont, J. S. "Octahedral Group." §3.10.D in *Applications of Finite Groups.* New York: Dover, p. 81, 1987.

Octahedral Number

A FIGURATE NUMBER which is the sum of two consecutive PYRAMIDAL NUMBERS,

$$O_n = P_{n-1} + P_n = \tfrac{1}{3}n(2n^2+1).$$

The first few are 1, 6, 19, 44, 85, 146, 231, 344, 489, 670, 891, 1156, ... (Sloane's A005900). The GENERATING FUNCTION for the octahedral numbers is

$$\frac{x(x+1)^2}{(x-1)^4} = x + 6x^2 + 19x^3 + 44x^4 + \ldots.$$

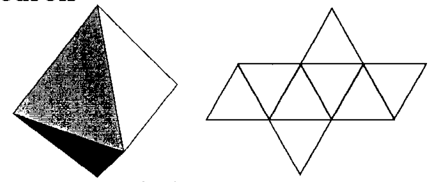
see also TRUNCATED OCTAHEDRAL NUMBER

References

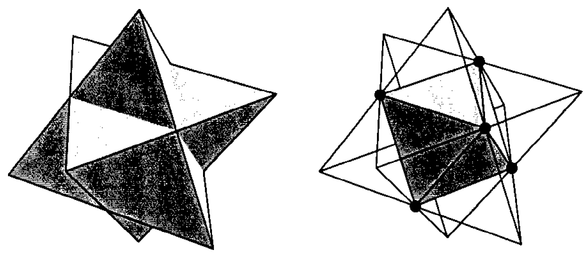
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 50, 1996.

Sloane, N. J. A. Sequence A005900/M4128 in "An On-Line Version of the Encyclopedia of Integer Sequences."

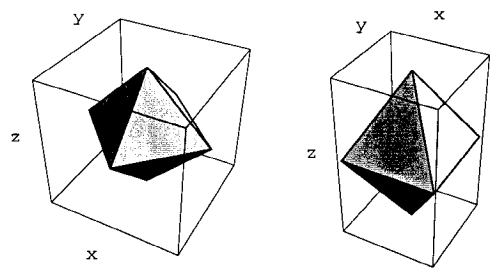
Octahedron



A PLATONIC SOLID (P_3) with six VERTICES, 12 EDGES, and eight equivalent EQUILATERAL TRIANGULAR faces ($8\{3\}$), given by the SCHLÄFLI SYMBOL $\{3,4\}$. It is also UNIFORM POLYHEDRON U_5 with the WYTHOFF SYMBOL $4\,|\,2\,3$. Its DUAL POLYHEDRON is the CUBE. Like the CUBE, it has the O_h OCTAHEDRAL GROUP of symmetries. The octahedron can be STELLATED to give the STELLA OCTANGULA.



The solid bounded by the two TETRAHEDRA of the STELLA OCTANGULA (left figure) is an octahedron (right figure; Ball and Coxeter 1987).

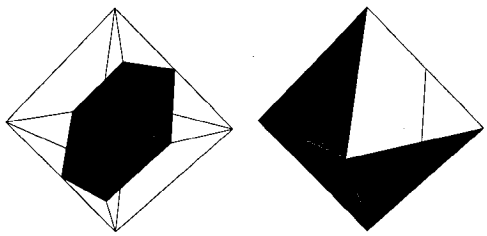


In one orientation (left figure), the VERTICES are given by $(\pm1,0,0)$, $(0,\pm1,0)$, $(0,0,\pm1)$. In another orientation (right figure), the vertices are $(\pm1,\pm1,0)$ and $(0,0,\pm\sqrt{3})$. In the latter, the constituent TRIANGLES are specified by

$$\begin{aligned}
T_1 &= \{(-1,-1,0),(1,-1,0),(0,0,\sqrt{3})\}\\
T_2 &= \{(-1,-1,0),(1,-1,0),(0,0,-\sqrt{3})\}\\
T_3 &= \{(-1,1,0),(1,1,0),(0,0,\sqrt{3})\}\\
T_4 &= \{(-1,1,0),(1,1,0),(0,0,-\sqrt{3})\}\\
T_5 &= \{(1,-1,0),(1,1,0),(0,0,\sqrt{3})\}\\
T_6 &= \{(-1,-1,0),(-1,1,0),(0,0,\sqrt{3})\}\\
T_7 &= \{(1,-1,0),(1,1,0),(0,0,-\sqrt{3})\}\\
T_8 &= \{(-1,-1,0),(-1,1,0),(0,0,-\sqrt{3})\}.
\end{aligned}$$

The face planes are $\pm x \pm y \pm z = 1$, so a solid octahedron is given by the equation

$$|x| + |y| + |z| \leq 1. \tag{1}$$



A plane PERPENDICULAR to a C_3 axis of an octahedron cuts the solid in a regular HEXAGONAL CROSS-SECTION (Holden 1991, pp. 22–23). Since there are four such axes, there are four possibly HEXAGONAL CROSS-SECTIONS. Faceted forms are the CUBOCTATRUNCATED CUBOCTAHEDRON and TETRAHEMIHEXAHEDRON.

Let an octahedron be length a on a side. The height of the top VERTEX from the square plane is also the CIRCUMRADIUS

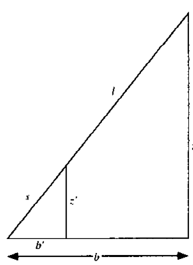
$$R = \sqrt{a^2 - d^2}, \tag{2}$$

where

$$d = \tfrac{1}{2}\sqrt{2}\,a \tag{3}$$

is the diagonal length, so

$$R = \sqrt{a^2 - \tfrac{1}{2}a^2} = \tfrac{1}{2}\sqrt{2}\,a \approx 0.70710a. \tag{4}$$



Now compute the INRADIUS.

$$\ell = \tfrac{1}{2}\sqrt{3}\,a \tag{5}$$

$$b = \tfrac{1}{2}a \tag{6}$$

$$s = \tfrac{1}{2}a\tan 30° = \frac{a}{2\sqrt{3}}, \tag{7}$$

so

$$\frac{s}{\ell} = \frac{1}{2\sqrt{3}}\frac{2}{\sqrt{3}} = \tfrac{1}{3}. \tag{8}$$

Now use similar TRIANGLES to obtain

$$b' = \frac{s}{\ell}b = \tfrac{1}{6}a \tag{9}$$

$$z' = \frac{s}{\ell}z = \frac{a}{3\sqrt{2}} \tag{10}$$

$$x = b - b' = \tfrac{1}{2}a - \tfrac{1}{6}a = \tfrac{1}{3}a, \tag{11}$$

so the INRADIUS is

$$r = \sqrt{x^2 + z'^2} = a\sqrt{\tfrac{1}{9} + \tfrac{1}{18}} = \tfrac{1}{6}\sqrt{6}\,a \approx 0.40824a. \tag{12}$$

The INTERRADIUS is

$$\rho = \tfrac{1}{2}a = 0.5a. \tag{13}$$

The AREA of one face is the AREA of an EQUILATERAL TRIANGLE

$$A = \tfrac{1}{4}\sqrt{3}\,a^2. \tag{14}$$

The volume is two times the volume of a square-base pyramid,

$$V = 2(\tfrac{1}{3}a^2R) = 2(\tfrac{1}{3})(a^2)(\tfrac{1}{2}\sqrt{2}\,a) = \tfrac{1}{3}\sqrt{2}\,a^3. \tag{15}$$

The DIHEDRAL ANGLE is

$$\alpha = \cos^{-1}(-\tfrac{1}{3}) \approx 70.528779°. \tag{16}$$

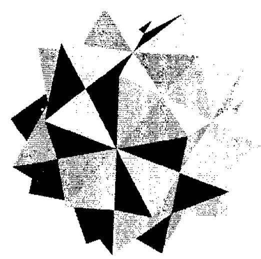
see also OCTAHEDRAL GRAPH, OCTAHEDRAL GROUP, OCTAHEDRON 5-COMPOUND, STELLA OCTANGULA, TRUNCATED OCTAHEDRON

References

Davie, T. "The Octahedron." `http://www.dcs.st-and.ac.uk/~ad/mathrecs/polyhedra/octahedron.html`.

Holden, A. *Shapes, Space, and Symmetry.* New York: Dover, 1991.

Octahedron 5-Compound



A POLYHEDRON COMPOUND composed of five OCTAHEDRA occupying the VERTICES of an ICOSAHEDRON. The 30 VERTICES of the compound form an ICOSIDODECAHEDRON (Ball and Coxeter 1987).

see also ICOSIDODECAHEDRON, OCTAHEDRON, POLYHEDRON COMPOUND

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 135 and 137, 1987.

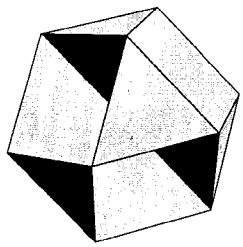
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 137–138, 1989.

Wenninger, M. J. *Polyhedron Models.* New York: Cambridge University Press, p. 43, 1989.

Octahemioctacron

The DUAL POLYHEDRON of the OCTAHEMIOCTAHEDRON.

Octahemioctahedron



The UNIFORM POLYHEDRON U_3, also called the OCTATETRAHEDRON, whose DUAL POLYHEDRON is the OCTAHEMIOCTACRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,3\,|\,3$. Its faces are $8\{3\}+4\{6\}$. It is a FACETED CUBOCTAHEDRON. For unit edge length, its CIRCUMRADIUS is

$$R = 1.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 103, 1989.

Octakaidecagon

see OCTADECAGON

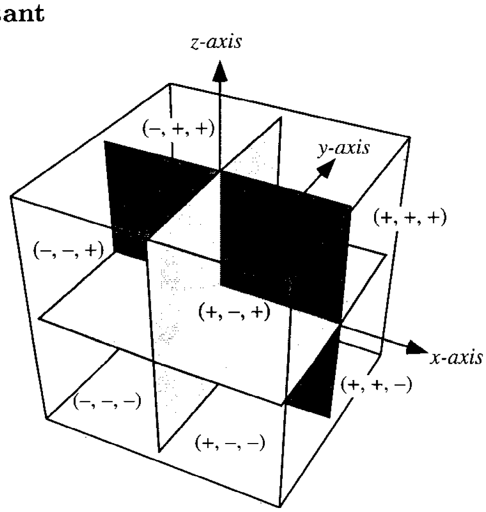
Octal

The base 8 notational system for representing REAL NUMBERS. The digits used are 0, 1, 2, 3, 4, 5, 6, and 7, so that 8_{10} (8 in base 10) is represented as 10_8 $(10 = 1 \cdot 8^1 + 0 \cdot 8^0)$ in base 8.

see also BASE (NUMBER), BINARY, DECIMAL, HEXADECIMAL, QUATERNARY, TERNARY

References
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 9–10, 1991.
Weisstein, E. W. "Bases." http://www.astro.virginia.edu/~eww6n/math/notebooks/Bases.m.

Octant



One of the eight regions of SPACE defined by the eight possible combinations of SIGNS $(\pm,\pm,\pm)$ for x, y, and z.

see also QUADRANT

Octatetrahedron

see OCTAHEMIOCTAHEDRON

Octic Surface

An ALGEBRAIC SURFACE of degree eight. The maximum number of ORDINARY DOUBLE POINTS known to exist on an octic surface is 168 (the ENDRASS OCTICS), although the rigorous upper bound is 174.

see also ALGEBRAIC SURFACE, ENDRASS OCTIC

Octillion

In the American system, 10^{27}.

see also LARGE NUMBER

Octodecillion

In the American system, 10^{57}.

see also LARGE NUMBER

Octonion

see CAYLEY NUMBER

Odd Function

An odd function is a function for which $f(x) = -f(-x)$. An EVEN FUNCTION times an odd function is odd.

Odd Number

An INTEGER of the form $N = 2n+1$, where n is an INTEGER. The odd numbers are therefore ..., -3, -1, 1, 3, 5, 7, ... (Sloane's A005408), which are also the GNOMIC NUMBERS. The GENERATING FUNCTION for the odd numbers is

$$\frac{x(1+x)}{(x-1)^2} = x + 3x^2 + 5x^3 + 7x^4 + \ldots.$$

Since the odd numbers leave a remainder of 1 when divided by two, $N \equiv 1 \pmod 2$ for odd N. Integers which are not odd are called EVEN.

see also EVEN NUMBER, GNOMIC NUMBER, NICOMACHUS'S THEOREM, ODD NUMBER THEOREM, ODD PRIME

References
Sloane, N. J. A. Sequence A005408/M2400 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Odd Number Theorem

The sum of the first n ODD NUMBERS is a SQUARE NUMBER:

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = 2\left[\frac{n(n+1)}{2}\right] - n$$
$$= n(n+1) - n = n^2.$$

see also NICOMACHUS'S THEOREM

Odd Order Theorem

see FEIT-THOMPSON THEOREM

Odd Prime

Any PRIME NUMBER other than 2 (which is the only EVEN PRIME).

see also PRIME NUMBER

Odd Sequence

A SEQUENCE of n 0s and 1s is called an odd sequence if each of the n SUMS $\sum_{i=1}^{n-k} a_i a_{i+k}$ for $k = 0, 1, \ldots, n-1$.

References

Guy, R. K. "Odd Sequences." §E38 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 238–239, 1994.

Odds

Betting odds are written in the form $r : s$ ("r to s") and correspond to the probability of winning $P = s/(r+s)$. Therefore, given a probability P, the odds of winning are $(1/P) - 1 : 1$.

see also FRACTION, RATIO, RATIONAL NUMBER

References

Kraitchik, M. "The Horses." §6.17 in *Mathematical Recreations.* New York: W. W. Norton, pp. 134–135, 1942.

ODE

see ORDINARY DIFFERENTIAL EQUATION

Offset Rings

see SURFACE OF REVOLUTION

Ogive

Any cumulative frequency curve.

see also HISTOGRAM

References

Kenney, J. F. and Keeping, E. S. "Ogive Curves." §2.7 in *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 29–31, 1962.

Oldknow Points

The PERSPECTIVE CENTERS of a triangle and the TANGENTIAL TRIANGLES of its inner and outer SODDY CIRCLES, given by

$$Ol = I + 2Ge$$
$$Ol' = I - 2Ge,$$

where I is the INCENTER and Ge is the GERGONNE POINT.

see also GERGONNE POINT, INCENTER, PERSPECTIVE CENTER, SODDY CIRCLES, TANGENTIAL TRIANGLE

References

Oldknow, A. "The Euler-Gergonne-Soddy Triangle of a Triangle." *Amer. Math. Monthly* **103**, 319–329, 1996.

Omega Constant

$$W(1) \equiv 0.5671432904\ldots, \tag{1}$$

where $W(x)$ is LAMBERT'S W-FUNCTION. It is available in *Mathematica*® (Wolfram Research, Champaign, IL) using the function `ProductLog[1]`. $W(1)$ can be considered a sort of "GOLDEN RATIO" for exponentials since

$$\exp[-W(1)] = W(1), \tag{2}$$

giving

$$\ln\left[\frac{1}{W(1)}\right] = W(1). \tag{3}$$

see also GOLDEN RATIO, LAMBERT'S W-FUNCTION

References

Corless, R. M.; Gonnet, G. H.; Hare, D. E. G.; and Jeffrey, D. J. "On Lambert's W Function." `ftp://watdragon.uwaterloo.ca/cs-archive/CS-93-03/W.ps.Z.`

Plouffe, S. "The Omega Constant or $W(1)$." `http://lacim.uqam.ca/piDATA/omega.txt.`

Omega Function

see LAMBERT'S W-FUNCTION

Omino

see POLYOMINO

Omnific Integer

The appropriate notion of INTEGER for SURREAL NUMBERS.

O'Nan Group

The SPORADIC GROUP $O'N$.

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/ON.html.`

Onduloid

see UNDULOID

One

see 1

One-Form

A linear, real-valued FUNCTION of VECTORS such that $\omega^1(\mathbf{v}) \mapsto \mathbb{R}$. VECTORS and one-forms are DUAL to each other because VECTORS are CONTRAVARIANT ("KETS": $|\psi\rangle$) and one-forms are COVARIANT VECTORS ("BRAS": $\langle\phi|$), so

$$\omega^1(\mathbf{v}) \equiv \mathbf{v}(\omega^1) \equiv \langle \omega^1, \mathbf{v} \rangle = \langle\phi|\psi\rangle.$$

The operation of applying the one-form to a VECTOR $\omega^1(\mathbf{v})$ is called CONTRACTION.

see also ANGLE BRACKET, BRA, DIFFERENTIAL k-FORM, KET

One-Mouth Theorem

Except for convex polygons, every SIMPLE POLYGON has at least one MOUTH.

see also MOUTH, PRINCIPAL VERTEX, TWO-EARS THEOREM

References

Toussaint, G. "Anthropomorphic Polygons." *Amer. Math. Monthly* **122**, 31–35, 1991.

One-Ninth Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $\lambda_{m,n}$ be CHEBYSHEV CONSTANTS. Schönhage (1973) proved that

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = \tfrac{1}{3}. \tag{1}$$

It was conjectured that

$$\Lambda \equiv \lim_{n\to\infty} (\lambda_{n,n})^{1/n} = \tfrac{1}{9}. \tag{2}$$

Carpenter *et al.* (1984) obtained

$$\Lambda = 0.1076539192\ldots \tag{3}$$

numerically. Gonchar and Rakhmanov (1980) showed that the limit exists and disproved the 1/9 conjecture, showing that Λ is given by

$$\Lambda = \exp\left[-\frac{\pi K(\sqrt{1-c^2}\,)}{K(c)}\right], \tag{4}$$

where K is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND, and $c = 0.9089085575485414\ldots$ is the PARAMETER which solves

$$K(k) = 2E(k), \tag{5}$$

and E is the complete ELLIPTIC INTEGRAL OF THE SECOND KIND. This gives the value for Λ computed by Carpenter *et al.* (1984) Λ is also given by the unique POSITIVE ROOT of

$$f(z) = \tfrac{1}{8}, \tag{6}$$

where

$$f(z) \equiv \sum_{j=1}^{\infty} a_j z^j \tag{7}$$

and

$$a_j = \left|\sum_{d|j} (-1)^d d\right| \tag{8}$$

(Gonchar and Rakhmanov 1980). a_j may also be computed by writing j as

$$j = 2^m {p_1}^{m_1} {p_2}^{m_2} \cdots {p_k}^{m_k}, \tag{9}$$

where $m \geq 0$ and $m_i \geq 1$, then

$$a_j = |2^{m+1} - 3| \frac{{p_1}^{m_1+1} - 1}{p_1 - 1} \frac{{p_2}^{m_2+1} - 1}{p_2 - 1} \cdots \frac{{p_k}^{m_k+1} - 1}{p_k - 1} \tag{10}$$

(Gonchar 1990). Yet another equation for Λ is due to Magnus (1986). Λ is the unique solution with $x \in (0,1)$ of

$$\sum_{k=0}^{\infty} (2k+1)^2 (-x)^{k(k+1)/2} = 0, \tag{11}$$

an equation which had been studied and whose root had been computed by Halphen (1886). It has therefore been suggested (Varga 1990) that the constant be called the HALPHEN CONSTANT. $1/\Lambda$ is sometimes called VARGA'S CONSTANT.

see also CHEBYSHEV CONSTANTS, HALPHEN CONSTANT, VARGA'S CONSTANT

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/onenin/onenin.html`.

Carpenter, A. J.; Ruttan, A.; and Varga, R. S. "Extended Numerical Computations on the '1/9' Conjecture in Rational Approximation Theory." In *Rational approximation and interpolation (Tampa, Fla., 1983)* (Ed. P. R. Graves-Morris, E. B. Saff, and R. S. Varga). New York: Springer-Verlag, pp. 383–411, 1984.

Cody, W. J.; Meinardus, G.; and Varga, R. S. "Chebyshev Rational Approximations to e^{-x} in $[0,+\infty)$ and Applications to Heat-Conduction Problems." *J. Approx. Th.* **2**, 50–65, 1969.

Dunham, C. B. and Taylor, G. D. "Continuity of Best Reciprocal Polynomial Approximation on $[0,\infty)$." *J. Approx. Th.* **30**, 71–79, 1980.

Gonchar, A. A. "Rational Approximations of Analytic Functions." *Amer. Math. Soc. Transl. Ser. 2* **147**, 25–34, 1990.

Gonchar, A. A. and Rakhmanov, E. A. "Equilibrium Distributions and Degree of Rational Approximation of Analytic Functions." *Math. USSR Sbornik* **62**, 305–348, 1980.

Magnus, A. P. "On Freud's Equations for Exponential Weights, Papers Dedicated to the Memory of Géza Freud." *J. Approx. Th.* **46**, 65–99, 1986.
Rahman, Q. I. and Schmeisser, G. "Rational Approximation to the Exponential Function." In *Padé and Rational Approximation, (Proc. Internat. Sympos., Univ. South Florida, Tampa, Fla., 1976)* (Ed. E. B. Saff and R. S. Varga). New York: Academic Press, pp. 189–194, 1977.
Schönhage, A. "Zur rationalen Approximierbarkeit von e^{-x} über $[0,\infty)$." *J. Approx. Th.* **7**, 395–398, 1973.
Varga, R. S. *Scientific Computations on Mathematical Problems and Conjectures.* Philadelphia, PA: SIAM, 1990.

One-to-One

Let f be a FUNCTION defined on a SET S and taking values in a set T. Then f is said to be one-to-one (a.k.a. an INJECTION or EMBEDDING) if, whenever $f(x) = f(y)$, it must be the case that $x = y$. In other words, f is one-to-one if it MAPS distinct objects to distinct objects.

If the function is a linear OPERATOR which assigns a unique MAP to each value in a VECTOR SPACE, it is called one-to-one. Specifically, given a VECTOR SPACE $\mathbb{V}$ with $\mathbf{X}, \mathbf{Y} \in \mathbb{V}$, then a TRANSFORMATION T defined on $\mathbb{V}$ is one-to-one if $T(\mathbf{X}) \neq T(\mathbf{Y})$ for all $\mathbf{X} \neq \mathbf{Y}$.

see also BIJECTION, ONTO

One-Way Function

Consider straight-line algorithms over a FINITE FIELD with q elements. Then the ϵ-straight line complexity $C_\epsilon(\phi)$ of a function ϕ is defined as the length of the shortest straight-line algorithm which computes a function f such that $f(x) = x$ is satisfied for at least $(1-\epsilon)q$ elements of F. A function ϕ is straight-line "one way" of range $0 \leq \delta \leq 1$ if ϕ satisfies the properties:

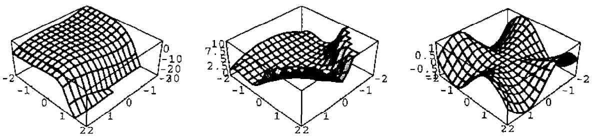
1. There exists an infinite set S of finite fields such that ϕ is defined in every $F \in S$ and ϵ is ONE-TO-ONE in every $F \in S$.
2. For every ϵ such that $0 \leq \epsilon \leq \delta$, $C_\epsilon(\phi^{-1})$ tends to infinity as the cardinality q of F approaches infinity.
3. For every ϵ such that $0 \leq \epsilon \leq \delta$, the "work function" η satisfies

$$\eta \equiv \liminf_{q\to\infty} \eta \equiv \liminf_{q\to\infty} \frac{\ln C\epsilon(\phi^{-1}) - \ln C\epsilon(\phi)}{\ln C\epsilon(\phi)} > 1.$$

It is not known if there is a one-way function with work factor $\eta > (\ln q)^3$.

References
Ziv, J. "In Search of a One-Way Function" §4.1 in *Open Problems in Communication and Computation* (Ed. T. M. Cover and B. Gopinath). New York: Springer-Verlag, pp. 104–105, 1987.

Only Critical Point in Town Test



If there is only one CRITICAL POINT at an EXTREMUM, the CRITICAL POINT must be the EXTREMUM for functions of one variable. There are exceptions for two variables, but none of degree ≤ 4. Such exceptions include

$$z = 3xe^y - x^3 - e^{3y}$$

$$z = x^2(1+y)^3 + y^2$$

$$z = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

(Wagon 1991). This latter function has discontinuous z_{xy} and z_{yx}, and $z_{yx}(0,0) = 1$ and $z_{xy}(0,0) = 1$.

References
Ash, A. M. and Sexton, H. "A Surface with One Local Minimum." *Math. Mag.* **58**, 147–149, 1985.
Calvert, B. and Vamanamurthy, M. K. "Local and Global Extrema for Functions of Several Variables." *J. Austral. Math. Soc.* **29**, 362–368, 1980.
Davies, R. Solution to Problem 1235. *Math. Mag.* **61**, 59, 1988.
Wagon, S. "Failure of the Only-Critical-Point-in-Town Test." §3.4 in *Mathematica in Action.* New York: W. H. Freeman, pp. 87–91 and 228, 1991.

Onto

Let f be a FUNCTION defined on a SET S and taking values in a set T. Then f is said to be onto (a.k.a. a SURJECTION) if, for any $t \in T$, there exists an $s \in S$ for which $t = f(s)$.

Let the function be an OPERATOR which MAPS points in the DOMAIN to every point in the RANGE and let $\mathbb{V}$ be a VECTOR SPACE with $\mathbf{X}, \mathbf{Y} \in \mathbb{V}$. Then a TRANSFORMATION T defined on $\mathbb{V}$ is onto if there is an $\mathbf{X} \in \mathbb{V}$ such that $T(\mathbf{X}) = \mathbf{Y}$ for all $\mathbf{Y}$.

see also BIJECTION, ONE-TO-ONE

Open Disk

An n-D open disk of RADIUS r is the collection of points of distance less than r from a fixed point in EUCLIDEAN n-space.

see also CLOSED DISK, DISK

Open Interval

An INTERVAL which does not include its LIMIT POINTS, denoted (a, b).

see also CLOSED INTERVAL, HALF-CLOSED INTERVAL

Open Map

A MAP which sends OPEN SETS to OPEN SETS.

see also OPEN MAPPING THEOREM

Open Mapping Theorem

There are several flavors of this theorem.

1. A continuous surjective linear mapping between BANACH SPACES is an OPEN MAP.
2. A nonconstant ANALYTIC FUNCTION on a DOMAIN D is an OPEN MAP.

References

Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics.* New York: Springer-Verlag, 1995.

Open Set

A SET is open if every point in the set has a NEIGHBORHOOD lying in the set. An open set of RADIUS r and center $\mathbf{x}_0$ is the set of all points $\mathbf{x}$ such that $|\mathbf{x}-\mathbf{x}_0|<r$, and is denoted $D_r(\mathbf{x}_0)$. In 1-space, the open set is an OPEN INTERVAL. In 2-space, the open set is a DISK. In 3-space, the open set is a BALL.

More generally, given a TOPOLOGY (consisting of a SET X and a collection of SUBSETS T), a SET is said to be open if it is in T. Therefore, while it is not possible for a set to be both finite and open in the TOPOLOGY of the REAL LINE (a single point is a CLOSED SET), it is possible for a more general topological SET to be both finite and open.

The complement of an open set is a CLOSED SET. It is possible for a set to be neither open nor CLOSED, e.g., the interval $(0,1]$.

see also BALL, CLOSED SET, EMPTY SET, OPEN INTERVAL

Operad

A system of parameter chain complexes used for MULTIPLICATION on differential GRADED ALGEBRAS up to HOMOTOPY.

Operand

A mathematical object upon which an OPERATOR acts. For example, in the expression 1×2, the MULTIPLICATION OPERATOR acts upon the operands 1 and 2.

see also OPERAD, OPERATOR

Operational Mathematics

The theory and applications of LAPLACE TRANSFORMS and other INTEGRAL TRANSFORMS.

References

Churchill, R. V. *Operational Mathematics, 3rd ed.* New York: McGraw-Hill, 1958.

Operations Research

A branch of mathematics which encompasses many diverse areas of minimization and optimization. Bronson (1982) describes operations research as being "concerned with the efficient allocation of scarce resources." It includes the CALCULUS OF VARIATIONS, CONTROL THEORY, CONVEX OPTIMIZATION THEORY, DECISION THEORY, GAME THEORY, LINEAR PROGRAMMING, MARKOV CHAINS, network analysis, OPTIMIZATION THEORY, queuing systems, etc. The more modern term for operations research is OPTIMIZATION THEORY.

see also CALCULUS OF VARIATIONS, CONTROL THEORY, CONVEX OPTIMIZATION THEORY, DECISION THEORY, GAME THEORY, LINEAR PROGRAMMING, MARKOV CHAIN, OPTIMIZATION THEORY, QUEUE

References

Bronson, R. *Schaum's Outline of Theory and Problems of Operations Research.* New York: McGraw-Hill, 1982.

Hiller, F. S. and Lieberman, G. J. *Introduction to Operations Research, 5th ed.* New York: McGraw-Hill, 1990.

Trick, M. "Michael Trick's Operations Research Page." `http://mat.gsia.cmu.edu`

Operator

An operator $A: f^{(n)}(I)\mapsto f(I)$ assigns to every function $f\in f^{(n)}(I)$ a function $A(f)\in f(I)$. It is therefore a mapping between two FUNCTION SPACES. If the range is on the REAL LINE or in the COMPLEX PLANE, the mapping is usually called a FUNCTIONAL instead.

see also ABSTRACTION OPERATOR, ADJOINT OPERATOR, ANTILINEAR OPERATOR, BIHARMONIC OPERATOR, BINARY OPERATOR, CASIMIR OPERATOR, CONVECTIVE OPERATOR, D'ALEMBERTIAN OPERATOR, DIFFERENCE OPERATOR, FUNCTIONAL ANALYSIS, HECKE OPERATOR, HERMITIAN OPERATOR, IDENTITY OPERATOR, LAPLACE-BELTRAMI OPERATOR, LINEAR OPERATOR, OPERAND, PERRON-FROBENIUS OPERATOR, PROJECTION OPERATOR, ROTATION OPERATOR, SCATTERING OPERATOR, SELF-ADJOINT OPERATOR, SPECTRUM (OPERATOR), THETA OPERATOR, WAVE OPERATOR

References

Gohberg, I.; Lancaster, P.; and Shivakuar, P. N. (Eds.). *Recent Developments in Operator Theory and Its Applications.* Boston, MA: Birkhäuser, 1996.

Hutson, V. and Pym, J. S. *Applications of Functional Analysis and Operator Theory.* New York: Academic Press, 1980.

Optimization Theory

see OPERATIONS RESEARCH

Or

A term in LOGIC which yields TRUE if any one of a sequence conditions is TRUE, and FALSE if *all* conditions are FALSE. A OR B is denoted $A|B$, $A+B$, or $A\vee B$. The symbol $\vee$ derives from the first letter of the Latin word "vel" meaning "or." The BINARY OR operator has the following TRUTH TABLE.

A	B	$A\vee B$
F	F	F
F	T	T
T	F	T
T	T	T

A product of ORs is called a DISJUNCTION and is denoted

$$\bigvee_{k=1}^{n} A_k.$$

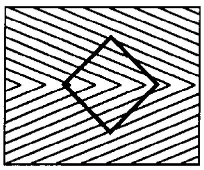
Two BINARY numbers can have the operation OR performed bitwise. This operation is sometimes denoted $A||B$.

see also AND, BINARY OPERATOR, LOGIC, NOT, PREDICATE, TRUTH TABLE, UNION, XOR

Orbifold

The object obtained by identifying any two points of a MAP which are equivalent under some symmetry of the MAP'S GROUP.

Orbison's Illusion



The illusion illustrated above in which the bounding RECTANGLE and inner SQUARE both appear distorted.

see also ILLUSION, MÜLLER-LYER ILLUSION, PONZO'S ILLUSION, VERTICAL-HORIZONTAL ILLUSION

References

Fineman, M. *The Nature of Visual Illusion.* New York: Dover, p. 153, 1996.

Orbit (Group)

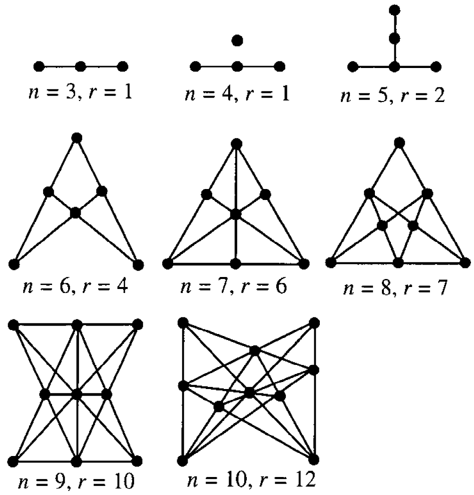
Given a PERMUTATION GROUP G on a set S, the orbit of an element $s \in S$ is the subset of S consisting of elements to which some element G can send s.

Orbit (Map)

The SEQUENCE generated by repeated application of a MAP. The MAP is said to have a closed orbit if it has a finite number of elements.

see also DYNAMICAL SYSTEM, SINK (MAP)

Orchard-Planting Problem



Also known as the TREE-PLANTING PROBLEM. Plant n trees so that there will be r straight rows with k trees in each row. The following table gives $\max(r)$ for various k. $k = 3$ is Sloane's A003035 and $k = 4$ is Sloane's A006065.

n	$k=3$	$k=4$	$k=5$
3	1	—	—
4	1	1	—
5	2	1	1
6	4	1	1
7	6	2	1
8	7	2	1
9	10	3	2
10	12	5	2
11	16	6	2
12	19	7	3
13	[22, 24]	≥ 9	3
14	[26, 27]	≥ 10	4
15	[31, 32]	≥ 12	≥ 6
16	37	≥ 15	≥ 6
17	[40, 42]	≥ 15	≥ 7
18	[46, 48]	≥ 18	≥ 9
19	[52, 54]	≥ 19	≥ 10
20	[57, 60]	≥ 21	≥ 11
21	[64, 67]		
22	[70, 73]		
23	[77, 81]		
24	[85, 88]		
25	[92, 96]		

Sylvester showed that

$$r(k=3) \geq \left\lfloor \tfrac{1}{6}(n-1)(n-2) \right\rfloor,$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION (Ball and Coxeter 1987). Burr, Grünbaum and Sloane (1974) have shown using cubic curves that

$$r(k=3) \leq 1 + \left\lfloor \tfrac{1}{6}n(n-3) \right\rfloor,$$

except for $n = 7$, 11, 16, and 19, and conjecture that the inequality is an equality with the exception of the preceding cases. For $n \geq 4$,

$$r(k=3) \geq \left\lfloor \tfrac{1}{3}\left[\tfrac{1}{2}n(n-1) - \left\lceil \tfrac{3}{7}n \right\rceil\right]\right\rfloor,$$

where $\lceil x \rceil$ is the CEILING FUNCTION.

see also ORCHARD VISIBILITY PROBLEM

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 104–105 and 129, 1987.

Burr, S. A. "Planting Trees." In *The Mathematical Gardner* (Ed. David Klarner). Boston, MA: Prindle, Weber, and Schmidt, pp. 90–99, 1981.

Dudeney, H. E. Problem 435 in *536 Puzzles & Curious Problems.* New York: Scribner, 1967.

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Jackson, J. *Rational Amusements for Winter Evenings.* London, 1821.

Macmillan, R. H. "An Old Problem." *Math. Gaz.* **30**, 109, 1946.

Sloane, N. J. A. Sequences A006065/M0290 and A003035/M0982 in "An On-Line Version of the Encyclopedia of Integer Sequences." `http://www.research.att.com/~njas/sequences/eisonline.html`.

Sloane, N. J. A. and Plouffe, S. Extended entry for M0982 in *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.

Orchard Visibility Problem

A tree is planted at each LATTICE POINT in a circular orchard which has CENTER at the ORIGIN and RADIUS r. If the radius of trees exceeds $1/r$ units, one is unable to see out of the orchard in any direction. However, if the RADII of the trees are $< 1/\sqrt{r^2+1}$, one can see out at certain ANGLES.

see also LATTICE POINT, ORCHARD-PLANTING PROBLEM, VISIBILITY

References

Honsberger, R. "The Orchard Problem." Ch. 4 in *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., pp. 43–52, 1973.

Order (Algebraic Curve)

The order of the POLYNOMIAL defining the curve.

Order (Algebraic Surface)

The order n of an ALGEBRAIC SURFACE is the order of the POLYNOMIAL defining a surface, which can be geometrically interpreted as the maximum number of points in which a line meets the surface.

Order	Surface
3	cubic surface
4	quartic surface
5	quintic surface
6	sextic surface
7	heptic surface
8	octic surface
9	nonic surface
10	decic surface

see also ALGEBRAIC SURFACE

References

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, p. 8, 1986.

Order (Conjugacy Class)

The number of elements of a GROUP in a given CONJUGACY CLASS.

Order (Difference Set)

Let G be GROUP of ORDER h and D be a set of k elements of G. If the set of differences $d_i - d_j$ contains every NONZERO element of G exactly λ times, then D is a (h, k, λ)-difference set in G of order $n = k - \lambda$.

Order (Field)

The number of elements in a FINITE FIELD.

Order (Group)

The number of elements in a GROUP G, denoted $|G|$. The order of an element g of a finite group G is the smallest POWER of n such that $g^n = I$, where I is the IDENTITY ELEMENT. In general, finding the order of the element of a group is at least as hard as factoring (Meijer 1996). However, the problem becomes significantly easier if $|G|$ and the factorization of $|G|$ are known. Under these circumstances, efficient ALGORITHMS are known (Cohen 1993).

see also ABELIAN GROUP, FINITE GROUP

References

Cohen, H. *A Course in Computational Algebraic Number Theory.* New York: Springer-Verlag, 1993.

Meijer, A. R. "Groups, Factoring, and Cryptography." *Math. Mag.* **69**, 103–109, 1996.

Order (Modulo)

For any INTEGER a which is not a multiple of a PRIME p, there exists a smallest exponent $h \geq 1$ such that $a^h \equiv 1 \pmod{p}$ IFF $h|k$. In that case, h is called the order of a modulo p.

see also CARMICHAEL FUNCTION

Order (Ordinary Differential Equation)

An ORDINARY DIFFERENTIAL EQUATION of order n is an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0.$$

Order (Permutation)

see PERMUTATION

Order (Polynomial)

The highest order POWER in a one-variable POLYNOMIAL is known as its order (or sometimes its DEGREE). For example, the POLYNOMIAL

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

is of order n.

Order Statistic

Given a sample of n variates $X_1, \ldots, X_n$, reorder them so that $X'_1 < X'_2 < \ldots < X'_n$. Then the ith order statistic $X^{(i)}$ is defined as X'_i, with the special cases

$$m_n = X^{\langle 1 \rangle} = \min_j(X_j)$$
$$M_n = X^{\langle n \rangle} = \max_j(X_j).$$

A ROBUST ESTIMATION technique based on linear combinations of order statistics is called an L-ESTIMATE.

see also EXTREME VALUE DISTRIBUTION, HINGE, MAXIMUM, MINIMUM, MODE, ORDINAL NUMBER

References
Balakrishnan, N. and Cohen, A. C. *Order Statistics and Inference.* New York: Academic Press, 1991.
David, H. A. *Order Statistics, 2nd ed.* New York: Wiley, 1981.
Gibbons, J. D. and Chakraborti, S. (Eds.). *Nonparametric Statistic Inference, 3rd ed. exp. rev.* New York: Marcel Dekker, 1992.

Order (Vertex)

The number of EDGES meeting at a given node in a GRAPH is called the order of that VERTEX.

Ordered Geometry

A GEOMETRY constructed without reference to measurement. The only primitive concepts are those of points and intermediacy. There are 10 AXIOMS underlying ordered GEOMETRY.

see also ABSOLUTE GEOMETRY, AFFINE GEOMETRY, GEOMETRY

Ordered Pair

A PAIR of quantities (a, b) where ordering is significant, so (a, b) is considered distinct from (b, a) for $a \neq b$.

see also PAIR

Ordered Tree

A ROOTED TREE in which the order of the subtrees is significant. There is a ONE-TO-ONE correspondence between ordered FORESTS with n nodes and BINARY TREES with n nodes.

see also BINARY TREE, FOREST, ROOTED TREE

Ordering

The number of "ARRANGEMENTS" in an ordering of n items is given by either a COMBINATION (order is ignored) or a PERMUTATION (order is significant).

see also ARRANGEMENT, COMBINATION, CUTTING, DERANGEMENT, PARTIAL ORDER, PERMUTATION, SORTING, TOTAL ORDER

Ordering Axioms

The four of HILBERT'S AXIOMS which concern the arrangement of points.

see also CONGRUENCE AXIOMS, CONTINUITY AXIOMS, HILBERT'S AXIOMS, INCIDENCE AXIOMS, PARALLEL POSTULATE

References
Hilbert, D. *The Foundations of Geometry, 2nd ed.* Chicago, IL: Open Court, 1980.
Iyanaga, S. and Kawada, Y. (Eds.). "Hilbert's System of Axioms." §163B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 544–545, 1980.

Ordinal Number

In informal usage, an ordinal number is an adjective which describes the numerical position of an object, e.g., first, second, third, etc.

In technical mathematics, an ordinal number is one of the numbers in Georg Cantor's extension of the WHOLE NUMBERS. The ordinal numbers are $0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+\omega, \omega+\omega+1, \ldots$. Cantor's "smallest" TRANSFINITE NUMBER ω is defined to be the earliest number greater than all WHOLE NUMBERS, and is denoted by Conway and Guy (1996) as $\omega = \{0, 1, \ldots |\}$. The notation of ordinal numbers can be a bit counter-intuitive, e.g., even though $1 + \omega = \omega$, $\omega + 1 > \omega$.

Ordinal numbers have some other rather peculiar properties. The sum of two ordinal numbers can take on two different values, the sum of three can take on five values. The first few terms of this sequence are 2, 5, 13, 33, 81, 193, 449, 33^2, $33 \cdot 81$, 81^2, $81 \cdot 193$, 192^2, ... (Conway and Guy 1996, Sloane's A005348). The sum of n ordinals has either $193^a 81^b$ or $33 \cdot 81^a$ possible answers for $n \geq 15$ (Conway and Guy 1996).

$r \times \omega$ is the same as ω, but $\omega \times r$ is equal to $\underbrace{\omega + \ldots + \omega}_{r}$.

ω^2 is larger than any number of the form $\omega \times r$, ω^3 is larger than ω^2, and so on.

There exist ordinal numbers which cannot be constructed from smaller ones by finite additions, multiplications, and exponentiations. These ordinals obey CANTOR'S EQUATION. The first such ordinal is

$$\epsilon_0 = \underbrace{\omega^{\omega^{\cdot^{\cdot^{\omega}}}}}_{\omega} = 1 + \omega + \omega^\omega + \omega^{\omega^\omega} + \ldots.$$

The next is

$$\epsilon_1 = (\epsilon_0 + 1) + \omega^{\epsilon_0+1} + \omega^{\omega\epsilon_0+1} + \ldots,$$

then follow ϵ_2, ϵ_3, ..., ϵ_ω, $\epsilon_{\omega+1}$, ..., $\epsilon_{\omega\times 2}$, ..., ϵ_{ω^2}, ϵ_{ω^ω}, ..., ϵ_{ϵ_0}, ϵ_{ϵ_0+1}, ..., $\epsilon_{\epsilon_0+\omega}$, ..., $\epsilon_{\epsilon_0+\omega^\omega}$, ..., $\epsilon_{\epsilon_0\times 2}$, ..., ϵ_{ϵ_1}, ..., ϵ_{ϵ_2}, ..., $\epsilon_{\epsilon_\omega}$, ..., $\epsilon_{\epsilon_{\epsilon_0}}$, ..., $\epsilon_{\epsilon_{\epsilon_1}}$, ..., $\epsilon_{\epsilon_{\epsilon_\omega}}$, ..., $\epsilon_{\epsilon_{\epsilon_{\epsilon_0}}}$, ... (Conway and Guy 1996).

see also AXIOM OF CHOICE, CANTOR'S EQUATION, CARDINAL NUMBER, ORDER STATISTIC, POWER SET, SURREAL NUMBER

References

Cantor, G. *Über unendliche, lineare Punktmannigfältigkeiten, Arbeiten zur Mengenlehre aus dem Jahren 1872–1884.* Leipzig, Germany: Teubner-Archiv zur Mathematik, 1884.

Conway, J. H. and Guy, R. K. "Cantor's Ordinal Numbers." In *The Book of Numbers.* New York: Springer-Verlag, pp. 266–267 and 274, 1996.

Sloane, N. J. A. Sequence A005348/M1435 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Ordinary Differential Equation

An ordinary differential equation (frequently abbreviated ODE) is an equality involving a function and its DERIVATIVES. An ODE of order n is an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where $y' = dy/dx$ is a first DERIVATIVE with respect to x and $y^{(n)} = d^n y/dx^n$ is an nth DERIVATIVE with respect to x. An ODE of order n is said to be linear if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = Q(x). \tag{2}$$

A linear ODE where $Q(x) = 0$ is said to be homogeneous. Confusingly, an ODE of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{3}$$

is also sometimes called "homogeneous."

Simple theories exist for first-order (INTEGRATING FACTOR) and second-order (STURM-LIOUVILLE THEORY) ordinary differential equations, and arbitrary ODEs with linear constant COEFFICIENTS can be solved when they are of certain factorable forms. Integral transforms such as the LAPLACE TRANSFORM can also be used to solve classes of linear ODEs. Morse and Feshbach (1953, pp. 667–674) give canonical forms and solutions for second-order ODEs.

While there are many general techniques for analytically solving classes of ODEs, the only practical solution technique for complicated equations is to use numerical methods (Milne 1970). The most popular of these is the RUNGE-KUTTA METHOD, but many others have been developed. A vast amount of research and huge numbers of publications have been devoted to the numerical solution of differential equations, both ordinary and PARTIAL (PDEs) as a result of their importance in fields as diverse as physics, engineering, economics, and electronics.

The solutions to an ODE satisfy EXISTENCE and UNIQUENESS properties. These can be formally established by PICARD'S EXISTENCE THEOREM for certain classes of ODEs. Let a system of first-order ODE be given by

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, t), \tag{4}$$

for $i = 1, \ldots, n$ and let the functions $f_i(x_1, \ldots, x_n, t)$, where $i = 1, \ldots, n$, all be defined in a DOMAIN D of the $(n+1)$-D space of the variables $x_1, \ldots, x_n, t$. Let these functions be continuous in D and have continuous first PARTIAL DERIVATIVES $\partial f_i/\partial x_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ in D. Let $(x_1^0, \ldots, x_n^0)$ be in D. Then there exists a solution of (4) given by

$$x_1 = x_1(t), \ldots, x_n = x_n(t) \tag{5}$$

for $t_0 - \delta < t < t_0 + \delta$ (where $\delta > 0$) satisfying the initial conditions

$$x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0. \tag{6}$$

Furthermore, the solution is unique, so that if

$$x_1 = x_1^*(t), \ldots, x_n = x_n^*(t) \tag{7}$$

is a second solution of (4) for $t_0 - \delta < t < t_0 + \delta$ satisfying (6), then $x_i(t) \equiv x_i^*(t)$ for $t_0 - \delta < t < t_0 + \delta$. Because every nth-order ODE can be expressed as a system of n first-order differential equations, this theorem also applies to the single nth-order ODE.

In general, an nth-order ODE has n linearly independent solutions. Furthermore, any linear combination of LINEARLY INDEPENDENT FUNCTIONS solutions is also a solution.

An exact FIRST-ORDER ODEs is one of the form

$$p(x, y)\,dx + q(x, y)\,dy = 0, \tag{8}$$

where

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \tag{9}$$

An equation of the form (8) with

$$\frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x} \tag{10}$$

is said to be nonexact. If

$$\frac{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}{q} = f(x) \tag{11}$$

in (8), it has an x-dependent integrating factor. If

$$\frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{xp - yq} = f(xy) \tag{12}$$

in (8), it has an xy-dependent integrating factor. If

$$\frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{p} = f(y) \tag{13}$$

in (8), it has a y-dependent integrating factor.

Other special first-order types include cross multiple equations

$$y f(xy)\, dx + x g(xy)\, dy = 0, \tag{14}$$

homogeneous equations

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \tag{15}$$

linear equations

$$\frac{dy}{dx} + p(x)y = q(x), \tag{16}$$

and separable equations

$$\frac{dy}{dx} = X(x)Y(y). \tag{17}$$

Special classes of SECOND-ORDER ODEs include

$$\frac{d^2y}{dx^2} = f(y, y') \tag{18}$$

(x missing) and

$$\frac{d^2y}{dx^2} = f(x, y') \tag{19}$$

(y missing). A second-order linear homogeneous ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{20}$$

for which

$$\frac{Q'(x) + 2P(x)Q(x)}{2[Q(x)]^{3/2}} = [\text{constant}] \tag{21}$$

can be transformed to one with constant coefficients.

The undamped equation of SIMPLE HARMONIC MOTION is

$$\frac{d^2y}{dx^2} + {\omega_0}^2 y = 0, \tag{22}$$

which becomes

$$\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + {\omega_0}^2 y = 0 \tag{23}$$

when damped, and

$$\frac{d^2y}{dx^2} + \beta\frac{dy}{dx} + {\omega_0}^2 y = A\cos(\omega t) \tag{24}$$

when both forced and damped.

SYSTEMS WITH CONSTANT COEFFICIENTS are of the form

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}(t) + \mathbf{p}(t). \tag{25}$$

The following are examples of important ordinary differential equations which commonly arise in problems of mathematical physics.

AIRY DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2} - xy = 0. \tag{26}$$

BERNOULLI DIFFERENTIAL EQUATION

$$\frac{dy}{dx} + p(x)y = q(x)y^n. \tag{27}$$

BESSEL DIFFERENTIAL EQUATION

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0. \tag{28}$$

CHEBYSHEV DIFFERENTIAL EQUATION

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \alpha^2 y = 0. \tag{29}$$

CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} + \alpha y = 0. \tag{30}$$

EULER DIFFERENTIAL EQUATION

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = S(x). \tag{31}$$

HERMITE DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0. \tag{32}$$

HILL'S DIFFERENTIAL EQUATION

$$\frac{d^2y}{dz^2} + \left[\theta_0 + 2\sum_{n=1}^{\infty}\theta_n\cos(2nz)\right] = 0. \tag{33}$$

HYPERGEOMETRIC DIFFERENTIAL EQUATION

$$x(x-1)\frac{d^2y}{dx^2}+[(1+\alpha+\beta)x-\gamma]\frac{dy}{dx}+\alpha\beta y=0. \quad (34)$$

JACOBI DIFFERENTIAL EQUATION

$$(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0. \quad (35)$$

LAGUERRE DIFFERENTIAL EQUATION

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+\lambda y=0. \quad (36)$$

LANE-EMDEN DIFFERENTIAL EQUATION

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0. \quad (37)$$

LEGENDRE DIFFERENTIAL EQUATION

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\alpha(\alpha+1)y=0. \quad (38)$$

LINEAR CONSTANT COEFFICIENTS

$$a_0\frac{d^ny}{dx^n}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny=p(x). \quad (39)$$

MALMSTÉN'S DIFFERENTIAL EQUATION

$$y''+\frac{r}{z}y'=\left(Az^m+\frac{s}{z^2}\right)y. \quad (40)$$

RICCATI DIFFERENTIAL EQUATION

$$\frac{dw}{dx}=q_0(x)+q_1(x)w+q_2(x)w^2. \quad (41)$$

RIEMANN P-DIFFERENTIAL EQUATION

$$\frac{d^2u}{dz^2}+\left[\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right]\frac{du}{dz}$$
$$+\left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-c)(b-a)}{z-b}\right.$$
$$\left.+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{u}{(z-a)(z-b)(z-c)}=0. \quad (42)$$

see also ADAMS' METHOD, GREEN'S FUNCTION, ISOCLINE, LAPLACE TRANSFORM, LEADING ORDER ANALYSIS, MAJORANT, ORDINARY DIFFERENTIAL EQUATION—FIRST-ORDER, ORDINARY DIFFERENTIAL EQUATION—SECOND-ORDER, PARTIAL DIFFERENTIAL EQUATION, RELAXATION METHODS, RUNGE-KUTTA METHOD, SIMPLE HARMONIC MOTION

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Ordinary Differential Equation—First-Order

Given a first-order ORDINARY DIFFERENTIAL EQUATION

$$\frac{dy}{dx}=F(x,y), \quad (1)$$

if $F(x,y)$ can be expressed using SEPARATION OF VARIABLES as

$$F(x,y)=X(x)Y(y), \quad (2)$$

then the equation can be expressed as

$$\frac{dy}{Y(y)}=X(x)\,dx \quad (3)$$

and the equation can be solved by integrating both sides to obtain

$$\int\frac{dy}{Y(y)}=\int X(x)\,dx. \quad (4)$$

Any first-order ODE of the form

$$\frac{dy}{dx}+p(x)y=q(x) \quad (5)$$

can be solved by finding an INTEGRATING FACTOR $\mu=\mu(x)$ such that

$$\frac{d}{dx}(\mu y)=\mu\frac{dy}{dx}+y\frac{d\mu}{dx}=\mu q(x). \quad (6)$$

Dividing through by μy yields

$$\frac{1}{y}\frac{dy}{dx}+\frac{1}{\mu}\frac{d\mu}{dx}=\frac{q(x)}{y}. \quad (7)$$

However, this condition enables us to explicitly determine the appropriate μ for arbitrary p and q. To accomplish this, take

$$p(x) = \frac{1}{\mu}\frac{d\mu}{dx} \qquad (8)$$

in the above equation, from which we recover the original equation (5), as required, in the form

$$\frac{1}{y}\frac{dy}{dx} + p(x) = \frac{q(x)}{y}. \qquad (9)$$

But we can integrate both sides of (8) to obtain

$$\int p(x)\,dx = \int \frac{d\mu}{\mu} = \ln \mu + c \qquad (10)$$

$$\mu = e^{\int p(x)\,dx}. \qquad (11)$$

Now integrating both sides of (6) gives

$$\mu y = \int \mu q(x)\,dx + c \qquad (12)$$

(with μ now a known function), which can be solved for y to obtain

$$y = \frac{\int \mu q(x)\,dx + c}{\mu} = \frac{\int e^{\int^x p(x')\,dx'} q(x)\,dx + c}{e^{\int^x p(x')\,dx'}}, \qquad (13)$$

where c is an arbitrary constant of integration.

Given an nth-order linear ODE with constant COEFFICIENTS

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0 y = Q(x), \qquad (14)$$

first solve the characteristic equation obtained by writing

$$y \equiv e^{rx} \qquad (15)$$

and setting $Q(x) = 0$ to obtain the n COMPLEX ROOTS.

$$r^n e^{rx} + a_{n-1}r^{n-1}e^{rx} + \ldots + a_1 r e^{rx} + a_0 e^{rx} = 0 \qquad (16)$$

$$r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = 0. \qquad (17)$$

Factoring gives the ROOTS r_i,

$$(r - r_1)(r - r_2)\cdots(r - r_n) = 0. \qquad (18)$$

For a nonrepeated REAL ROOT r, the corresponding solution is

$$y = e^{rx}. \qquad (19)$$

If a REAL ROOT r is repeated k times, the solutions are degenerate and the linearly independent solutions are

$$y = e^{rx}, y = xe^{rx}, \ldots, y = x^{k-1}e^{rx}. \qquad (20)$$

Complex ROOTS always come in COMPLEX CONJUGATE pairs, $r_\pm = a \pm ib$. For nonrepeated COMPLEX ROOTS, the solutions are

$$y = e^{ax}\cos(bx), y = e^{ax}\sin(bx). \qquad (21)$$

If the COMPLEX ROOTS are repeated k times, the linearly independent solutions are

$$y = e^{ax}\cos(bx), y = e^{ax}\sin(bx), \ldots, \\ y = x^{k-1}e^{ax}\cos(bx), y = x^{k-1}e^{ax}\sin(bx). \qquad (22)$$

Linearly combining solutions of the appropriate types with arbitrary multiplicative constants then gives the complete solution. If initial conditions are specified, the constants can be explicitly determined. For example, consider the sixth-order linear ODE

$$(\tilde{D} - 1)(\tilde{D} - 2)^3(\tilde{D}^2 + \tilde{D} + 1)y = 0, \qquad (23)$$

which has the characteristic equation

$$(r - 1)(r - 2)^3(r^2 + r + 1) = 0. \qquad (24)$$

The roots are 1, 2 (three times), and $(-1 \pm \sqrt{3}\,i)/2$, so the solution is

$$y = Ae^x + Be^{2x} + Cxe^{2x} + Dx^2e^{3x} + Ee^{-x/2}\cos(\tfrac{1}{2}\sqrt{3}\,x) \\ + Fe^{-x}\sin(\tfrac{1}{2}\sqrt{3}\,x). \qquad (25)$$

If the original equation is nonhomogeneous ($Q(x) \neq 0$), now find the particular solution y^* by the method of VARIATION OF PARAMETERS. The general solution is then

$$y(x) = \sum_{i=1}^{n} c_i y_i(x) + y^*(x), \qquad (26)$$

where the solutions to the linear equations are $y_1(x)$, $y_2(x)$, ..., $y_n(x)$, and $y^*(x)$ is the particular solution.

see also INTEGRATING FACTOR, ORDINARY DIFFERENTIAL EQUATION—FIRST-ORDER EXACT, SEPARATION OF VARIABLES, VARIATION OF PARAMETERS

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Ordinary Differential Equation—First-Order Exact

Consider a first-order ODE in the slightly different form

$$p(x, y)\,dx + q(x, y)\,dy = 0. \qquad (1)$$

Such an equation is said to be exact if

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \qquad (2)$$

This statement is equivalent to the requirement that a CONSERVATIVE FIELD exists, so that a scalar potential can be defined. For an exact equation, the solution is

$$\int_{(x_0,y_0)}^{(x,y)} p(x,y)\,dx + q(x,y)\,dy = c, \tag{3}$$

where c is a constant.

A first-order ODE (1) is said to be inexact if

$$\frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x}. \tag{4}$$

For a nonexact equation, the solution may be obtained by defining an INTEGRATING FACTOR μ of (6) so that the new equation

$$\mu p(x,y)\,dx + \mu q(x,y)\,dy = 0 \tag{5}$$

satisfies

$$\frac{\partial}{\partial y}(\mu p) = \frac{\partial}{\partial x}(\mu q), \tag{6}$$

or, written out explicitly,

$$p\frac{\partial \mu}{\partial y} + \mu\frac{\partial p}{\partial y} = q\frac{\partial \mu}{\partial x} + \mu\frac{\partial p}{\partial x}. \tag{7}$$

This transforms the nonexact equation into an exact one. Solving (7) for μ gives

$$\mu = \frac{q\frac{\partial \mu}{\partial x} - p\frac{\partial \mu}{\partial y}}{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}. \tag{8}$$

Therefore, if a function μ satisfying (8) can be found, then writing

$$P(x,y) = \mu p \tag{9}$$
$$Q(x,y) = \mu q \tag{10}$$

in equation (5) then gives

$$P(x,y)\,dx + Q(x,y)\,dy = 0, \tag{11}$$

which is then an exact ODE. Special cases in which μ can be found include x-dependent, xy-dependent, and y-dependent integrating factors.

Given an inexact first-order ODE, we can also look for an INTEGRATING FACTOR $\mu(x)$ so that

$$\frac{\partial \mu}{\partial y} = 0. \tag{12}$$

For the equation to be exact in μp and μq, the equation for a first-order nonexact ODE

$$p\frac{\partial \mu}{\partial y} + \mu\frac{\partial p}{\partial y} = q\frac{\partial \mu}{\partial x} + \mu\frac{\partial p}{\partial x} \tag{13}$$

becomes

$$\mu\frac{\partial p}{\partial y} = q\frac{\partial \mu}{\partial x} + \mu\frac{\partial p}{\partial x}. \tag{14}$$

Solving for $\partial\mu/\partial x$ gives

$$\frac{\partial \mu}{\partial x} = \mu(x)\frac{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}{q} \equiv f(x,y)\mu(x), \tag{15}$$

which will be integrable if

$$f(x,y) \equiv \frac{\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}}{q} = f(x), \tag{16}$$

in which case

$$\frac{d\mu}{\mu} = f(x)\,dx, \tag{17}$$

so that the equation is integrable

$$\mu(x) = e^{\int f(x)\,dx}, \tag{18}$$

and the equation

$$[\mu p(x,y)]dx + [\mu q(x,y)]dy = 0 \tag{19}$$

with known $\mu(x)$ is now exact and can be solved as an exact ODE.

Given in an exact first-order ODE, look for an INTEGRATING FACTOR $\mu(x,y) = g(xy)$. Then

$$\frac{\partial \mu}{\partial x} = \frac{\partial g}{\partial x}y \tag{20}$$

$$\frac{\partial \mu}{\partial y} = \frac{\partial g}{\partial y}x. \tag{21}$$

Combining these two,

$$\frac{\partial \mu}{\partial x} = \frac{y}{x}\frac{\partial \mu}{\partial y}. \tag{22}$$

For the equation to be exact in μp and μq, the equation for a first-order nonexact ODE

$$p\frac{\partial \mu}{\partial y} + \mu\frac{\partial p}{\partial y} = q\frac{\partial \mu}{\partial x} + \mu\frac{\partial p}{\partial x} \tag{23}$$

becomes

$$\frac{\partial \mu}{\partial y}\left(p - \frac{y}{x}q\right) = \left(\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}\right)\mu. \tag{24}$$

Therefore,

$$\frac{1}{x}\frac{\partial \mu}{\partial y} = \frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{xp - yq}\mu. \tag{25}$$

Define a new variable

$$t(x,y) \equiv xy, \tag{26}$$

then $\partial t/\partial y = x$, so

$$\frac{\partial\mu}{\partial t} = \frac{\partial\mu}{\partial y}\frac{\partial y}{\partial t} = \frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{xp - yq}\mu(t) \equiv f(x,y)\mu(t). \quad (27)$$

Now, if

$$f(x,y) \equiv \frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{xp - yq} = f(xy) = f(t), \quad (28)$$

then

$$\frac{\partial\mu}{\partial t} = f(t)\mu(t), \quad (29)$$

so that

$$\mu = e^{\int f(t)\,dt} \quad (30)$$

and the equation

$$[\mu p(x,y)]\,dx + [\mu q(x,y)]\,dy = 0 \quad (31)$$

is now exact and can be solved as an exact ODE.

Given an inexact first-order ODE, assume there exists an integrating factor

$$\mu = f(y), \quad (32)$$

so $\partial\mu/\partial x = 0$. For the equation to be exact in μp and μq, equation (7) becomes

$$\frac{\partial\mu}{\partial y} = \frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{p}\mu = f(x,y)\mu(y). \quad (33)$$

Now, if

$$f(x,y) \equiv \frac{\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}}{p} = f(y), \quad (34)$$

then

$$\frac{d\mu}{\mu} = f(y)\,dy, \quad (35)$$

so that

$$\mu(y) = e^{\int f(y)\,dy}, \quad (36)$$

and the equation

$$\mu p(x,y)\,dx + \mu q(x,y)\,dy = 0 \quad (37)$$

is now exact and can be solved as an exact ODE.

Given a first-order ODE of the form

$$yf(xy)\,dx + xg(xy)\,dy = 0, \quad (38)$$

define

$$v \equiv xy. \quad (39)$$

Then the solution is

$$\begin{cases} \ln x = \int \frac{g(v)\,dv}{c[g(v)-f(v)]} + c & \text{for } g(v) \neq f(v) \\ xy = c & \text{for } g(v) = f(v). \end{cases} \quad (40)$$

If

$$\frac{dy}{dx} = F(x,y) = G(v), \quad (41)$$

where

$$v \equiv \frac{y}{x}, \quad (42)$$

then letting

$$y \equiv xv \quad (43)$$

gives

$$\frac{dy}{dx} = x\,dv/dx + v \quad (44)$$

$$x\frac{dv}{dx} + v = G(v). \quad (45)$$

This can be integrated by quadratures, so

$$\ln x = \int \frac{dv}{f(v) - v} + c \qquad \text{for } f(v) \neq v \quad (46)$$

$$y = cx \qquad \text{for} f(v) = v. \quad (47)$$

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Ordinary Differential Equation—Second-Order

An ODE

$$y'' + P(x)y' + Q(x)y = 0 \quad (1)$$

has singularities for finite $x = x_0$ under the following conditions: (a) If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$, but $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ remain finite as $x \to x_0$, then x_0 is called a regular or nonessential singular point. (b) If $P(x)$ diverges faster than $(x - x_0)^{-1}$ so that $(x - x_0)P(x) \to \infty$ as $x \to x_0$, or $Q(x)$ diverges faster than $(x - x_0)^{-2}$ so that $(x - x_0)^2Q(x) \to \infty$ as $x \to x_0$, then x_0 is called an irregular or essential singularity.

Singularities of equation (1) at infinity are investigated by making the substitution $x \equiv z^{-1}$, so $dx = -z^{-2}\,dz$, giving

$$\frac{dy}{dx} = -z^2\frac{dy}{dz} \quad (2)$$

$$\frac{d^2y}{dx^2} = -z^2\frac{d}{dz}\left(-z^2\frac{dy}{dz}\right) = -z^2\left(-2z\frac{dy}{dz} - z^2\frac{d^2y}{dz^2}\right)$$
$$= 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}. \quad (3)$$

Then (1) becomes

$$z^4\frac{d^2y}{dz^2} + [2z^3 - z^2P(z)]\frac{dy}{dz} + Q(z)y = 0. \quad (4)$$

Case (a): If

$$\alpha(z) \equiv \frac{2z - P(z)}{z^2} \quad (5)$$

$$\beta(z) \equiv \frac{Q(z)}{z^4} \quad (6)$$

remain finite at $x = \pm\infty$ ($y = 0$), then the point is ordinary. Case (b): If either $\alpha(z)$ diverges no more rapidly than $1/z$ or $\beta(z)$ diverges no more rapidly than $1/z^2$, then the point is a regular singular point. Case (c): Otherwise, the point is an irregular singular point.

Morse and Feshbach (1953, pp. 667–674) give the canonical forms and solutions for second-order ODEs classified by types of singular points.

For special classes of second-order linear ordinary differential equations, variable COEFFICIENTS can be transformed into constant COEFFICIENTS. Given a second-order linear ODE with variable COEFFICIENTS

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \quad (7)$$

Define a function $z \equiv y(x)$,

$$\frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz} \quad (8)$$

$$\frac{d^2y}{dx^2} = \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \frac{d^2z}{dx^2}\frac{dy}{dz} \quad (9)$$

$$\left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \left[\frac{d^2z}{dx^2} + p(x)\frac{dz}{dx}\right]\frac{dy}{dz} + q(x)y = 0 \quad (10)$$

$$\frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + p(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \left[\frac{q(x)}{\left(\frac{dz}{dx}\right)^2}\right] y \equiv \frac{d^2y}{dz^2} + A\frac{dy}{dz} + By = 0. \quad (11)$$

This will have constant COEFFICIENTS if A and B are not functions of x. But we are free to set B to an arbitrary POSITIVE constant for $q(x) \geq 0$ by defining z as

$$z \equiv B^{-1/2} \int [q(x)]^{1/2}\, dx. \quad (12)$$

Then

$$\frac{dz}{dx} = B^{-1/2}[q(x)]^{1/2} \quad (13)$$

$$\frac{d^2z}{dx^2} = \tfrac{1}{2}B^{-1/2}[q(x)]^{-1/2}q'(x), \quad (14)$$

and

$$A = \frac{\frac{1}{2}B^{-1/2}[q(x)]^{-1/2}q'(x) + B^{-1/2}p(x)[q(x)]^{1/2}}{B^{-1}q(x)} = \frac{q'(x) + 2p(x)q(x)}{2[q(x)]^{3/2}}B^{1/2}. \quad (15)$$

Equation (11) therefore becomes

$$\frac{d^2y}{dz^2} + \frac{q'(x) + 2p(x)q(x)}{2[q(x)]^{3/2}}B^{1/2}\frac{dy}{dz} + By = 0, \quad (16)$$

which has constant COEFFICIENTS provided that

$$A \equiv \frac{q'(x) + 2p(x)q(x)}{2[q(x)]^{3/2}}B^{1/2} = [\text{constant}]. \quad (17)$$

Eliminating constants, this gives

$$A' \equiv \frac{q'(x) + 2p(x)q(x)}{[q(x)]^{3/2}} = [\text{constant}]. \quad (18)$$

So for an ordinary differential equation in which A' is a constant, the solution is given by solving the second-order linear ODE with constant COEFFICIENTS

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + By = 0 \quad (19)$$

for z, where z is defined as above.

A linear second-order homogeneous differential equation of the general form

$$y''(x) + P(x)y' + Q(x)y = 0 \quad (20)$$

can be transformed into standard form

$$z''(x) + q(x)z = 0 \quad (21)$$

with the first-order term eliminated using the substitution

$$\ln y \equiv \ln z - \tfrac{1}{2}\int P(x)\, dx. \quad (22)$$

Then

$$\frac{y'}{y} = \frac{z'}{z} - \tfrac{1}{2}P(x) \quad (23)$$

$$\frac{yy'' - y'^2}{y^2} = \frac{zz'' - z'^2}{z^2} - \tfrac{1}{2}P'(x) \quad (24)$$

$$\frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{z''}{z} - \frac{z'^2}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2}P'(x) \quad (25)$$

$$\frac{y''}{y} = \left[\frac{z'}{z} - \tfrac{1}{2}P(x)\right]^2 + \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2}P'(x) = \frac{z'^2}{z^2} - \frac{z'}{z}P(x) + \tfrac{1}{4}P^2(x) + \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2}P'(x), \quad (26)$$

so

$$\frac{y''}{y} + P(x)\frac{y'}{y} + Q(x) = -\frac{z'}{z}P(x) + \tfrac{1}{4}P^2(x) + \frac{z''}{z} - \tfrac{1}{2}P'(x) + P(x)\left[\frac{z'}{z} - \tfrac{1}{2}P(x)\right] + Q(x) = \frac{z''}{z} - \tfrac{1}{2}P'(x) - \tfrac{1}{4}P^2(x) + Q(x) = 0. \quad (27)$$

Therefore,

$$z'' + [Q(x) - \tfrac{1}{2}P'(x) - \tfrac{1}{4}P^2(x)]z \equiv z''(x) + q(x)z = 0, \quad (28)$$

where

$$q(x) \equiv Q(x) - \tfrac{1}{2}P'(x) - \tfrac{1}{4}P^2(x). \quad (29)$$

If $Q(x) = 0$, then the differential equation becomes

$$y'' + P(x)y' = 0, \quad (30)$$

which can be solved by multiplying by

$$\exp\left[\int^x P(x')\,dx'\right] \quad (31)$$

to obtain

$$0 = \frac{d}{dx}\left\{\exp\left[\int^x P(x')\,dx'\right]\frac{dy}{dx}\right\} \quad (32)$$

$$c_1 = \exp\left[\int^x P(x')\,dx'\right]\frac{dy}{dx} \quad (33)$$

$$y = c_1 \int^x \frac{dx}{\exp\left[\int^x P(x')\,dx'\right]} + c_2. \quad (34)$$

If one solution (y_1) to a second-order ODE is known, the other (y_2) may be found using the REDUCTION OF ORDER method. From the ABEL'S IDENTITY

$$\frac{dW}{W} = -P(x)\,dx, \quad (35)$$

where

$$W \equiv y_1 y_2' - y_1' y_2 \quad (36)$$

$$\int_a^x \frac{dW}{W} = \int_a^x P(x')\,dx' \quad (37)$$

$$\ln\left[\frac{W(x)}{W(a)}\right] = \int_a^x P(x')\,dx' \quad (38)$$

$$W(x) = W(a)\exp\left[-\int_a^x P(x')\,dx'\right]. \quad (39)$$

But

$$W \equiv y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right). \quad (40)$$

Combining (39) and (40) yields

$$\frac{d}{dx}\left(\frac{y_2}{y1}\right) = W(a)\frac{\exp[-\int_a^x P(x')\,dx']}{y_1^2} \quad (41)$$

$$y_2(x) = y_1(x)W(a)\int_b^x \frac{\exp[-\int_a^{x'} P(x'')\,dx'']}{[y_1(x')]^2}\,dx'. \quad (42)$$

Disregarding $W(a)$, since it is simply a multiplicative constant, and the constants a and b, which will contribute a solution which is not linearly independent of y_1,

$$y_2(x) = y_1(x)\int^x \frac{\exp\left[-\int^{x'} P(x'')\,dx''\right]}{[y_1(x')]^2}\,dx'. \quad (43)$$

If $P(x) = 0$, this simplifies to

$$y_2(x) = y_1(x)\int^x \frac{dx'}{[y_1(x')]^2}. \quad (44)$$

For a nonhomogeneous second-order ODE in which the x term does not appear in the function $f(x, y, y')$,

$$\frac{d^2y}{dx^2} = f(y, y'), \quad (45)$$

let $v \equiv y'$, then

$$\frac{dv}{dx} = f(v, y) = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}. \quad (46)$$

So the first-order ODE

$$v\frac{dv}{dy} = f(y, v), \quad (47)$$

if linear, can be solved for v as a linear first-order ODE. Once the solution is known,

$$\frac{dy}{dx} = v(y) \quad (48)$$

$$\int \frac{dy}{v(y)} = \int dx. \quad (49)$$

On the other hand, if y is missing from $f(x, y, y')$,

$$\frac{d^2y}{dx^2} = f(x, y'), \quad (50)$$

let $v \equiv y'$, then $v' = y''$, and the equation reduces to

$$v' = f(x, v), \quad (51)$$

which, if linear, can be solved for v as a linear first-order ODE. Once the solution is known,

$$y = \int v(x)\,dx. \quad (52)$$

see also ABEL'S IDENTITY, ADJOINT OPERATOR

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Ordinary Differential Equation—System with Constant Coefficients

To solve the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}(t) + \mathbf{p}(t), \tag{1}$$

where A is a MATRIX and $\mathbf{x}$ and $\mathbf{p}$ are VECTORS, first consider the homogeneous case with $\mathbf{p} = \mathbf{0}$. Then the solutions to

$$\frac{d\mathbf{x}}{dt} = \mathsf{A}\mathbf{x}(t) \tag{2}$$

are given by

$$\mathbf{x}(t) = e^{\mathsf{A}t}\mathbf{x}(\mathbf{t}). \tag{3}$$

But, by the MATRIX DECOMPOSITION THEOREM, the MATRIX EXPONENTIAL can be written as

$$e^{\mathsf{A}t} = \mathsf{uDu}^{-1}, \tag{4}$$

where the EIGENVECTOR MATRIX is

$$\mathsf{u} = [\,\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n\,] \tag{5}$$

and the EIGENVALUE MATRIX is

$$\mathsf{D} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}. \tag{6}$$

Now consider

$$\begin{aligned} e^{\mathsf{A}t}\mathsf{u} &= \mathsf{uDu}^{-1}\mathsf{u} = \mathsf{uD} \\ &= \begin{bmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \\ &= \begin{bmatrix} u_{11}e^{\lambda_1 t} & \cdots & u_{n1}e^{\lambda_n t} \\ u_{11}e^{\lambda_1 t} & \cdots & u_{n2}e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ u_{n1}e^{\lambda_1 t} & \cdots & u_{n2}e^{\lambda_n t} \end{bmatrix}. \end{aligned} \tag{7}$$

The individual solutions are then

$$\mathbf{x}_i = (e^{\mathsf{A}t}\mathsf{u}) \cdot \hat{\mathbf{e}}_i = \mathbf{u}_i e^{\lambda_i t}, \tag{8}$$

so the homogeneous solution is

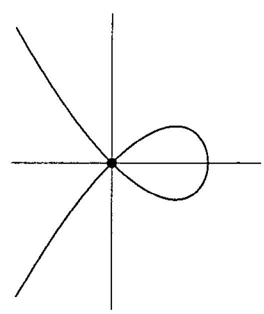
$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{u}_i e^{\lambda_i t}, \tag{9}$$

where the c_is are arbitrary constants.

The general procedure is therefore

1. Find the EIGENVALUES of the MATRIX A ($\lambda_1, \ldots, \lambda_n$) by solving the CHARACTERISTIC EQUATION.
2. Determine the corresponding EIGENVECTORS $\mathbf{u}_1, \ldots, \mathbf{u}_n$.
3. Compute
$$\mathbf{x}_i \equiv e^{\lambda_i t}\mathbf{u}_i \tag{10}$$
for $i = 1, \ldots, n$. Then the VECTORS $\mathbf{x}_i$ which are REAL are solutions to the homogeneous equation. If A is a 2×2 matrix, the COMPLEX vectors $\mathbf{x}_j$ correspond to REAL solutions to the homogeneous equation given by $\Re(\mathbf{x}_j)$ and $\Im(\mathbf{x}_j)$.
4. If the equation is nonhomogeneous, find the particular solution given by
$$\mathbf{x}^*(t) = \mathsf{X}(t) \int \mathsf{X}^{-1}(t)\mathbf{p}(t)\,dt, \tag{11}$$
where the MATRIX X is defined by
$$\mathsf{X}(t) \equiv [\,\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n\,]. \tag{12}$$
If the equation is homogeneous so that $\mathbf{p}(t) = \mathbf{0}$, then look for a solution of the form
$$\mathbf{x} = \boldsymbol{\xi} e^{\lambda t}. \tag{13}$$
This leads to an equation
$$(\mathsf{A} - \lambda\mathsf{I})\boldsymbol{\xi} = \mathbf{0}, \tag{14}$$
so ξ is an EIGENVECTOR and λ an EIGENVALUE.
5. The general solution is
$$\mathbf{x}(t) = \mathbf{x}^*(t) + \sum_{i=1}^{n} c_i \mathbf{x}_i. \tag{15}$$

Ordinary Double Point



A RATIONAL DOUBLE POINT of CONIC DOUBLE POINT type, known as "A_1." An ordinary DOUBLE POINT is called a NODE. The above plot shows the curve $x^3 - x^2 + y^2 = 0$, which has an ordinary double point at the ORIGIN.

A surface in complex 3-space admits at most finitely many ordinary double points. The maximum possible number of ordinary double points $\mu(d)$ for a surface of degree $d = 1, 2, \ldots$, are 0, 1, 4, 16, 31, 65, $93 \leq \mu(7) \leq 104$, $168 \leq \mu(8) \leq 174$, $216 \leq \mu(8) \leq 246$, $345 \leq \mu(10) \leq 360$, $425 \leq \mu(11) \leq 480$, $576 \leq \mu(12) \leq 645 \ldots$ (Sloane's A046001; Chmutov 1992, Endraß 1995). The fact that $\mu(5) = 31$ was proved by Beauville (1980), and $\mu(6) = 65$ was proved by Jaffe and Ruberman (1994). For $d \geq 3$, the following inequality holds:

$$\mu(d) \leq \tfrac{1}{2}[d(d-1)-3]$$

(Endraß 1995). Examples of ALGEBRAIC SURFACES having the maximum (known) number of ordinary double points are given in the following table.

d	$\mu(d)$	Surface
3	4	Cayley cubic
4	16	Kummer surface
5	31	dervish
6	65	Barth sextic
8	168	Endraß octic
10	345	Barth decic

see also ALGEBRAIC SURFACE, BARTH DECIC, BARTH SEXTIC, CAYLEY CUBIC, CUSP, DERVISH, ENDRASS OCTIC, KUMMER SURFACE, RATIONAL DOUBLE POINT

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Ordinary Line

Given an arrangement of $n \geq 3$ points, a LINE containing just two of them is called an ordinary line. Moser (1958) proved that at least $3n/7$ lines must be ordinary (Guy 1989, p. 903).

see also GENERAL POSITION, NEAR-PENCIL, ORDINARY POINT, SPECIAL POINT, SYLVESTER GRAPH

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Guy, R. K. "Unsolved Problems Come of Age." *Amer. Math. Monthly* **96**, 903–909, 1989.

Ordinary Point

A POINT which lies on at least one ORDINARY LINE.

see also ORDINARY LINE, SPECIAL POINT, SYLVESTER GRAPH

References

Guy, R. K. "Unsolved Problems Come of Age." *Amer. Math. Monthly* **96**, 903–909, 1989.

Ordinate

The y- (vertical) axis of a GRAPH.

see also ABSCISSA, x-AXIS, y-AXIS, z-AXIS

Ore's Conjecture

Define the HARMONIC MEAN of the DIVISORS of n

$$H(n) \equiv \frac{\tau(n)}{\sum_{d|n} \frac{1}{d}},$$

where $\tau(n)$ is the TAU FUNCTION (the number of DIVISORS of n). If n is a PERFECT NUMBER, $H(n)$ is an INTEGER. Ore conjectured that if n is ODD, then $H(n)$ is not an INTEGER. This implies that no ODD PERFECT NUMBERS exist.

see also HARMONIC DIVISOR NUMBER, HARMONIC MEAN, PERFECT NUMBER, TAU FUNCTION

Ore Number

see HARMONIC DIVISOR NUMBER

Ore's Theorem

If a GRAPH G has $n \geq 3$ VERTICES such that every pair of the n VERTICES which are not joined by an EDGE has a sum of VALENCES which is $\geq n$, then G is HAMILTONIAN.

see also HAMILTONIAN GRAPH

Orientable Surface

A REGULAR SURFACE $M \subset \mathbb{R}^n$ is called orientable if each TANGENT SPACE M_p has a COMPLEX STRUCTURE $J_p : M_p \to M_p$ such that $p \to J_p$ is a continuous function.

see also NONORIENTABLE SURFACE, REGULAR SURFACE

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Orientation (Plane Curve)

A curve has positive orientation if a region R is on the left when traveling around the outside of R, or on the right when traveling around the inside of R.

Orientation-Preserving

A nonsingular linear MAP $A : \mathbb{R}^n \to \mathbb{R}^n$ is orientation-preserving if $\det(A) > 0$.

see also ORIENTATION-REVERSING, ROTATION

Orientation-Reversing

A nonsingular linear MAP $A : \mathbb{R}^n \to \mathbb{R}^n$ is orientation-reversing if $\det(A) < 0$.

see also ORIENTATION-PRESERVING

Orientation (Vectors)

Let θ be the ANGLE between two VECTORS. If $0 < \theta < \pi$, the VECTORS are positively oriented. If $\pi < \theta < 2\pi$, the vectors are negatively oriented.

Two vectors in the plane

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

are positively oriented IFF the DETERMINANT

$$D \equiv \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} > 0,$$

and are negatively oriented IFF the DETERMINANT $D < 0$.

Origami

The Japanese art of paper folding to make 3-dimensional objects. CUBE DUPLICATION and TRISECTION of an ANGLE can be solved using origami, although they cannot be solved using the traditional rules for GEOMETRIC CONSTRUCTIONS.

see also FOLDING, GEOMETRIC CONSTRUCTION, STOMACHION, TANGRAM

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Origin

The central point ($r = 0$) in POLAR COORDINATES, or the point with all zero coordinates $(0, \ldots, 0)$ in CARTESIAN COORDINATES. In 3-D, the x-AXIS, y-AXIS, and z-AXIS meet at the origin.

see also OCTANT, QUADRANT, x-AXIS, y-AXIS, z-AXIS

Ornstein's Theorem

An important result in ERGODIC THEORY. It states that any two "Bernoulli schemes" with the same MEASURE-THEORETIC ENTROPY are MEASURE-THEORETICALLY ISOMORPHIC.

see also ERGODIC THEORY, ISOMORPHISM, MEASURE THEORY

Orr's Theorem

If

$$(1-z)^{\alpha+\beta+\gamma-1/2}\,{}_2F_1(2\alpha, 2\beta; 2\gamma; z) = \sum a_n z^n, \quad (1)$$

where ${}_2F_1(a, b; c; z)$ is a HYPERGEOMETRIC FUNCTION, then

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z)\,{}_2F_1(\gamma-\alpha+\tfrac{1}{2},\gamma-\beta+\tfrac{1}{2};\gamma+1;z) \\ = \sum_{(\gamma+\frac{1}{2})_n/(\gamma+1)_n} a_n z^n. \quad (2) \end{aligned}$$

Furthermore, if

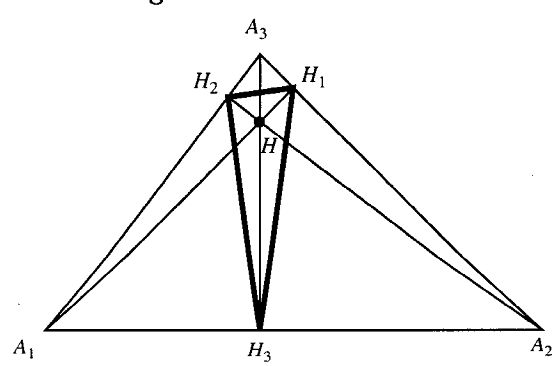
$$(1-z)^{\alpha+\beta-\gamma-1/2}\,{}_2F_1(2\alpha-1, 2\beta; 2\gamma-1; z) = \sum a_n z^n, \quad (3)$$

then

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z)\Gamma(\gamma-\alpha+\tfrac{1}{2},\gamma-\beta-\tfrac{1}{2};\gamma;z) \\ = \sum_{(\gamma-\frac{1}{2})_n/(\gamma)_n} a_n z^n, \quad (4) \end{aligned}$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

Orthic Triangle



Given a TRIANGLE $\Delta A_1A_2A_3$, the TRIANGLE $\Delta H_1H_2H_3$ with VERTICES at the feet of the ALTITUDES (perpendiculars from a point to the sides) is called the orthic triangle. The three lines A_iH_i are CONCURRENT at the ORTHOCENTER H of $\Delta A_1A_2A_3$.

The centroid of the orthic triangle has TRIANGLE CENTER FUNCTION

$$\alpha = a^2 \cos(B - C)$$

(Casey 1893, Kimberling 1994). The ORTHOCENTER of the orthic triangle has TRIANGLE CENTER FUNCTION

$$\alpha = \cos(2A)\cos(B - C)$$

(Casey 1893, Kimberling 1994). The SYMMEDIAN POINT of the orthic triangle has TRIANGLE CENTER FUNCTION

$$\alpha = \tan A \cos(B - C)$$

(Casey 1893, Kimberling 1994).

see also ALTITUDE, FAGNANO'S PROBLEM, ORTHOCENTER, PEDAL TRIANGLE, SCHWARZ'S TRIANGLE PROBLEM, SYMMEDIAN POINT

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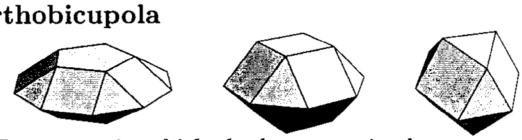
Orthobicupola



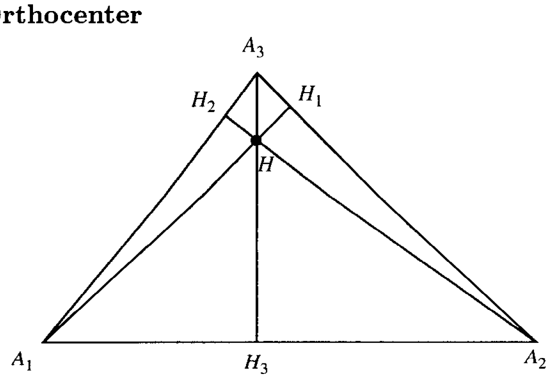
A BICUPOLA in which the bases are in the same orientation.

see also PENTAGONAL ORTHOBICUPOLA, SQUARE ORTHOBICUPOLA, TRIANGULAR ORTHOBICUPOLA

Orthobirotunda

A BIROTUNDA in which the bases are in the same orientation.

Orthocenter



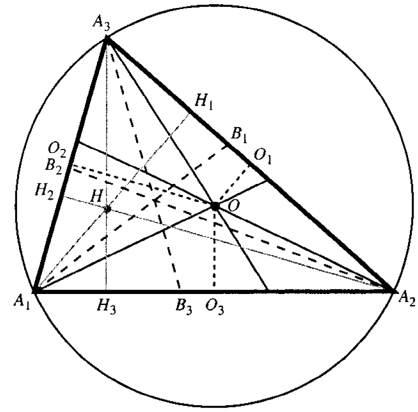
The intersection H of the three ALTITUDES of a TRIANGLE is called the orthocenter. Its TRILINEAR COORDINATES are

$$\cos B \cos C : \cos C \cos A : \cos A \cos B. \tag{1}$$

If the TRIANGLE is not a RIGHT TRIANGLE, then (1) can be divided through by $\cos A \cos B \cos C$ to give

$$\sec A : \sec B : \sec C. \tag{2}$$

If the triangle is ACUTE, the orthocenter is in the interior of the triangle. In a RIGHT TRIANGLE, the orthocenter is the VERTEX of the RIGHT ANGLE.



The CIRCUMCENTER O and orthocenter H are ISOGONAL CONJUGATE points. The orthocenter lies on the EULER LINE.

$$a_1{}^2 + a_2{}^2 + a_3{}^2 + \overline{A_1H}^2 + \overline{A_2H}^2 + \overline{A_3H}^2 = 12R^2 \tag{3}$$

$$\overline{A_1H} + \overline{A_2H} + \overline{A_3H} = 2(r + R), \tag{4}$$

$$\overline{A_1H}^2 + \overline{A_2H}^2 + \overline{A_3H}^2 = 4R^2 - 4Rr, \tag{5}$$

where r is the INRADIUS and R is the CIRCUMRADIUS (Johnson 1929, p. 191).

Any HYPERBOLA circumscribed on a TRIANGLE and passing through the orthocenter is RECTANGULAR, and has its center on the NINE-POINT CIRCLE (Falisse 1920, Vandeghen 1965).

see also CENTROID (TRIANGLE), CIRCUMCENTER, EULER LINE, INCENTER, ORTHIC TRIANGLE, ORTHOCENTRIC COORDINATES, ORTHOCENTRIC QUADRILATERAL, ORTHOCENTRIC SYSTEM, POLAR CIRCLE

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Orthocentric Coordinates

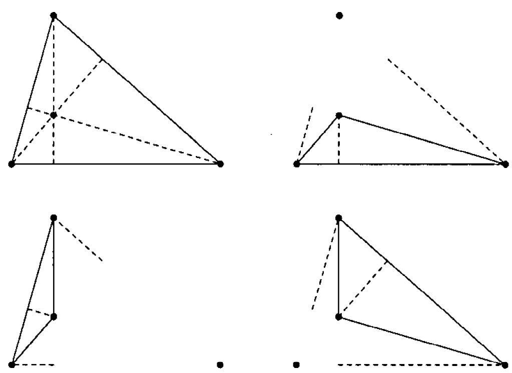
Coordinates defined by an ORTHOCENTRIC SYSTEM.

see also TRILINEAR COORDINATES

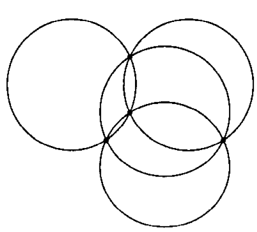
Orthocentric Quadrilateral

If two pairs of opposite sides of a COMPLETE QUADRILATERAL are pairs of PERPENDICULAR lines, the QUADRILATERAL is said to be orthocentric. In such a case, the remaining sides are also PERPENDICULAR.

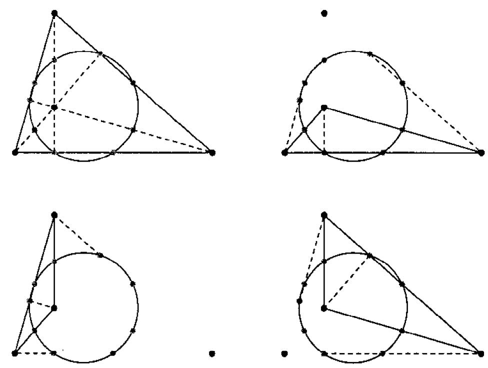
Orthocentric System



A set of four points, one of which is the ORTHOCENTER of the other three. In an orthocentric system, each point is the ORTHOCENTER of the TRIANGLE of the other three, as illustrated above. The INCENTER and EXCENTERS of a TRIANGLE are an orthocentric system. The centers of the CIRCUMCIRCLES of an orthocentric system form another orthocentric system congruent to the first. The sum of the squares of any nonadjacent pair of connectors of an orthocentric system equals the square of the DIAMETER of the CIRCUMCIRCLE. Orthocentric systems are used to define ORTHOCENTRIC COORDINATES.



The four CIRCUMCIRCLES of points in an orthocentric system taken three at a time (illustrated above) have equal RADIUS.



The four triangles of an orthocentric system have a common NINE-POINT CIRCLE, illustrated above.

see also ANGLE BISECTOR, CIRCUMCIRCLE, CYCLIC QUADRANGLE, NINE-POINT CIRCLE, ORTHIC TRIANGLE, ORTHOCENTER, ORTHOCENTRIC SYSTEM, POLAR CIRCLE

References

Altshiller-Court, N. *College Geometry: A Second Course in Plane Geometry for Colleges and Normal Schools, 2nd ed.* New York: Barnes and Noble, pp. 109–114, 1952.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 165–176, 1929.

Orthocupolarotunda

A CUPOLAROTUNDA in which the bases are in the same orientation.

see also GYROCUPOLAROTUNDA, PENTAGONAL ORTHOCUPOLARONTUNDA

Orthodrome

see GREAT CIRCLE

Orthogonal Array

An orthogonal array OA(k, s) is a $k \times s^2$ ARRAY with entries taken from an s-set S having the property that in any two rows, each ordered pair of symbols from S occurs exactly once.

References

Colbourn, C. J. and Dinitz, J. H. (Eds.) *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, p. 111, 1996.

Orthogonal Basis

A BASIS of vectors $\mathbf{x}$ which satisfy

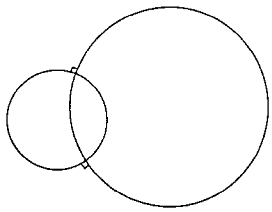
$$x_j x_k = C_{jk}\delta_{jk}$$

$$x^\mu x_\nu = C^\mu_\nu \delta^\mu_\nu,$$

where C_{jk}, C^μ_ν are constants (not necessarily equal to 1) and δ_{jk} is the KRONECKER DELTA.

see also BASIS, ORTHONORMAL BASIS

Orthogonal Circles



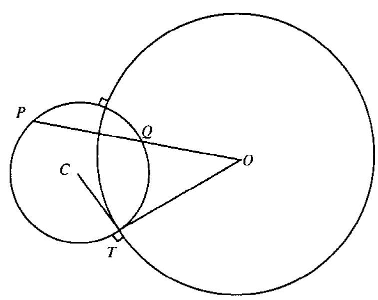
Orthogonal circles are ORTHOGONAL CURVES, i.e., they cut one another at RIGHT ANGLES. Two CIRCLES with equations

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{1}$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \tag{2}$$

are orthogonal if

$$2gg' + 2ff' = c + c'. \tag{3}$$



A theorem of Euclid states that, for the orthogonal circles in the above diagram,

$$OP \times OQ = OT^2 \tag{4}$$

(Dixon 1991, p. 65).

References
Dixon, R. *Mathographics.* New York: Dover, pp. 65–66, 1991.
Euclid. *The Thirteen Books of the Elements, 2nd ed. unabridged, Vol. 3: Books X–XIII* New York: Dover, p. 36, 1956.
Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., p. xxiv, 1995.

Orthogonal Curves

Two intersecting curves which are PERPENDICULAR at their INTERSECTION are said to be orthogonal.

Orthogonal Functions

Two functions $f(x)$ and $g(x)$ are orthogonal on the interval $a \leq x \leq b$ if

$$\langle f(x)|g(x)\rangle \equiv \int_a^b f(x)g(x)\,dx = 0.$$

see also ORTHOGONAL POLYNOMIALS, ORTHONORMAL FUNCTIONS

Orthogonal Group

see GENERAL ORTHOGONAL GROUP, LIE-TYPE GROUP, ORTHOGONAL ROTATION GROUP, PROJECTIVE GENERAL ORTHOGONAL GROUP, PROJECTIVE SPECIAL ORTHOGONAL GROUP, SPECIAL ORTHOGONAL GROUP

References
Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#orth`.

Orthogonal Group Representations

Two representations of a GROUP χ_i and χ_j are said to be orthogonal if

$$\sum_R \chi_i(R)\chi_j(R) = 0$$

for $i \neq j$, where the sum is over all elements R of the representation.

see also GROUP

Orthogonal Lines

Two or more LINES or LINE SEGMENTS which are PERPENDICULAR are said to be orthogonal.

Orthogonal Matrix

Any ROTATION can be given as a composition of rotations about three axes (EULER'S ROTATION THEOREM), and thus can be represented by a 3×3 MATRIX operating on a VECTOR,

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{1}$$

We wish to place conditions on this matrix so that it is consistent with an ORTHOGONAL TRANSFORMATION (basically, a ROTATION or ROTOINVERSION).

In a ROTATION, a VECTOR must keep its original length, so it must be true that

$$x_i'x_i' = x_ix_i \tag{2}$$

for $i = 1, 2, 3$, where EINSTEIN SUMMATION is being used. Therefore, from the transformation equation,

$$(a_{ij}x_j)(a_{ik}x_k) = x_ix_i. \tag{3}$$

This can be rearranged to

$$\begin{aligned} a_{ij}(x_ja_{ik})x_k &= a_{ij}(a_{ik}x_j)x_k \\ &= a_{ij}a_{ik}x_jx_k = x_ix_i. \end{aligned} \tag{4}$$

In order for this to hold, it must be true that

$$a_{ij}a_{ik} = \delta_{jk} \tag{5}$$

for $j, k = 1, 2, 3$, where δ_{ij} is the KRONECKER DELTA. This is known as the ORTHOGONALITY CONDITION, and it guarantees that

$$\mathsf{A}^{-1} = \mathsf{A}^{\mathrm{T}}, \tag{6}$$

and

$$\mathsf{A}^{\mathrm{T}}\mathsf{A} = \mathsf{I}, \tag{7}$$

where A^{T} is the MATRIX TRANSPOSE and I is the IDENTITY MATRIX. Equation (7) is the identity which gives the orthogonal matrix its name. Orthogonal matrices have special properties which allow them to be manipulated and identified with particular ease.

Let A and B be two orthogonal matrices. By the ORTHOGONALITY CONDITION, they satisfy

$$a_{ij}a_{ik} = \delta_{jk}, \tag{8}$$

and

$$b_{ij}b_{ik} = \delta_{jk}, \tag{9}$$

where δ_{ij} is the KRONECKER DELTA. Now

$$\begin{aligned} c_{ij}c_{ik} &= (ab)_{ij}(ab)_{jk} = a_{is}b_{sj}a_{it}b_{tk} = a_{is}a_{it}b_{sj}b_{tk} \\ &= \delta_{st}b_{sj}b_{tk} = b_{tj}b_{tk} = \delta_{jk}, \end{aligned} \tag{10}$$

so the product $\mathsf{C} \equiv \mathsf{AB}$ of two orthogonal matrices is also orthogonal.

The EIGENVALUES of an orthogonal matrix must satisfy one of the following:

1. All EIGENVALUES are 1.
2. One EIGENVALUE is 1 and the other two are -1.
3. One EIGENVALUE is 1 and the other two are COMPLEX CONJUGATES of the form $e^{i\theta}$ and $e^{-i\theta}$.

An orthogonal MATRIX A is classified as proper (corresponding to pure ROTATION) if

$$\det(\mathsf{A}) = 1, \tag{11}$$

where $\det(\mathsf{A})$ is the DETERMINANT of A, or improper (corresponding to inversion with possible rotation; ROTOINVERSION) if

$$\det(\mathsf{A}) = -1. \tag{12}$$

see also EULER'S ROTATION THEOREM, ORTHOGONAL TRANSFORMATION, ORTHOGONALITY CONDITION, ROTATION, ROTATION MATRIX, ROTOINVERSION

References

Arfken, G. "Orthogonal Matrices." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 191–205, 1985.

Goldstein, H. "Orthogonal Transformations." §4–2 in *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, 132–137, 1980.

Orthogonal Polynomials

Orthogonal polynomials are classes of POLYNOMIALS $\{p_n(x)\}$ over a range $[a, b]$ which obey an ORTHOGONALITY relation

$$\int_a^b w(x)p_m(x)p_n(x)\,dx = \delta_{mn}c_n, \tag{1}$$

where $w(x)$ is a WEIGHTING FUNCTION and δ is the KRONECKER DELTA. If $c_m = 1$, then the POLYNOMIALS are not only orthogonal, but orthonormal.

Orthogonal polynomials have very useful properties in the solution of mathematical and physical problems. Just as FOURIER SERIES provide a convenient method of expanding a periodic function in a series of linearly independent terms, orthogonal polynomials provide a natural way to solve, expand, and interpret solutions to many types of important DIFFERENTIAL EQUATIONS. Orthogonal polynomials are especially easy to generate using GRAM-SCHMIDT ORTHONORMALIZATION. Abramowitz and Stegun (1972, pp. 774–775) give a table of common orthogonal polynomials.

Type	Interval	$w(x)$	c_n
Chebyshev First Kind	$[-1, 1]$	$(1-x^2)^{-1/2}$	$\begin{cases} \frac{1}{2}\pi & \text{for } n = 0 \\ \pi & \text{otherwise} \end{cases}$
Chebyshev Second Kind	$[-1, 1]$	$\sqrt{1-x^2}$	$\frac{1}{2}\pi$
Hermite	$(-\infty, \infty)$	e^{-x^2}	$\sqrt{\pi}\,2^n n!$
Jacobi	$(-1, 1)$	$(1-x)^\alpha(1+x)^\beta$	h_n
Laguerre	$[0, \infty)$	e^{-x}	1
Laguerre (Associated)	$[0, \infty)$	$x^k e^{-x}$	$\frac{(n+k)!}{n!}$
Legendre	$[-1, 1]$	1	$\frac{2}{2n+1}$
Ultraspherical	$[-1, 1]$	$(1-x^2)^{\alpha-1/2}$	$\begin{cases} \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2} & \text{for } \alpha \neq 0 \\ \frac{2\pi}{n^2} & \text{for } \alpha = 0 \end{cases}$

In the above table, the normalization constant is the value of

$$c_n \equiv \int w(x)[p_n(x)]^2\,dx \tag{2}$$

and

$$h_n \equiv \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}, \tag{3}$$

where $\Gamma(z)$ is a GAMMA FUNCTION.

The ROOTS of orthogonal polynomials possess many rather surprising and useful properties. For instance, let $x_1 < x_2 < \ldots < x_n$ be the ROOTS of the $p_n(x)$ with $x_0 = a$ and $x_{n+1} = b$. Then each interval $[x_\nu, x_{\nu+1}]$ for $\nu = 0, 1, \ldots, n$ contains exactly one ROOT of $p_{n+1}(x)$. Between two ROOTS of $p_n(x)$ there is at least one ROOT of $p_m(x)$ for $m > n$.

Let c be an arbitrary REAL constant, then the POLYNOMIAL

$$p_{n+1}(x) - cp_n(x) \tag{4}$$

has $n+1$ distinct REAL ROOTS. If $c > 0$ ($c < 0$), these ROOTS lie in the interior of $[a, b]$, with the exception of the greatest (least) ROOT which lies in $[a, b]$ only for

$$c \leq \frac{p_{n+1}(b)}{p_n(b)} \qquad \left(c \geq \frac{p_{n+1}(a)}{p_n(a)}\right). \tag{5}$$

The following decomposition into partial fractions holds

$$\frac{p_n(x)}{p_{n+1}(x)} = \sum_{\nu=0}^{n} \frac{l_\nu}{x - \xi}, \tag{6}$$

where $\{\xi_\nu\}$ are the ROOTS of $p_{n+1}(x)$ and

$$\begin{aligned} l_\nu &= \frac{p_n(\xi_\nu)}{p'_{n+1}(\xi_\nu)} \\ &= \frac{p'_{n+1}(\xi_\nu)p_n(\xi_\nu) - p'_n(\xi_\nu)' p_{n+1}(\xi_\nu)}{[p'_{n+1}(\xi_\nu)]^2} > 0. \end{aligned} \tag{7}$$

Another interesting property is obtained by letting $\{p_n(x)\}$ be the orthonormal set of POLYNOMIALS associated with the distribution $d\alpha(x)$ on $[a, b]$. Then the CONVERGENTS R_n/S_n of the CONTINUED FRACTION

$$\frac{1}{A_1x + B_1} - \frac{C_2}{A_2x + B_2} - \frac{C_3}{A_3x + B_3} - \ldots - \frac{C_n}{A_nx + B_n} + \ldots \tag{8}$$

are given by

$$\begin{aligned} R_n &= R_n(x) \\ &= c_0{}^{-3/2}\sqrt{c_0c_2 - c_1{}^2} \int_a^b \frac{p_n(x) - p_n(t)}{x - t}\, d\alpha(t) \end{aligned} \tag{9}$$

$$S_n = S_n(x) = \sqrt{c_0}\, p_n(x), \tag{10}$$

where $n = 0, 1, \ldots$ and

$$c_n = \int_a^b x^n\, d\alpha(x). \tag{11}$$

Furthermore, the ROOTS of the orthogonal polynomials $p_n(x)$ associated with the distribution $d\alpha(x)$ on the interval $[a, b]$ are REAL and distinct and are located in the interior of the interval $[a, b]$.

see also CHEBYSHEV POLYNOMIAL OF THE FIRST KIND, CHEBYSHEV POLYNOMIAL OF THE SECOND KIND, GRAM-SCHMIDT ORTHONORMALIZATION, HERMITE POLYNOMIAL, JACOBI POLYNOMIAL, KRAWTCHOUK POLYNOMIAL, LAGUERRE POLYNOMIAL, LEGENDRE POLYNOMIAL, ORTHOGONAL FUNCTIONS, SPHERICAL HARMONIC, ULTRASPHERICAL POLYNOMIAL, ZERNIKE POLYNOMIAL

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Orthogonal Polynomials." Ch. 22 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 771–802, 1972.

Arfken, G. "Orthogonal Polynomials." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 520–521, 1985.

Iyanaga, S. and Kawada, Y. (Eds.). "Systems of Orthogonal Functions." Appendix A, Table 20 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1477, 1980.

Nikiforov, A. F.; Uvarov, V. B.; and Suslov, S. S. *Classical Orthogonal Polynomials of a Discrete Variable.* New York: Springer-Verlag, 1992.

Sansone, G. *Orthogonal Functions.* New York: Dover, 1991.

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 44–47 and 54–55, 1975.

Orthogonal Projection

A PROJECTION of a figure by parallel rays. In such a projection, tangencies are preserved. Parallel lines project to parallel lines. The ratio of lengths of parallel segments is preserved, as is the ratio of areas.

Any TRIANGLE can be positioned such that its shadow under an orthogonal projection is EQUILATERAL. Also, the MEDIANS of a TRIANGLE project to the MEDIANS of the image TRIANGLE. ELLIPSES project to ELLIPSES, and any ELLIPSE can be projected to form a CIRCLE. The center of an ELLIPSE projects to the center of the image ELLIPSE. The CENTROID of a TRIANGLE projects to the CENTROID of its image. Under an ORTHOGONAL TRANSFORMATION, the MIDPOINT ELLIPSE can be transformed into a CIRCLE INSCRIBED in an EQUILATERAL TRIANGLE.

SPHEROIDS project to ELLIPSES (or CIRCLE in the DEGENERATE case).

see also PROJECTION

Orthogonal Rotation Group

Orthogonal rotation groups are LIE GROUPS. The orthogonal rotation group $O_3(n)$ is the set of $n \times n$ REAL ORTHOGONAL MATRICES.

The orthogonal rotation group $O_3^-(n)$ is the set of $n \times n$ REAL ORTHOGONAL MATRICES (having $n(n-1)/2$ independent parameters) with DETERMINANT -1.

The orthogonal rotation group $O_3^+(n)$ is the set of $n \times n$ REAL ORTHOGONAL MATRICES, having $n(n-1)/2$ independent parameters, with DETERMINANT $+1$. $O_3^+(n)$ is

HOMEOMORPHIC with $SU(2)$. Its elements can be written using EULER ANGLES and ROTATION MATRICES as

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{2}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3}$$

$$R_z(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

References

Arfken, G. "Orthogonal Group, O_3^+." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 252–253, 1985.

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#orth`.

Orthogonal Tensors

Orthogonal CONTRAVARIANT and COVARIANT satisfy

$$g_{ik}g^{ij} = \delta_k^j,$$

where δ_j^k is the KRONECKER DELTA.

see also CONTRAVARIANT TENSOR, COVARIANT TENSOR

Orthogonal Transformation

Any linear transformation

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + x_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}$$

satisfying the ORTHOGONALITY CONDITION

$$a_{ij}a_{ik} = \delta_{jk},$$

where EINSTEIN SUMMATION has been used and δ_{ij} is the KRONECKER DELTA, is called an orthogonal transformation.

Orthogonal transformations correspond to rigid ROTATIONS (or ROTOINVERSIONS), and may be represented using ORTHOGONAL MATRICES. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then $\det(A) = \pm 1$.

see also AFFINE TRANSFORMATION, ORTHOGONAL MATRIX, ORTHOGONALITY CONDITION, ROTATION, ROTOINVERSION

References

Goldstein, H. "Orthogonal Transformations." §4–2 in *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, 132–137, 1980.

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 104, 1993.

Orthogonal Vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ whose DOT PRODUCT is $\mathbf{u}\cdot\mathbf{v} = 0$ (i.e., the vectors are PERPENDICULAR) are said to be orthogonal. The definition can be extended to three or more vectors which are mutually PERPENDICULAR.

see also DOT PRODUCT, PERPENDICULAR

Orthogonality Condition

A linear transformation

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + x_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned}$$

is said to be an ORTHOGONAL TRANSFORMATION if it satisfies the orthogonality condition

$$a_{ij}a_{ik} = \delta_{jk},$$

where EINSTEIN SUMMATION has been used and δ_{ij} is the KRONECKER DELTA.

see also ORTHOGONAL TRANSFORMATION

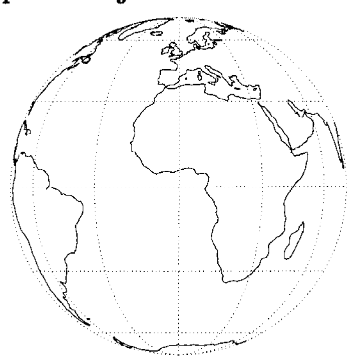
References

Goldstein, H. "Orthogonal Transformations." §4–2 in *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, 132–137, 1980.

Orthogonality Theorem

see GROUP ORTHOGONALITY THEOREM

Orthographic Projection



A projection from infinity which preserves neither AREA nor angle.

$$x = \cos\phi\sin(\lambda - \lambda_0) \tag{1}$$

$$y = \cos\phi_1\sin\phi - \sin\phi_1\cos\phi\cos(\lambda - \lambda_0). \tag{2}$$

The inverse FORMULAS are

$$\phi = \sin^{-1}\left(\cos c\sin\phi_1 + \frac{y\sin c\cos\phi_1}{\rho}\right) \tag{3}$$

$$\lambda = \lambda_0 + \tan^{-1}\left(\frac{x\sin c}{\rho\cos\phi_1\cos c - y\sin\phi_1\sin c}\right), \tag{4}$$

where

$$\rho = \sqrt{x^2 + y^2} \quad (5)$$
$$c = \sin^{-1} \rho. \quad (6)$$

References
Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 145–153, 1987.

Orthologic

Two TRIANGLES $A_1B_1C_1$ and $A_2B_2C_2$ are orthologic if the perpendiculars from the VERTICES A_1, B_1, C_1 on the sides B_2C_2, A_2C_2, and A_2B_2 pass through one point. This point is known as the orthology center of TRIANGLE 1 with respect to TRIANGLE 2.

Orthonormal Basis

A BASIS of VECTORS $\mathbf{x}$ which satisfy

$$x_j x_k = \delta_{jk}$$

and

$$x^\mu x_\nu = \delta^\mu_\nu,$$

where δ_{jk} is the KRONECKER DELTA. An orthonormal basis is a normalized ORTHOGONAL BASIS.

see also BASIS, ORTHOGONAL BASIS

Orthonormal Functions

A pair of functions ϕ_i and ϕ_j are orthonormal if they are ORTHOGONAL and each normalized. These two conditions can be succinctly written as

$$\int_a^b \phi_i(x)\phi_j(x)w(x)\,dx = \delta_{ij},$$

where $w(x)$ is a WEIGHTING FUNCTION and δ_{ij} is the KRONECKER DELTA.

see also ORTHOGONAL POLYNOMIALS

Orthonormal Vectors

UNIT VECTORS which are ORTHOGONAL are said to be orthonormal.

see also ORTHOGONAL VECTORS

Orthopole

If perpendiculars are dropped on any line from the vertices of a TRIANGLE, then the perpendiculars to the opposite sides from their FEET are CONCURRENT at a point called the orthopole.

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 247, 1929.

Orthoptic Curve

An ISOPTIC CURVE formed from the locus of TANGENTS meeting at RIGHT ANGLES. The orthoptic of a PARABOLA is its DIRECTRIX. The orthoptic of a central CONIC was investigated by Monge and is a CIRCLE concentric with the CONIC SECTION. The orthoptic of an ASTROID is a CIRCLE.

Curve	Orthoptic
astroid	quadrifolium
cardioid	circle or limaçon
deltoid	circle
logarithmic spiral	equal logarithmic spiral
parabola	directrix

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 58 and 207, 1972.

Orthotomic

Given a source S and a curve γ, pick a point on γ and find its tangent T. Then the LOCUS of reflections of S about tangents T is the orthotomic curve (also known as the secondary CAUSTIC). The INVOLUTE of the orthotomic is the CAUSTIC. For a parametric curve $(f(t), g(t))$ with respect to the point (x_0, y_0), the orthotomic is

$$x = x_0 - \frac{2g'[f'(g-y_0) - g'(f-x_0)]}{f'^2 + g'^2}$$
$$y = y_0 + \frac{2f'[f'(g-y_0) - g'(f-x_0)]}{f'^2 + g'^2}.$$

see also CAUSTIC, INVOLUTE

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, p. 60, 1972.

Orthotope

A PARALLELOTOPE whose edges are all mutually PERPENDICULAR. The orthotope is a generalization of the RECTANGLE and RECTANGULAR PARALLELEPIPED.

see also RECTANGLE, RECTANGULAR PARALLELEPIPED

References
Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 122–123, 1973.

Osborne's Rule

The prescription that a TRIGONOMETRY identity can be converted to an analogous identity for HYPERBOLIC FUNCTIONS by expanding, exchanging trigonometric functions with their hyperbolic counterparts, and then flipping the sign of each term involving the product of two HYPERBOLIC SINES. For example, given the identity

$$\cos(x-y) = \cos x \cos y + \sin x \sin y,$$

Osborne's rule gives the corresponding identity

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y.$$

see also HYPERBOLIC FUNCTIONS, TRIGONOMETRY

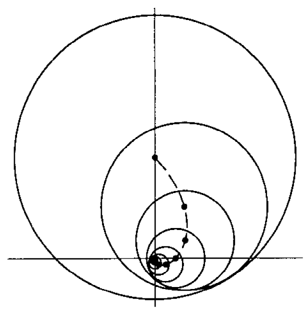
Oscillation

The variation of a FUNCTION which exhibits SLOPE changes, also called the SALTUS of a function.

Oscillation Land

see CAROTID-KUNDALINI FUNCTION

Osculating Circle



The CIRCLE which shares the same TANGENT as a curve at a given point. The RADIUS OF CURVATURE of the osculating circle is

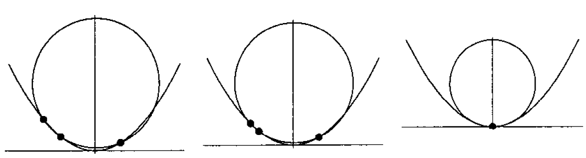
$$\rho(t) = \frac{1}{|\kappa(t)|}, \tag{1}$$

where κ is the CURVATURE, and the center is

$$x = f - \frac{(f'^2 + g'^2)g'}{f'g'' - f''g'} \tag{2}$$

$$y = g + \frac{(f'^2 + g'^2)g'}{f'g'' - f''g'}, \tag{3}$$

i.e., the centers of the osculating circles to a curve form the EVOLUTE to that curve.



In addition, let $C(t_1, t_2, t_3)$ denote the CIRCLE passing through three points on a curve $(f(t), g(t))$ with $t_1 < t_2 < t_3$. Then the osculating circle C is given by

$$C = \lim_{t_1, t_2, t_3 \to t} C(t_1, t_2, t_3) \tag{4}$$

(Gray 1993).

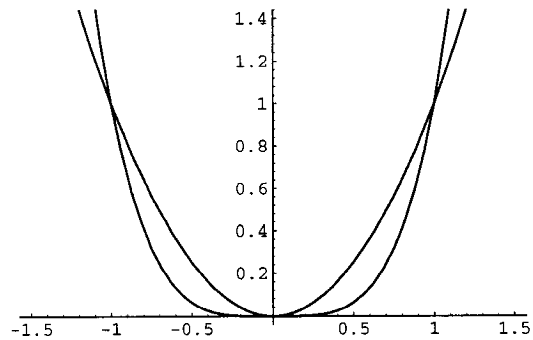
see also CURVATURE, EVOLUTE, RADIUS OF CURVATURE, TANGENT

References

Gardner, M. "The Game of Life, Parts I–III." Chs. 20–22 in *Wheels, Life, and other Mathematical Amusements.* New York: W. H. Freeman, pp. 221, 237, and 243, 1983.

Gray, A. "Osculating Circles to Plane Curves." §5.6 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 90–95, 1993.

Osculating Curves



An osculating curve to $f(x)$ at x_0 is tangent at that point and has the same CURVATURE. It therefore satisfies

$$y^{(k)}(x_0) = f^{(k)}(x_0)$$

for $k = 0, 1, 2$. The point of tangency is called a TACNODE. The simplest example of osculating curves are x^2 and x^4, which osculate at the point $x_0 = 0$.

see also TACNODE

Osculating Interpolation

see HERMITE'S INTERPOLATING FUNDAMENTAL POLYNOMIAL

Osculating Plane

The PLANE spanned by the three points $\mathbf{x}(t)$, $\mathbf{x}(t + h_1)$, and $\mathbf{x}(t + h_2)$ on a curve as $h_1, h_2 \to 0$. Let $\mathbf{z}$ be a point on the osculating plane, then

$$[(\mathbf{z} - \mathbf{x}), \mathbf{x}', \mathbf{x}''] = 0,$$

where $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$ denotes the SCALAR TRIPLE PRODUCT. The osculating plane passes through the tangent. The intersection of the osculating plane with the NORMAL PLANE is known as the PRINCIPAL NORMAL VECTOR. The VECTORS $\mathbf{T}$ and $\mathbf{N}$ (TANGENT VECTOR and NORMAL VECTOR) span the osculating plane.

see also NORMAL VECTOR, OSCULATING SPHERE, SCALAR TRIPLE PRODUCT, TANGENT VECTOR

Osculating Sphere

The center of any SPHERE which has a contact of (at least) first-order with a curve C at a point P lies in the normal plane to C at P. The center of any SPHERE which has a contact of (at least) second-order with C at point P, where the CURVATURE $\kappa > 0$, lies on the polar axis of C corresponding to P. All these SPHERES intersect the OSCULATING PLANE of C at P along a circle of curvature at P. The osculating sphere has center

$$\mathbf{a} = \mathbf{x} + \rho\hat{\mathbf{N}} + \frac{\dot{\rho}}{\tau}\hat{\mathbf{B}}$$

where $\hat{\mathbf{N}}$ is the unit NORMAL VECTOR, $\hat{\mathbf{B}}$ is the unit BINORMAL VECTOR, ρ is the RADIUS OF CURVATURE, and τ is the TORSION, and RADIUS

$$R = \sqrt{\rho^2 + \left(\frac{\dot{\rho}}{\tau}\right)^2},$$

and has contact of (at least) third order with C.

see also CURVATURE, OSCULATING PLANE, RADIUS OF CURVATURE, SPHERE, TORSION (DIFFERENTIAL GEOMETRY)

References
Kreyszig, E. *Differential Geometry.* New York: Dover, pp. 54–55, 1991.

Osedelec Theorem

For an n-D MAP, the LYAPUNOV CHARACTERISTIC EXPONENTS are given by

$$\sigma_i = \lim_{N\to\infty} \ln|\lambda_i(N)|$$

for $i = 1, \ldots, n$, where λ_i is the LYAPUNOV CHARACTERISTIC NUMBER.

see also LYAPUNOV CHARACTERISTIC EXPONENT, LYAPUNOV CHARACTERISTIC NUMBER

Ostrowski's Inequality

Let $f(x)$ be a monotonic function integrable on $[a, b]$ and let $f(a), f(b) \leq 0$ and $|f(a)| \geq |f(b)|$, then if g is a REAL function integrable on $[a, b]$,

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq |f(a)| \max_{a\leq\xi\leq b} \left|\int_a^\xi g(x)\,dx\right|.$$

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1100, 1979.

Ostrowski's Theorem

Let $\mathsf{A} = a_{ij}$ be a MATRIX with POSITIVE COEFFICIENTS and λ_0 be the POSITIVE EIGENVALUE in the FROBENIUS THEOREM, then the $n-1$ EIGENVALUES $\lambda_j \neq \lambda_0$ satisfy the INEQUALITY

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max_{i,j} a_{ij}$$
$$m = \min_{i,j} a_{ij}$$

and $i, j = 1, 2, \ldots, n$.

see also FROBENIUS THEOREM

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1121, 1980.

Otter's Tree Enumeration Constants

see TREE

Outdegree

The number of outward directed EDGES from a given VERTEX in a DIRECTED GRAPH.

see also DIRECTED GRAPH, INDEGREE, LOCAL DEGREE

Outer Automorphism Group

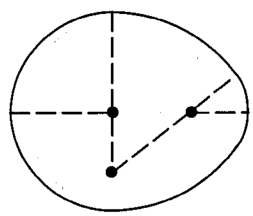
A particular type of AUTOMORPHISM GROUP which exists only for GROUPS. For a GROUP G, the outer automorphism group is the QUOTIENT GROUP $\mathrm{Aut}(G)/\mathrm{Inn}(G)$, which is the AUTOMORPHISM GROUP of G modulo its INNER AUTOMORPHISM GROUP.

see also AUTOMORPHISM GROUP, INNER AUTOMORPHISM GROUP, QUOTIENT GROUP

Outer Product

see DIRECT PRODUCT (TENSOR)

Oval



An oval is a curve resembling a squashed CIRCLE but, unlike the ELLIPSE, without a precise mathematical definition. The word oval derived from the Latin word "ovus" for egg. Unlike ellipses, ovals sometimes have only a single axis of reflection symmetry (instead of two).

Ovals can be constructed with a COMPASS by joining together arcs of different radii such that the centers of the arcs lie on a line passing through the join point (Dixon 1991). Albrecht Dürer used this method to design a Roman letter font.

see also CARTESIAN OVALS, CASSINI OVALS, EGG, ELLIPSE, OVOID, SUPERELLIPSE

References
Critchlow, K. *Time Stands Still.* London: Gordon Fraser, 1979.
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., 1989.
Dixon, R. *Mathographics.* New York: Dover, pp. 3–11, 1991.
Dixon, R. "The Drawing Out of an Egg." *New Sci.*, July 29, 1982.
Pedoe, D. *Geometry and the Liberal Arts.* London: Peregrine, 1976.

Oval of Descartes

see CARTESIAN OVALS

Ovals of Cassini

see CASSINI OVALS

Overlapping Resonance Method

see RESONANCE OVERLAP METHOD

Oversampling

A signal sampled at a frequency higher than the NYQUIST FREQUENCY is said to be oversampled β times, where the oversampling ratio is defined as

$$\beta \equiv \frac{\nu_{\text{sampling}}}{\nu_{\text{Nyquist}}}.$$

see also NYQUIST FREQUENCY, NYQUIST SAMPLING

Ovoid

An egg-shaped curve. Lockwood (1967) calls the NEGATIVE PEDAL CURVE of an ELLIPSE with ECCENTRICITY $e \leq 1/2$ an ovoid.

see also OVAL

References

Lockwood, E. H. *A Book of Curves.* Cambridge, England: Cambridge University Press, p. 157, 1967.

P

p-adic Number

A p-adic number is an extension of the FIELD of RATIONAL NUMBERS such that CONGRUENCES MODULO POWERS of a fixed PRIME p are related to proximity in the so called "p-adic metric."

Any NONZERO RATIONAL NUMBER x can be represented by

$$x = \frac{p^a r}{s}, \tag{1}$$

where p is a PRIME NUMBER, r and s are INTEGERS not DIVISIBLE by p, and a is a unique INTEGER. Then define the p-adic absolute value of x by

$$|x|_p = p^{-a}. \tag{2}$$

Also define the p-adic value

$$|0|_p = 0. \tag{3}$$

As an example, consider the FRACTION

$$\tfrac{140}{297} = 2^2 \cdot 3^{-3} \cdot 5 \cdot 7 \cdot 11^{-1}. \tag{4}$$

It has p-adic absolute values given by

$$|\tfrac{140}{297}|_2 = \tfrac{1}{4} \tag{5}$$
$$|\tfrac{140}{297}|_3 = 27 \tag{6}$$
$$|\tfrac{140}{297}|_5 = \tfrac{1}{5} \tag{7}$$
$$|\tfrac{140}{297}|_7 = \tfrac{1}{7} \tag{8}$$
$$|\tfrac{140}{297}|_{11} = 11. \tag{9}$$

The p-adic absolute value satisfies the relations

1. $|x|_p \geq 0$ for all x,
2. $|x|_p = 0$ IFF $x = 0$,
3. $|xy|_p = |x|_p\,|y|_p$ for all x and y,
4. $|x+y|_p \leq |x|_p + |y|_p$ for all x and y (the TRIANGLE INEQUALITY), and
5. $|x+y|_p \leq \max(|x|_p, |y|_p)$ for all x and y (the STRONG TRIANGLE INEQUALITY).

In the above, relation 4 follows trivially from relation 5, but relations 4 and 5 are relevant in the more general VALUATION THEORY.

The p-adics were probably first introduced by Hensel in 1902 in a paper which was concerned with the development of algebraic numbers in POWER SERIES. p-adic numbers were then generalized to VALUATIONS by Kürschák in 1913. In the early 1920s, Hasse formulated the LOCAL-GLOBAL PRINCIPLE (now usually called the HASSE PRINCIPLE), which is one of the chief applications of LOCAL FIELD theory. Skolem's p-adic method, which is used in attacking certain DIOPHANTINE EQUATIONS, is another powerful application of p-adic numbers. Another application is the theorem that the HARMONIC NUMBERS H_n are never INTEGERS (except for H_1). A similar application is the proof of the VON STAUDT-CLAUSEN THEOREM using the p-adic valuation, although the technical details are somewhat difficult. Yet another application is provided by the MAHLER-LECH THEOREM.

Every RATIONAL x has an "essentially" unique p-adic expansion ("essentially" since zero terms can always be added at the beginning)

$$x = \sum_{j=m}^{\infty} a_j p^j, \tag{10}$$

with m an INTEGER, a_j the INTEGERS between 0 and $p-1$ inclusive, and where the sum is convergent with respect to p-adic valuation. If $x \neq 0$ and $a_m \neq 0$, then the expansion is unique. Burger and Struppeck (1996) show that for p a PRIME and n a POSITIVE INTEGER,

$$|n!|_p = p^{-(n-A_p(n))/(p-1)}, \tag{11}$$

where the p-adic expansion of n is

$$n = a_0 + a_1 p + a_2 p^2 + \ldots + a_L P^L, \tag{12}$$

and

$$A_p(n) = a_0 + a_1 + \ldots + a_L. \tag{13}$$

For sufficiently large n,

$$|n!|_p \leq p^{-n/(2p-2)}. \tag{14}$$

The p-adic valuation on $\mathbb{Q}$ gives rise to the p-adic metric

$$d(x,y) = |x-y|_p, \tag{15}$$

which in turn gives rise to the p-adic topology. It can be shown that the rationals, together with the p-adic metric, do not form a COMPLETE METRIC SPACE. The completion of this space can therefore be constructed, and the set of p-adic numbers $\mathbb{Q}_p$ is defined to be this completed space.

Just as the REAL NUMBERS are the completion of the RATIONALS $\mathbb{Q}$ with respect to the usual absolute valuation $|x-y|$, the p-adic numbers are the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|x-y|_p$. The p-adic numbers are useful in solving DIOPHANTINE EQUATIONS. For example, the equation $X^2 = 2$ can easily be shown to have no solutions in the field of 2-adic numbers (we simply take the valuation of both sides). Because the 2-adic numbers contain the rationals as a subset, we can immediately see that the equation has no solutions in the RATIONALS. So we have an immediate proof of the irrationality of $\sqrt{2}$.

This is a common argument that is used in solving these types of equations: in order to show that an equation has no solutions in $\mathbb{Q}$, we show that it has no solutions in a FIELD EXTENSION. For another example, consider $X^2+1=0$. This equation has no solutions in $\mathbb{Q}$ because it has no solutions in the reals $\mathbb{R}$, and $\mathbb{Q}$ is a subset of $\mathbb{R}$.

Now consider the converse. Suppose we have an equation that does have solutions in $\mathbb{R}$ and in all the $\mathbb{Q}_p$. Can we conclude that the equation has a solution in $\mathbb{Q}$? Unfortunately, in general, the answer is no, but there are classes of equations for which the answer is yes. Such equations are said to satisfy the HASSE PRINCIPLE.

see also AX-KOCHEN ISOMORPHISM THEOREM, DIOPHANTINE EQUATION, HARMONIC NUMBER, HASSE PRINCIPLE, LOCAL FIELD, LOCAL-GLOBAL PRINCIPLE, MAHLER-LECH THEOREM, PRODUCT FORMULA, VALUATION, VALUATION THEORY, VON STAUDT-CLAUSEN THEOREM

References
Burger, E. B. and Struppeck, T. "Does $\sum_{n=0}^{\infty}\frac{1}{n!}$ Really Converge? Infinite Series and p-adic Analysis." *Amer. Math. Monthly* **103**, 565–577, 1996.
Cassels, J. W. S. and Scott, J. W. *Local Fields.* Cambridge, England: Cambridge University Press, 1986.
Gouvêa, F. Q. *P-adic Numbers: An Introduction, 2nd ed.* New York: Springer-Verlag, 1997.
Koblitz, N. *P-adic Numbers, P-adic Analysis, and Zeta-Functions, 2nd ed.* New York: Springer-Verlag, 1984.
Mahler, K. *P-adic Numbers and Their Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1981.

P-Circle

see SPIEKER CIRCLE

p-Element

see SEMISIMPLE

p-Good Path

A LATTICE PATH from one point to another is p-good if it lies completely below the line

$$y=(p-1)x.$$

Hilton and Pederson (1991) show that the number of p-good paths from $(1,\ q-1)$ to $(k,\ n-k)$ under the condition $2\le k\le n-p+1\le p(k-1)$ is

$$\binom{n-q}{k-1}-\sum_{j=1}^{\ell}{}_pd_{qj}\binom{n-pj}{k-j},$$

where $\binom{a}{b}$ is a BINOMIAL COEFFICIENT, and

$$\ell\equiv\left\lfloor\frac{n-k}{p-1}\right\rfloor,$$

where $\lfloor x\rfloor$ is the FLOOR FUNCTION.

see also CATALAN NUMBER, LATTICE PATH, SCHRÖDER NUMBER

References
Hilton, P. and Pederson, J. "Catalan Numbers, Their Generalization, and Their Uses." *Math. Intel.* **13**, 64–75, 1991.

p-Group

A FINITE GROUP of ORDER p^a for p a PRIME is called a p-group. Sylow proved that every GROUP of this form has a POWER-commutator representation on n generators defined by

$$a_i^p=\prod_{k=i+1}^{n}a_k^{\beta(i,k)}\tag{1}$$

for $0\le\beta(i,k)<p$, $1\le i\le n$ and

$$[a_j,a_i]=\prod_{k=j+1}^{n}a_k^{\beta(i,j,k)}\tag{2}$$

for $0\le\beta(i,j,k)<p$, $1\le i<j\le n$. If p is PRIME and $f(p)$ the number of GROUPS of order p^m, then

$$f(p)=p^{Am^2},\tag{3}$$

where

$$\lim_{m\to\infty}A=\tfrac{2}{27}\tag{4}$$

(Higman 1960a,b).

see also FINITE GROUP

References
Higman, G. "Enumerating p-Groups. I. Inequalities." *Proc. London Math. Soc.* **10**, 24–30, 1960a.
Higman, G. "Enumerating p-Groups. II. Problems Whose Solution is PORC." *Proc. London Math. Soc.* **10**, 566–582, 1960b.

p'-Group

X is a p'-group if p does not divide the ORDER of X.

p-Layer

The p-layer of H, $L_{p'}(H)$ is the unique minimal NORMAL SUBGROUP of H which maps onto $E(H/O_{p'}(H))$.

see also B_p-THEOREM, $L_{p'}$-BALANCE THEOREM, SIGNALIZER FUNCTOR THEOREM

P-Polynomial

see HOMFLY POLYNOMIAL

P-Problem

A problem is assigned to the P (POLYNOMIAL time) class if the number of steps is bounded by a POLYNOMIAL.

see also COMPLEXITY THEORY, NP-COMPLETE PROBLEM, NP-HARD PROBLEM, NP-PROBLEM

References

Borwein, J. M. and Borwein, P. B. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, 1987.

Greenlaw, R.; Hoover, H. J.; and Ruzzo, W. L. *Limits to Parallel Computation: P-Completeness Theory.* Oxford, England: Oxford University Press, 1995.

p-Series

A shorthand name for a POWER SERIES with a NEGATIVE exponent, $\sum_{k=1}^{\infty} k^{-p}$, where $p > 0$.

see also POWER SERIES, RIEMANN ZETA FUNCTION

p-Signature

Diagonalize a form over the rationals to

$$\operatorname{diag}[p^a \cdot A, p^b \cdot B, \ldots],$$

where all the entries are INTEGERS and A, B, ... are RELATIVELY PRIME to p. Then the p-signature of the form (for $p \neq -1, 2$) is

$$p^a + p^b + \ldots + 4k \pmod 8,$$

where k is the number of ANTISQUARES. For $p = -1$, the p-signature is SYLVESTER'S SIGNATURE.

see also SIGNATURE (QUADRATIC FORM)

P-Symbol

A symbol employed in a formal PROPOSITIONAL CALCULUS.

References

Nidditch, P. H. *Propositional Calculus.* New York: Free Press of Glencoe, p. 1, 1962.

P-Value

The PROBABILITY that a variate would assume a value greater than or equal to the observed value strictly by chance: $P(z \geq z_{\text{observed}})$.

see also ALPHA VALUE, SIGNIFICANCE

Paasche's Index

The statistical INDEX

$$P_P \equiv \frac{\sum p_n q_n}{\sum p_0 q_n},$$

where p_n is the price per unit in period n and q_n is the quantity produced in period n.

see also INDEX

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, p. 65, 1962.

Packing

The placement of objects so that they touch in some specified manner, often inside a container with specified properties.

see also BOX-PACKING THEOREM, CIRCLE PACKING, GROEMER PACKING, HYPERSPHERE PACKING, KEPLER PROBLEM, KISSING NUMBER PACKING DENSITY, POLYHEDRON PACKING, SPACE-FILLING POLYHEDRON, SPHERE PACKING

References

Eppstein, D. "Covering and Packing." `http://www.ics.uci.edu/~eppstein/junkyard/cover.html`.

Packing Density

The fraction of a volume filled by a given collection of solids.

see also HYPERSPHERE PACKING, PACKING, SPHERE PACKING

Padé Approximant

Approximants derived by expanding a function as a ratio of two POWER SERIES and determining both the NUMERATOR and DENOMINATOR COEFFICIENTS. Padé approximations are usually superior to TAYLOR EXPANSIONS when functions contain POLES, because the use of RATIONAL FUNCTIONS allows them to be well-represented.

The Padé approximant $R_{L/0}$ corresponds to the MACLAURIN SERIES. When it exists, the $R_{L/M} \equiv [L/M]$ Padé approximant to any POWER SERIES

$$A(x) = \sum_{j=0}^{\infty} a_j x^j \tag{1}$$

is unique. If $A(x)$ is a TRANSCENDENTAL FUNCTION, then the terms are given by the TAYLOR SERIES about x_0

$$a_n = \frac{1}{n!} A^{(n)}(x_0). \tag{2}$$

The COEFFICIENTS are found by setting

$$A(x) - \frac{P_L(x)}{Q_M(x)} = 0 \tag{3}$$

and equating COEFFICIENTS. $Q_M(x)$ can be multiplied by an arbitrary constant which will rescale the other COEFFICIENTS, so an addition constraint can be applied. The conventional normalization is

$$Q_M(0) = 1. \tag{4}$$

Expanding (3) gives

$$P_L(x) = p_0 + p_1 x + \ldots + p_L x^L \tag{5}$$

$$Q_M(x) = 1 + q_1 x + \ldots + q_M x^M. \tag{6}$$

Padé Approximant

These give the set of equations

$$a_0 = p_0 \tag{7}$$

$$a_1 + a_0 q_1 = p_1 \tag{8}$$

$$a_2 + a_1 q_1 + a_0 q_2 = p_2 \tag{9}$$

$$\vdots$$

$$a_L + a_{L-1} q_1 + \ldots + a_0 q_L = p_L \tag{10}$$

$$a_{L+1} + a_L q_1 + \ldots + a_{L-M+1} q_M = 0 \tag{11}$$

$$\vdots$$

$$q_{L+M} + a_{L+M-1} q_1 + \ldots + a_L q_M = 0, \tag{12}$$

where $a_n = 0$ for $n < 0$ and $q_j = 0$ for $j > M$. Solving these directly gives

$$[L/M] = \frac{\begin{vmatrix} a_{L-m+1} & a_{L-m+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{vmatrix}}{\begin{vmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}}, \tag{13}$$

where sums are replaced by a zero if the lower index exceeds the upper. Alternate forms are

$$\begin{aligned} [L/M] &= \sum_{j=0}^{L-M} a_j x^j + x^{L-M+1} \mathbf{w}_{L/M}^{\mathrm{T}} \mathsf{W}_{L/M}^{-1} \mathbf{w}_{L/M} \\ &= \sum_{j=0}^{L+n} a_j x^j + x^{L+n+1} \mathbf{w}_{(L+M)/M}^{\mathrm{T}} \mathsf{W}_{L/M}^{-1} \mathbf{w}_{(L+n)/M} \end{aligned}$$

for

$$\mathsf{W}_{L/M} = \begin{bmatrix} a_{L-M+1} - x a_{L-M+2} & \cdots & a_L - x a_{L+1} \\ \vdots & \ddots & \vdots \\ a_L - x a_{L+1} & \cdots & a_{L+M-1} - x a_{L+M} \end{bmatrix} \tag{14}$$

$$\mathbf{w}_{L/M} = \begin{bmatrix} a_{L-M+1} \\ a_{L-M+2} \\ \vdots \\ a_L \end{bmatrix}, \tag{15}$$

and $0 \le n \le M$.

The first few Padé approximants for e^x are

$$\begin{aligned}
\exp_{0/0}(x) &= 1 \\
\exp_{0/1}(x) &= \frac{1}{1-x} \\
\exp_{0/2}(x) &= \frac{2}{2-2x+x^2} \\
\exp_{0/3}(x) &= \frac{6}{6-6x+3x^2-x^3} \\
\exp_{1/0}(x) &= 1+x \\
\exp_{1/1}(x) &= \frac{2+x}{2-x} \\
\exp_{1/2}(x) &= \frac{6+2x}{6-4x+x^2} \\
\exp_{1/3}(x) &= \frac{24+6x}{24-18x+6x^2-x^3} \\
\exp_{2/0}(x) &= \frac{2+2x+x^2}{2} \\
\exp_{2/1}(x) &= \frac{6+4x+x^2}{6-2x} \\
\exp_{2/2}(x) &= \frac{12+6x+x^2}{12-6x+x^2} \\
\exp_{2/3}(x) &= \frac{60+24x+3x^2}{60-36x+9x^2-x^3} \\
\exp_{3/0}(x) &= \frac{6+6x+3x^2+x^3}{6} \\
\exp_{3/1}(x) &= \frac{24+18x+16x^2+x^3}{24-6x} \\
\exp_{3/2}(x) &= \frac{60+36x+9x^2+x^3}{60-24x+3x^2} \\
\exp_{3/3}(x) &= \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3}.
\end{aligned}$$

Two-term identities include

$$\frac{P_{L+1}(x)}{Q_{M+1}(x)} - \frac{P'_L(x)}{Q'_M(x)} = \frac{{C_{(L+1)/(M+1)}}^2 x^{L+M+1}}{Q_{M+1}(x) Q'_M(x)} \tag{16}$$

$$\frac{P_{L+1}(x)}{Q_M(x)} - \frac{P'_L(x)}{Q'_M(x)} = \frac{C_{(L+1)/M} C_{(L+1)/(M+1)} x^{L+M+1}}{Q_M(x) Q'_M(x)} \tag{17}$$

$$\frac{P_L(x)}{Q_{M+1}(x)} - \frac{P'_L(x)}{Q'_M(x)} = \frac{C_{L/(M+1)} C_{(L+1)/(M+1)} x^{L+M+1}}{Q_M(x) Q'_M(x)} \tag{18}$$

$$\frac{P_L(x)}{Q_{M+1}(x)} - \frac{P'_{L+1}(x)}{Q'_M} = \frac{{C_{(L+1)/(M+1)}}^2 x^{L+M+2}}{Q_{M+1} Q'_M} \tag{19}$$

$$\frac{P_{L+1}}{Q_M(x)} - \frac{P'_{L-1}(x)}{Q'_M(x)} = \frac{C_{L/(M+1)} C_{(L+1)/M} x^{L+M} + C_{L/M} C_{(L+1)/(M+1)} x^{L+M+1}}{Q_M(x) Q'_M(x)} \tag{20}$$

$$\frac{P_L(x)}{Q_{M+1}(x)} - \frac{P_L'(x)}{Q_{M-1}'(x)} = \frac{C_{L/(M+1)}C_{(L+1)/M}x^{L+M} - C_{L/M}C_{(L+1)/(M+1)}x^{L+M+1}}{Q_{M+1}(x)Q_{M-1}'(x)}, \tag{21}$$

where C is the C-DETERMINANT. Three-term identities can be derived using the FROBENIUS TRIANGLE IDENTITIES (Baker 1975, p. 32).

A five-term identity is

$$S_{(L+1)/M}S_{(L-1)/M} - S_{L/(M+1)}S_{L/(M-1)} = S_{L/M}{}^2. \tag{22}$$

Cross ratio identities include

$$\frac{(R_{L/M} - R_{L/(M+1)})(R_{(L+1)/M} - R_{(L+1)/(M+1)})}{(R_{L/M} - R_{(L+1)/M})(R_{L/(M+1)} - R_{(L+1)/(M+1)})} = \frac{C_{L/(M+1)}C_{(L+2)/(M+1)}}{C_{(L+1)/M}C_{(L+1)/(M+2)}} \tag{23}$$

$$\frac{(R_{L/M} - R_{(L+1)/(M+1)})(R_{(L+1)/M} - R_{L/(M+1)})}{(R_{L/M} - R_{L/(M+1)})(R_{(L+1)/M} - R_{(L+1)/(M+1)})} = \frac{C_{(L+1)/(M+1)}{}^2 x}{C_{L/(M+1)}C_{(L+2)/(M+1)}} \tag{24}$$

$$\frac{(R_{L/M} - R_{(L+1)/(M+1)})(R_{(L+1)/M} - R_{L/(M+1)})}{(R_{L/M} - R_{(L+1)/M})(R_{L/(M+1)} - R_{(L+1)/(M+1)})} = \frac{C_{(L+1)/(M+1)}{}^2 x}{C_{(L+1)/M}C_{(L+1)/(M+2)}} \tag{25}$$

$$\frac{(R_{L/M} - R_{(L+1)/(M-1)})(R_{L/(M+1)} - R_{(L+1)/M})}{(R_{L/M} - R_{L/(M+1)})(R_{(L+1)/(M+1)} - R_{(L+1)/M})} = \frac{C_{(L+1)/M}C_{(L+1)/(M+1)}x}{C_{L/(M+1)}C_{(L+2)/M}} \tag{26}$$

$$\frac{(R_{L/M} - R_{(L-1)/(M+1)})(R_{(L+1)/M} - R_{L/(M+1)})}{(R_{L/M} - R_{(L+1)/M})(R_{(L-1)/(M+1)} - R_{L/(M+1)})} = \frac{C_{L/(M+1)}C_{(L+1)/(M+1)}x}{C_{(L+1)/M}C_{L/(M+2)}}. \tag{27}$$

see also C-DETERMINANT, ECONOMIZED RATIONAL APPROXIMATION, FROBENIUS TRIANGLE IDENTITIES

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Padé Conjecture

If $P(z)$ is a POWER series which is regular for $|z| \leq 1$ except for m POLES within this CIRCLE and except for $z = +1$, at which points the function is assumed continuous when only points $|z| \leq 1$ are considered, then at least a subsequence of the $[N, N]$ PADÉ APPROXIMANTS are uniformly bounded in the domain formed by removing the interiors of small circles with centers at these POLES and uniformly continuous at $z = +1$ for $|z| \leq 1$.

see also PADÉ APPROXIMANT

References

Baker, G. A. Jr. "The Padé Conjecture and Some Consequences." §II.D in *Advances in Theoretical Physics, Vol. 1* (Ed. K. A. Brueckner). New York: Academic Press, pp. 23–27, 1965.

Padovan Sequence

The INTEGER SEQUENCE defined by the RECURRENCE RELATION

$$P(n) = P(n-2) + P(n-3)$$

with the initial conditions $P(0) = P(1) = P(2) = 1$. The first few terms are 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, ... (Sloane's A000931). The ratio $\lim_{n\to\infty} P(n)/P(n-1)$ is called the PLASTIC CONSTANT.

see also PERRIN SEQUENCE, PLASTIC CONSTANT

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Painlevé Property

Following the work of Fuchs in classifying first-order ORDINARY DIFFERENTIAL EQUATIONS, Painlevé studied second-order ODEs of the form

$$\frac{d^2y}{dx^2} = F(y', y, x),$$

where F is ANALYTIC in x and rational in y and y'. Painlevé found 50 types whose only movable SINGULARITIES are ordinary POLES. This characteristic is known as the Painlevé property. Six of the transcendents define new transcendents known as PAINLEVÉ TRANSCENDENTS, and the remaining 44 can be integrated in terms of classical transcendents, quadratures, or the PAINLEVÉ TRANSCENDENTS.

see also PAINLEVÉ TRANSCENDENTS

Painlevé Transcendents

$$y'' = 6y^2 + x \tag{1}$$

$$y'' = 2y^3 + xy + \alpha \tag{2}$$

$$y'' = \frac{y'^2}{y} - \frac{1}{xy'} + \alpha y^3 + \frac{\beta}{xy^2} + \frac{\gamma}{x} + \frac{\delta}{y}. \tag{3}$$

Transcendents 4–6 do not have known first integrals, but all transcendents have first integrals for special values of their parameters except (1). Painlevé found the above transcendents (1) to (3), and the rest were investigated by his students. The sixth transcendent was found by Gambier and contains the other five as limiting cases.

see also PAINLEVÉ PROPERTY

Pair

A SET of two numbers or objects linked in some way are said to be a pair. The pair a and b are usually denoted $(a,\ b)$. In certain circumstances, pairs are also called BROTHERS or TWINS.

see also AMICABLE PAIR, AUGMENTED AMICABLE PAIR, BROWN NUMBERS, FRIENDLY PAIR, HEXAD, HOMOGENEOUS NUMBERS, IMPULSE PAIR, IRREGULAR PAIR, LAX PAIR, LONG EXACT SEQUENCE OF A PAIR AXIOM, MONAD, ORDERED PAIR, PERKO PAIR, QUADRUPLET, QUASIAMICABLE PAIR, QUINTUPLET, REDUCED AMICABLE PAIR, SMITH BROTHERS, TRIAD, TRIPLET, TWIN PEAKS, TWIN PRIMES, TWINS, UNITARY AMICABLE PAIR, WILF-ZEILBERGER PAIR

Pair Sum

Given an AMICABLE PAIR (m,n), the quantity

$$\sigma(m) = \sigma(n) = s(m) + s(n) = m + n$$

is called the pair sum, where $\sigma(n)$ is the DIVISOR FUNCTION and $s(n)$ is the RESTRICTED DIVISOR FUNCTION.

see also AMICABLE PAIR

Paired t-Test

Given two paired sets X_i and Y_i of n measured values, the paired t-test determines if they differ from each other in a significant way. Let

$$\begin{aligned}\hat{X}_i &= (X_i - \bar{X}_i)\\ \hat{Y}_i &= (Y_i - \bar{Y}_i),\end{aligned}$$

then define t by

$$t = (\bar{X} - \bar{Y})\sqrt{\frac{n(n-1)}{\sum_{i=1}^n (\hat{X}_i - \hat{Y}_i)^2}}.$$

This statistic has $n-1$ DEGREES OF FREEDOM.

A table of STUDENT'S t-DISTRIBUTION confidence interval can be used to determine the significance level at which two distributions differ.

see also FISHER SIGN TEST, HYPOTHESIS TESTING, STUDENT'S t-DISTRIBUTION, WILCOXON SIGNED RANK TEST

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Paley Class

The Paley class of a POSITIVE INTEGER $m \equiv 0 \pmod 4$ is defined as the set of all possible QUADRUPLES (k,e,q,n) where

$$m = 2^e(q^n + 1),$$

q is an ODD PRIME, and

$$k = \begin{cases} 0 & \text{if } q = 0\\ 1 & \text{if } q^n - 3 \equiv 0 \pmod 4\\ 2 & \text{if } q^n - 1 \equiv 0 \pmod 4\\ \text{undefined} & \text{otherwise.}\end{cases}$$

see also HADAMARD MATRIX, PALEY CONSTRUCTION

Paley Construction

HADAMARD MATRICES H_n can be constructed using GALOIS FIELD GF(p^m) when $p = 4l - 1$ and m is ODD. Pick a representation r RELATIVELY PRIME to p. Then by coloring white $\lfloor (p-1)/2 \rfloor$ (where $\lfloor x \rfloor$ is the FLOOR FUNCTION) distinct equally spaced RESIDUES mod p (r^0, r, r^2, …; r^0, r^2, r^4, …; etc.) *in addition to* 0, a HADAMARD MATRIX is obtained if the POWERS of r (mod p) run through $< \lfloor (p-1)/2 \rfloor$. For example,

$$n = 12 = 11^1 + 1 = 2(5+1) = 2^2(2+1)$$

is of this form with $p = 11 = 4 \times 3 - 1$ and $m = 1$. Since $m = 1$, we are dealing with GF(11), so pick $p = 2$ and compute its RESIDUES (mod 11), which are

$$\begin{aligned}p^0 &\equiv 1\\ p^1 &\equiv 2\\ p^2 &\equiv 4\\ p^3 &\equiv 8\\ p^4 &\equiv 16 \equiv 5\\ p^5 &\equiv 10\\ p^6 &\equiv 20 \equiv 9\\ p^7 &\equiv 18 \equiv 7\\ p^8 &\equiv 14 \equiv 3\\ p^9 &\equiv 6\\ p^{10} &\equiv 12 \equiv 1.\end{aligned}$$

Picking the first $\lfloor 11/2 \rfloor = 5$ RESIDUES and adding 0 gives: 0, 1, 2, 4, 5, 8, which should then be colored in the MATRIX obtained by writing out the RESIDUES increasing to the left and up along the border (0 through $p-1$, followed by ∞), then adding horizontal and vertical coordinates to get the residue to place in each square.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \infty \\ 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \infty \\ 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \infty \\ 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \infty \\ 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & \infty \\ 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & \infty \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & 3 & \infty \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & \infty \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 & \infty \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 & \infty \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \infty \end{bmatrix}$$

H_{16} can be trivially constructed from $\mathsf{H}_4 \otimes \mathsf{H}_4$. H_{20} cannot be built up from smaller MATRICES, so use $n = 20 = 19 + 1 = 2(3^2 + 1) = 2^2(2^2 + 1)$. Only the first form can be used, with $p = 19 = 4 \times 5 - 1$ and $m = 1$. We therefore use GF(19), and color 9 RESIDUES plus 0 white. H_{24} can be constructed from $\mathsf{H}_2 \otimes \mathsf{H}_{12}$.

Now consider a more complicated case. For $n = 28 = 3^3 + 1 = 2(13 + 1)$, the only form having $p = 4l - 1$ is the first, so use the GF(3^3) field. Take as the modulus the IRREDUCIBLE POLYNOMIAL $x^3 + 2x + 1$, written 1021. A four-digit number can always be written using only three digits, since $1000 - 1021 \equiv 0012$ and $2000 - 2012 \equiv 0021$. Now look at the moduli starting with 10, where each digit is considered separately. Then

$x^0 \equiv 1$	$x^1 \equiv 10$	$x^2 \equiv 100$
$x^3 \equiv 1000 \equiv 12$	$x^4 \equiv 120$	$x^5 \equiv 1200 \equiv 212$
$x^6 \equiv 2120 \equiv 111$	$x^7 \equiv 1100 \equiv 122$	$x^8 \equiv 1220 \equiv 202$
$x^9 \equiv 2020 \equiv 11$	$x^{10} \equiv 110$	$x^{11} \equiv 1100 \equiv 112$
$x^{12} \equiv 1120 \equiv 102$	$x^{13} \equiv 1020 \equiv 2$	$x^{14} \equiv 20$
$x^{15} \equiv 200$	$x^{16} \equiv 2000 \equiv 21$	$x^{17} \equiv 210$
$x^{18} \equiv 2100 \equiv 121$	$x^{19} \equiv 1210 \equiv 222$	$x^{20} \equiv 2220 \equiv 211$
$x^{21} \equiv 2110 \equiv 101$	$x^{22} \equiv 101 \equiv 22$	$x^{23} \equiv 220$
$x^{24} \equiv 2200 \equiv 221$	$x^{25} \equiv 2210 \equiv 201$	$x^{26} \equiv 2010 \equiv 1$

Taking the alternate terms gives white squares as 000, 001, 020, 021, 022, 100, 102, 110, 111, 120, 121, 202, 211, and 221.

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Paley's Theorem

Proved in 1933. If q is an ODD PRIME or $q = 0$ and n is any POSITIVE INTEGER, then there is a HADAMARD MATRIX of order

$$m = 2^e(q^n + 1),$$

where e is any POSITIVE INTEGER such that $m \equiv 0 \pmod 4$. If m is of this form, the matrix can be constructed with a PALEY CONSTRUCTION. If m is divisible by 4 but not of the form (1), the PALEY CLASS is undefined. However, HADAMARD MATRICES have been shown to exist for all $m \equiv 0 \pmod 4$ for $m < 428$.

see also HADAMARD MATRIX, PALEY CLASS, PALEY CONSTRUCTION

Palindrome Number

see PALINDROMIC NUMBER

Palindromic Number

A symmetrical number which is written in some base b as $a_1 a_2 \ldots a_2 a_1$. The first few are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, 121, ... (Sloane's A002113).

The first few n for which the PRONIC NUMBER P_n is palindromic are 1, 2, 16, 77, 538, 1621, ... (Sloane's A028336), and the first few palindromic numbers which are PRONIC are 2, 6, 272, 6006, 289982, ... (Sloane's A028337). The first few numbers whose squares are palindromic are 1, 2, 3, 11, 22, 26, ... (Sloane's A002778), and the first few palindromic squares are 1, 4, 9, 121, 484, 676, ... (Sloane's A002779).

see also DEMLO NUMBER, PALINDROMIC NUMBER CONJECTURE, REVERSAL

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Palindromic Number Conjecture

Apply the 196-ALGORITHM, which consists of taking any POSITIVE INTEGER of two digits or more, reversing the digits, and adding to the original number. Now sum the two and repeat the procedure with the sum. Of the first 10,000 numbers, only 251 do not produce a PALINDROMIC NUMBER in ≤ 23 steps (Gardner 1979).

It was therefore conjectured that *all* numbers will eventually yield a PALINDROMIC NUMBER. However, the conjecture has been proven false for bases which are a POWER of 2, and seems to be false for base 10 as well. Among the first 100,000 numbers, 5,996 numbers apparently never generate a PALINDROMIC NUMBER (Gruenberger 1984). The first few are 196, 887, 1675, 7436, 13783, 52514, 94039, 187088, 1067869, 10755470, ... (Sloane's A006960).

It is conjectured, but not proven, that there are an infinite number of palindromic PRIMES. With the exception of 11, palindromic PRIMES must have an ODD number of digits.

see also 196-ALGORITHM

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Pancake Cutting

see CIRCLE CUTTING

Pancake Theorem

The 2-D version of the HAM SANDWICH THEOREM.

Pandiagonal Square

see PANMAGIC SQUARE

Pandigital

A decimal INTEGER which contains each of the digits from 0 to 9.

Panmagic Square

8	17	1	15	24
11	25	9	18	2
19	3	12	21	10
22	6	20	4	13
5	14	23	7	16

If all the diagonals (including those obtained by "wrapping around" the edges) of a MAGIC SQUARE, as well as the usual rows, columns, and *main diagonals* sum to the MAGIC CONSTANT, the square is said to be a PANMAGIC SQUARE (also called DIABOLICAL SQUARE, NASIK SQUARE, or PANDIAGONAL SQUARE). No panmagic squares exist of order 3 or any order $4k+2$ for k an INTEGER. The Siamese method for generating MAGIC SQUARES produces panmagic squares for orders $6k \pm 1$ with ordinary vector (2, 1) and break vector $(1, -1)$.

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

The LO SHU is not panmagic, but it is an ASSOCIATIVE MAGIC SQUARE. Order four squares can be panmagic or ASSOCIATIVE, but not both. Order five squares are the smallest which can be both ASSOCIATIVE and panmagic, and 16 distinct ASSOCIATIVE panmagic squares exist, one of which is illustrated above (Gardner 1988).

The number of distinct panmagic squares of order 1, 2, ... are 1, 0, 0, 384, 3600, 0, ... (Sloane's A027567, Hunter and Madachy 1975). Panmagic squares are related to HYPERCUBES.

see also ASSOCIATIVE MAGIC SQUARE, HYPERCUBE, FRANKLIN MAGIC SQUARE, MAGIC SQUARE

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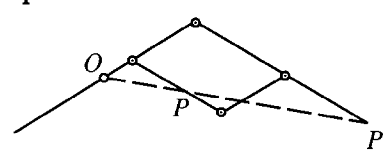
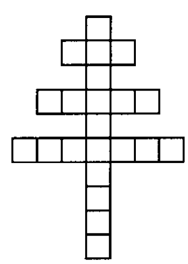
Pantograph



A LINKAGE invented in 1630 by Christoph Scheiner for making a scaled copy of a given figure. The linkage is pivoted at O; hinges are denoted $\odot$. By placing a PENCIL at P (or P'), a DILATED image is obtained at P' (or P).

see also LINKAGE

Papal Cross



see also CROSS

Pappus's Centroid Theorem

The SURFACE AREA of a SURFACE OF REVOLUTION is given by

$$S_{\text{solid of rotation}} = [\text{perimenter}] \times [\text{distance traveled by centroid}],$$

and the VOLUME of a SOLID OF REVOLUTION is given by

$$V_{\text{solid of rotation}} = [\text{cross-section area}] \times [\text{distance traveled by centroid}].$$

see also CENTROID (GEOMETRIC), CROSS-SECTION, PERIMETER, SOLID OF REVOLUTION, SURFACE AREA, SURFACE OF REVOLUTION, TOROID, TORUS

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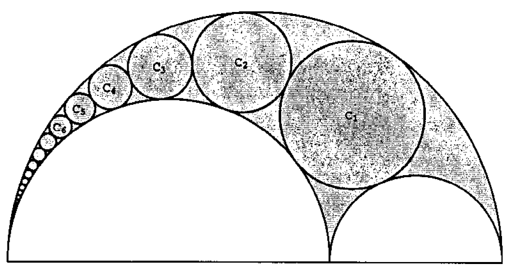
Pappus Chain



In the ARBELOS, construct a chain of TANGENT CIRCLES starting with the CIRCLE TANGENT to the two small interior semicircles and the large exterior one. Then the distance from the center of the first INSCRIBED CIRCLE to the bottom line is twice the CIRCLE'S RADIUS, from the second CIRCLE is four times the RADIUS, and for the nth CIRCLE is $2n$ times the RADIUS. The centers of the CIRCLES lie on an ELLIPSE, and the DIAMETER of the nth CIRCLE C_n is $(1/n)$th PERPENDICULAR distance to the base of the SEMICIRCLE. This result was known to Pappus, who referred to it as an ancient theorem (Hood 1961, Cadwell 1966, Gardner 1979, Bankoff 1981). The simplest proof is via INVERSIVE GEOMETRY.

If $r \equiv AB/AC$, then the radius of the nth circle in the pappus chain is

$$r_n = \frac{(1-r)r}{2[n^2(1-r)^2+r]}.$$

This equation can be derived by iteratively solving the QUADRATIC FORMULA generated by DESCARTES CIRCLE THEOREM for the radius of the SODDY CIRCLE. This general result simplifies to $r_n = 1/(6+n^2)$ for $r = 2/3$ (Gardner 1979). Further special cases when $AC = 1 + AB$ are considered by Gaba (1940).

If B divides AC in the GOLDEN RATIO ϕ, then the circles in the chain satisfy a number of other special properties (Bankoff 1955).

see also ARBELOS, COXETER'S LOXODROMIC SEQUENCE OF TANGENT CIRCLES, SODDY CIRCLES, STEINER CHAIN

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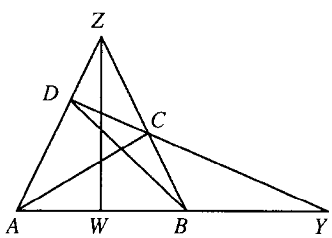
Pappus-Guldinus Theorem

see PAPPUS'S CENTROID THEOREM

Pappus's Harmonic Theorem



AW, AB, and AY in the above figure are in a HARMONIC RANGE.

see also CEVA'S THEOREM, MENELAUS' THEOREM

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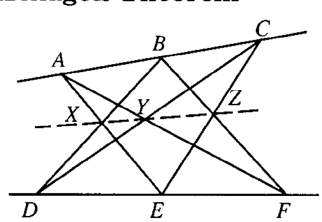
Pappus's Hexagon Theorem



If A, B, and C are three points on one LINE, D, E, and F are three points on another LINE, and AE meets BD at X, AF meets CD at Y, and BF meets CE at Z, then the three points X, Y, and Z are COLLINEAR. Pappus's hexagon theorem is essentially its own dual according to the DUALITY PRINCIPLE of PROJECTIVE GEOMETRY.

see also CAYLEY-BACHARACH THEOREM, DESARGUES' THEOREM, DUALITY PRINCIPLE, PASCAL'S THEOREM, PROJECTIVE GEOMETRY

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Pappus's Theorem

There are several THEOREMS that generally are known by the generic name "Pappus's Theorem."

see also PAPPUS'S CENTROID THEOREM, PAPPUS CHAIN, PAPPUS'S HARMONIC THEOREM, PAPPUS'S HEXAGON THEOREM

Parabiaugmented Dodecahedron

see JOHNSON SOLID

Parabiaugmented Hexagonal Prism

see JOHNSON SOLID

Parabiaugmented Truncated Dodecahedron

see JOHNSON SOLID

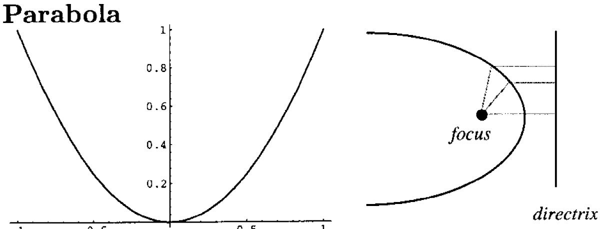
Parabidiminished Rhombicosidodecahedron

see JOHNSON SOLID

Parabigyrate Rhombicosidodecahedron

see JOHNSON SOLID

Parabola



The set of all points in the PLANE equidistant from a given LINE (the DIRECTRIX) and a given point not on the line (the FOCUS).

The parabola was studied by Menaechmus in an attempt to achieve CUBE DUPLICATION. Menaechmus solved the problem by finding the intersection of the two parabolas $x^2 = y$ and $y^2 = 2x$. Euclid wrote about the parabola, and it was given its present name by Apollonius. Pascal considered the parabola as a projection of a CIRCLE, and Galileo showed that projectiles falling under uniform gravity follow parabolic paths. Gregory and Newton considered the CATACAUSTIC properties of a parabola which bring parallel rays of light to a focus (MacTutor Archive).

For a parabola opening to the right, the equation in CARTESIAN COORDINATES is

$$\sqrt{(x-p)^2+y^2} = x+p \tag{1}$$

$$(x-p)^2+y^2 = (x+p)^2 \tag{2}$$

$$x^2-2px+p^2+y^2 = x^2+2px+p^2 \tag{3}$$

$$y^2 = 4px. \tag{4}$$

If the VERTEX is at (x_0, y_0) instead of (0, 0), the equation is

$$(y-y_0)^2 = 4p(x-x_0). \tag{5}$$

If the parabola opens upwards,

$$x^2 = 4py \tag{6}$$

(which is the form shown in the above figure at left). The quantity $4p$ is known as the LATUS RECTUM. In POLAR COORDINATES,

$$r = \frac{2a}{1-\cos\theta}. \tag{7}$$

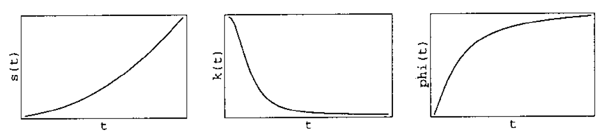
In PEDAL COORDINATES with the PEDAL POINT at the FOCUS, the equation is

$$p^2 = ar. \tag{8}$$

The parametric equations for the parabola are

$$x = 2at \tag{9}$$

$$y = at^2. \tag{10}$$



The CURVATURE, ARC LENGTH, and TANGENTIAL ANGLE are

$$\kappa(t) = \frac{1}{2(1+t^2)^{3/2}} \tag{11}$$

$$s(t) = t\sqrt{1+t^2} + \sinh^{-1} t \tag{12}$$

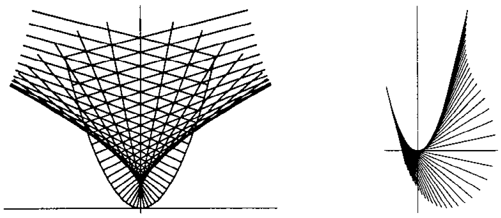
$$\phi(t) = \tan^{-1} t. \tag{13}$$

The TANGENT VECTOR of the parabola is

$$x_T(t) = \frac{1}{\sqrt{1+t^2}} \tag{14}$$

$$y_T(t) = \frac{t}{\sqrt{1+t^2}}. \tag{15}$$

The plots below show the normal and tangent vectors to a parabola.



see also CONIC SECTION, ELLIPSE, HYPERBOLA, QUADRATIC CURVE, REFLECTION PROPERTY, TSCHIRNHAUSEN CUBIC PEDAL CURVE

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Parabola Caustic

The CAUSTIC of a PARABOLA with rays PERPENDICULAR to the axis of the PARABOLA is TSCHIRNHAUSEN CUBIC.

Parabola Evolute

Given a PARABOLA

$$y = x^2, \tag{1}$$

the parametric equation and its derivatives are

$$\begin{array}{lll} x = t & x' = t & y' = 2t \\ y = t^2 & x'' = 0 & y'' = 2. \end{array} \tag{2}$$

The RADIUS OF CURVATURE is

$$R = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(1+4t^2)^{3/2}}{2}. \tag{3}$$

The TANGENT VECTOR is

$$\hat{\mathbf{T}} = \frac{1}{\sqrt{1+4t^2}} \begin{bmatrix} 1 \\ 2t \end{bmatrix}, \tag{4}$$

so the parametric equations of the evolute are

$$\xi = -4t^3 \tag{5}$$

$$\eta = \tfrac{1}{2} + 3t^2, \tag{6}$$

and

$$-\tfrac{1}{4}\xi = t^3 \tag{7}$$

$$\tfrac{1}{3}(\eta - \tfrac{1}{2}) = t^2 \tag{8}$$

$$\tfrac{1}{3}(\eta - \tfrac{1}{2}) = (-\tfrac{1}{4}\xi)^{2/3} \tag{9}$$

$$\tfrac{1}{3}(\eta - h) = \left(-\frac{2\xi}{8}\right)^{2/3} = \tfrac{1}{4}(2\xi)^{2/3}. \tag{10}$$

The EVOLUTE is therefore

$$\eta = \tfrac{3}{4}(2\xi)^{2/3} + \tfrac{1}{2}. \tag{11}$$

This is known as NEILE'S PARABOLA and is a SEMICUBICAL PARABOLA. From a point above the evolute three normals can be drawn to the PARABOLA, while only one normal can be drawn to the PARABOLA from a point below the EVOLUTE.

see also NEILE'S PARABOLA, PARABOLA, SEMICUBICAL PARABOLA

Parabola Inverse Curve

The INVERSE CURVE for a PARABOLA given by

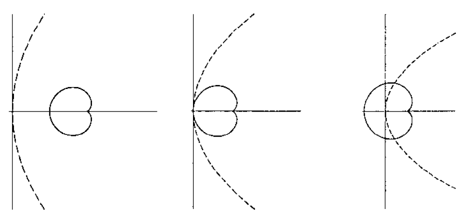
$$x = at^2 \tag{1}$$

$$y = 2at \tag{2}$$

with INVERSION CENTER (x_0, y_0) and INVERSION RADIUS k is

$$x = x_0 + \frac{k(at^2 - x_0)}{(at^2 + x_0)^2 + (2at - y_0)^2} \tag{3}$$

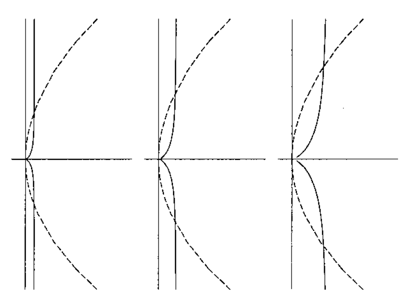
$$y = y_0 + \frac{k(2at - y_0)}{(at^2 + x_0)^2 + (2at - y_0)^2}. \tag{4}$$



For $(x_0, y_0) = (a, 0)$ at the FOCUS, the INVERSE CURVE is the CARDIOID

$$x = a + \frac{k(t^2 - 1)}{a(1+t^2)^2} \tag{5}$$

$$y = \frac{2kt}{a(1+t^2)^2}. \tag{6}$$

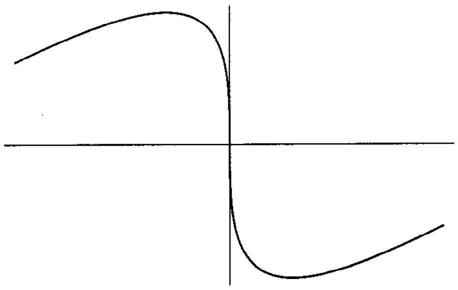


For $(x_0, y_0) = (0, 0)$ at the VERTEX, the INVERSE CURVE is the CISSOID OF DIOCLES

$$x = \frac{k}{a(4+t^2)} \tag{7}$$

$$y = \frac{2k}{at(4+t^2)}. \tag{8}$$

Parabola Involute



$$\frac{d\mathbf{r}}{dt} = \begin{bmatrix} 1 \\ 2t \end{bmatrix} \tag{1}$$

$$\hat{\mathbf{T}} = \frac{1}{\sqrt{1+4t^2}} \begin{bmatrix} 1 \\ 2t \end{bmatrix} \tag{2}$$

$$ds^2 = |d\mathbf{r}|^2 = (1+4t^2)\,dt^2 \tag{3}$$

$$ds = \sqrt{1+4t^2}\,dt \tag{4}$$

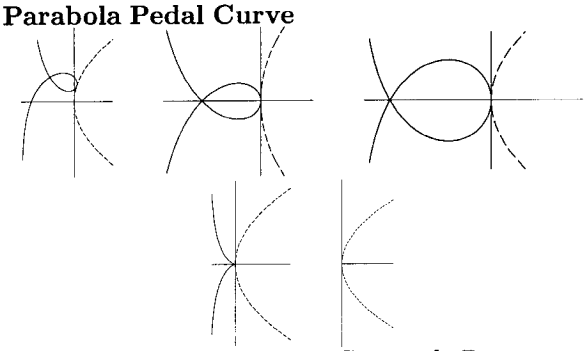
$$s = \int \sqrt{1+4t^2}\,dt = \tfrac{1}{2}t\sqrt{1+4t^2} + \tfrac{1}{4}\sinh^{-1}(2t). \tag{5}$$

So the equation of the INVOLUTE is

$$\mathbf{r}_i = \mathbf{r} - s\hat{\mathbf{T}} = \begin{bmatrix} t \\ t^2 \end{bmatrix} - \frac{\frac{1}{2}t\sqrt{1+4t^2} + \frac{1}{4}\sinh^{-1}(2t)}{\sqrt{1+4t^2}} \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$= \frac{1}{2\sqrt{1+4t^2}} \begin{bmatrix} t - \frac{1}{2}\sinh^{-1}(2t) \\ -\sinh^{-1}(2t) \end{bmatrix}. \tag{6}$$

Parabola Pedal Curve

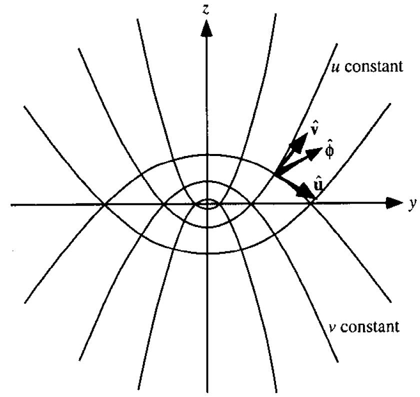


On the DIRECTRIX, the PEDAL CURVE of a PARABOLA is a STROPHOID (top left). On the foot of the DIRECTRIX, it is a RIGHT STROPHOID (top middle). On reflection of the FOCUS in the DIRECTRIX, it is a MACLAURIN TRISECTRIX (top right). On the VERTEX, it is a CISSOID OF DIOCLES (bottom left). On the FOCUS, it is a straight line (bottom right).

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 94–97, 1972.

Parabolic Coordinates



A system of CURVILINEAR COORDINATES in which two sets of coordinate surfaces are obtained by revolving the parabolas of PARABOLIC CYLINDRICAL COORDINATES about the x-AXIS, which is then relabeled the z-AXIS. There are several notational conventions. Whereas (u, v, θ) is used in this work, Arfken (1970) uses (ξ, η, φ).

The equations for the parabolic coordinates are

$$x = uv\cos\theta \tag{1}$$

$$y = uv\sin\theta \tag{2}$$

$$z = \tfrac{1}{2}(u^2 - v^2), \tag{3}$$

where $u \in [0, \infty)$, $v \in [0, \infty)$, and $\theta \in [0, 2\pi)$. To solve for u, v, and θ, examine

$$\begin{aligned} x^2 + y^2 + z^2 &= u^2v^2 + \tfrac{1}{4}(u^4 - 2u^2v^2 + v^4) \\ &= \tfrac{1}{4}(u^4 + 2u^2v^2 + v^4) \\ &= \tfrac{1}{4}(u^2 + v^2)^2, \end{aligned} \tag{4}$$

so

$$\sqrt{x^2+y^2+z^2} = \tfrac{1}{2}(u^2 + v^2) \tag{5}$$

and

$$\sqrt{x^2+y^2+z^2} + z = u^2 \tag{6}$$

$$\sqrt{x^2+y^2+z^2} - z = v^2. \tag{7}$$

We therefore have

$$u = \sqrt{\sqrt{x^2+y^2+z^2} + z} \tag{8}$$

$$v = \sqrt{\sqrt{x^2+y^2+z^2} - z} \tag{9}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{10}$$

The SCALE FACTORS are

$$h_u = \sqrt{u^2 + v^2} \tag{11}$$

$$h_v = \sqrt{u^2 + v^2} \tag{12}$$

$$h_\theta = uv. \tag{13}$$

The LINE ELEMENT is

$$ds^2 = (u^2+v^2)(du^2+dv^2)+u^2v^2\,d\theta^2, \tag{14}$$

and the VOLUME ELEMENT is

$$dV = uv(u^2+v^2)\,du\,dv\,d\theta. \tag{15}$$

The LAPLACIAN is

$$\begin{aligned}\nabla^2 f &= \frac{1}{uv(u^2+v^2)}\left[\frac{\partial}{\partial u}\left(uv\frac{\partial f}{\partial u}\right)+\frac{\partial}{\partial v}\left(uv\frac{\partial f}{\partial v}\right)\right] + \frac{1}{u^2v^2}\frac{\partial^2 f}{\partial\theta^2}\\ &= \frac{1}{u^2+v^2}\left[\frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial u}\right)+\frac{1}{v}\frac{\partial}{\partial v}\left(v\frac{\partial f}{\partial v}\right)\right]+\frac{1}{u^2v^2}\frac{\partial^2 f}{\partial\theta^2}\\ &= \frac{1}{u^2+v^2}\left(\frac{1}{u}\frac{\partial f}{\partial u}+\frac{\partial^2 f}{\partial u^2}+\frac{1}{v}\frac{\partial f}{\partial v}+\frac{\partial^2 f}{\partial v^2}\right)+\frac{1}{u^2v^2}\frac{\partial^2 f}{\partial\theta^2}.\end{aligned} \tag{16}$$

The HELMHOLTZ DIFFERENTIAL EQUATION is SEPARABLE in parabolic coordinates.

see also CONFOCAL PARABOLOIDAL COORDINATES, HELMHOLTZ DIFFERENTIAL EQUATION—PARABOLIC COORDINATES, PARABOLIC CYLINDRICAL COORDINATES

References

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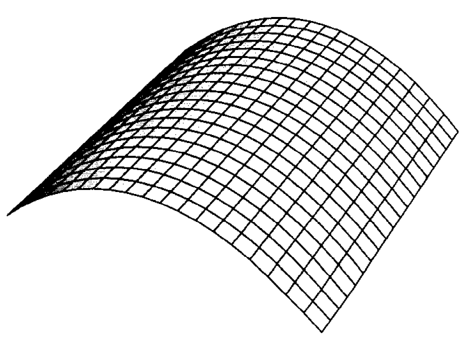
Parabolic Cyclide

A CYCLIDE formed by inversion of a STANDARD TORUS when the sphere of inversion is tangent to the torus.

see also PARABOLIC HORN CYCLIDE, PARABOLIC RING CYCLIDE, PARABOLIC SPINDLE CYCLIDE

Parabolic Cylinder



A QUADRATIC SURFACE given by the equation

$$x^2+2rz=0.$$

Parabolic Cylinder Function

These functions are sometimes called WEBER FUNCTIONS. Whittaker and Watson (1990, p. 347) define the parabolic cylinder functions as solutions to the WEBER DIFFERENTIAL EQUATION

$$\frac{d^2D_n(z)}{dz^2}+(n+\tfrac{1}{2}-\tfrac{1}{4}z^2)D_n(z)=0. \tag{1}$$

The two independent solutions are given by $D_n(z)$ and $D_{-n-1}(ze^{i\pi/2})$, where

$$D_n(z) = 2^{n/2+1/4}z^{-1/2}W_{n/2+1/4,-1/4}(\tfrac{1}{2}z^2) \tag{2}$$

$$= \frac{\Gamma(\frac{1}{2})2^{n/2+1/4}z^{-1/2}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)}M_{n/2+1/4,-1/4}(\tfrac{1}{2}z^2) + \frac{\Gamma(-\frac{1}{2})2^{n/2+1/4}z^{-1/2}}{\Gamma(-\frac{1}{2}n)}M_{n/2+1/4,1/4}(\tfrac{1}{2}z^2). \tag{3}$$

Here, $W_{a,b}(z)$ is a WHITTAKER FUNCTION and $M_{a,b}(z) = {}_1F_1(a;b;z)$ are CONFLUENT HYPERGEOMETRIC FUNCTIONS.

Abramowitz and Stegun (1972, p. 686) define the parabolic cylinder functions as solutions to

$$y''+(ax^2+bx+c)=0. \tag{4}$$

This can be rewritten by COMPLETING THE SQUARE,

$$y''+\left[a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c\right]y=0. \tag{5}$$

Now letting

$$u = x+\frac{b}{2a} \tag{6}$$

$$du = dx \tag{7}$$

gives

$$\frac{d^2y}{du^2}+(au^2+d)y=0 \tag{8}$$

where

$$d \equiv \frac{b^2}{4a}+c. \tag{9}$$

Equation (4) has the two standard forms

$$y''-(\tfrac{1}{4}x^2+a)y=0 \tag{10}$$

$$y''+(\tfrac{1}{4}x^2-a)y=0. \tag{11}$$

For a general a, the EVEN and ODD solutions to (10) are

$$y_1(x) = e^{-x^2/4}\,{}_1F_1(\tfrac{1}{2}a+\tfrac{1}{4};\tfrac{1}{2};\tfrac{1}{2}x^2) \tag{12}$$

$$y_2(x) = xe^{-x^2/4}\,{}_1F_1(\tfrac{1}{2}a+\tfrac{3}{4};\tfrac{3}{2};\tfrac{1}{2}x^2), \tag{13}$$

where ${}_1F_1(a;b;z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION. If $y(a,x)$ is a solution to (10), then (11) has solutions

$$y(\pm ia, xe^{\mp i\pi/4}), y(\pm ia, -xe^{\mp i\pi/4}). \tag{14}$$

Abramowitz and Stegun (1972, p. 687) define standard solutions to (10) as

$$U(a,x) = \cos[\pi(\tfrac{1}{4}+\tfrac{1}{2}a)]Y_1 - \sin[\pi(\tfrac{1}{4}+\tfrac{1}{2}a)]Y_2 \tag{15}$$

$$V(a,x) = \frac{\sin[\pi(\frac{1}{4}+\frac{1}{2}a)]Y_1 + \cos[\pi(\frac{1}{4}+\frac{1}{2}a)]Y_2}{\Gamma(\frac{1}{2}-a)}, \tag{16}$$

where

$$Y_1 \equiv \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{a/2+1/4}}y_1 = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{a/2+1/4}}e^{-x^2/4}{}_1F_1(\tfrac{1}{2}a+\tfrac{1}{4};\tfrac{1}{2};\tfrac{1}{2}x^2) \tag{17}$$

$$Y_2 \equiv \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{a/2+1/4}}y_2 = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{a/2+1/4}}xe^{-x^2/4}{}_1F_1(\tfrac{1}{2}a+\tfrac{3}{4};\tfrac{3}{2};\tfrac{1}{2}x^2). \tag{18}$$

In terms of Whittaker and Watscn's functions,

$$U(a,x) = D_{-a-1/2}(x) \tag{19}$$

$$V(a,x) = \frac{\Gamma(\frac{1}{2}+a)[\sin(\pi a)D_{-a-1/2}(x) + D_{-a-1/2}(-x)]}{\pi}. \tag{20}$$

For NONNEGATIVE INTEGER n, the solution D_n reduces to

$$D_n(x) = 2^{-n/2}e^{-x^2/4}H_n\left(\frac{x}{\sqrt{2}}\right) = e^{-x^2/4}\mathrm{He}_n(x), \tag{21}$$

where $H_n(x)$ is a HERMITE POLYNOMIAL and He_n is a modified HERMITE POLYNOMIAL.

The parabolic cylinder functions D_ν satisfy the RECURRENCE RELATIONS

$$D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0 \tag{22}$$

$$D'_\nu(z) + \tfrac{1}{2}zD_\nu(z) - \nu D_{\nu-1}(z) = 0. \tag{23}$$

The parabolic cylinder function for integral n can be defined in terms of an integral by

$$D_n(z) = \frac{1}{\pi}\int_0^\pi \sin(n\theta - z\sin\theta)\,d\theta \tag{24}$$

(Watson 1966, p. 308), which is similar to the ANGER FUNCTION. The result

$$\int_{-\infty}^{\infty} D_m(x)D_n(x)\,dx = \delta_{mn}n!\sqrt{2\pi}, \tag{25}$$

where δ_{ij} is the KRONECKER DELTA, can also be used to determine the COEFFICIENTS in the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n D_n \tag{26}$$

as

$$a_n = \frac{1}{n!\sqrt{2\pi}}\int_{-\infty}^{\infty} D_n(t)f(t)\,dt. \tag{27}$$

For ν real,

$$\int_0^\infty [D_\nu(t)]^2\,dt = \pi^{1/2}2^{-3/2}\frac{\phi_0(\frac{1}{2}-\frac{1}{2}\nu) - \phi_0(-\frac{1}{2}\nu)}{\Gamma(-\nu)} \tag{28}$$

(Gradshteyn and Ryzhik 1980, p. 885, 7.711.3), where $\Gamma(z)$ is the GAMMA FUNCTION and $\phi_0(z)$ is the POLYGAMMA FUNCTION of order 0.

see also ANGER FUNCTION, BESSEL FUNCTION, DARWIN'S EXPANSIONS, HH FUNCTION, STRUVE FUNCTION

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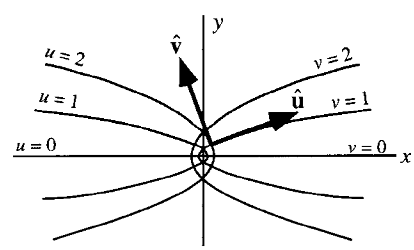
Parabolic Cylindrical Coordinates



A system of CURVILINEAR COORDINATES. There are several different conventions for the orientation and designation of these coordinates. Arfken (1970) defines coordinates (ξ, η, z) such that

$$x = \xi\eta \tag{1}$$
$$y = \tfrac{1}{2}(\eta^2 - \xi^2) \tag{2}$$
$$z = z. \tag{3}$$

In this work, following Morse and Feshbach (1953), the coordinates (u, v, z) are used instead. In this convention, the traces of the coordinate surfaces of the xy-PLANE are confocal PARABOLAS with a common axis. The u curves open into the NEGATIVE x-AXIS; the v curves open into the POSITIVE x-AXIS. The u and v curves intersect along the y-AXIS.

$$x = \tfrac{1}{2}(u^2 - v^2) \tag{4}$$
$$y = uv \tag{5}$$
$$z = z, \tag{6}$$

where $u \in [0, \infty)$, $v \in [0, \infty)$, and $z \in (-\infty, \infty)$. The SCALE FACTORS are

$$h_1 = \sqrt{u^2 + v^2} \tag{7}$$
$$h_2 = \sqrt{u^2 + v^2} \tag{8}$$
$$h_3 = 1. \tag{9}$$

LAPLACE'S EQUATION is

$$\nabla^2 f = \frac{1}{u^2 + v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial z^2}. \tag{10}$$

The HELMHOLTZ DIFFERENTIAL EQUATION is SEPARABLE in parabolic cylindrical coordinates.

see also CONFOCAL PARABOLOIDAL COORDINATES, HELMHOLTZ DIFFERENTIAL EQUATION—PARABOLIC CYLINDRICAL COORDINATES, PARABOLIC COORDINATES

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Arfken, G. "Parabolic Cylinder Coordinates (ξ, η, z)." §2.8 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, p. 97, 1970.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 658, 1953.

Parabolic Fixed Point

A FIXED POINT of a LINEAR TRANSFORMATION for which the rescaled variables satisfy

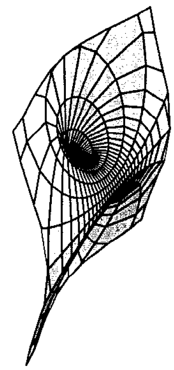
$$(\delta - \alpha)^2 + 4\beta\gamma = 0.$$

see also ELLIPTIC FIXED POINT (MAP), HYPERBOLIC FIXED POINT (MAP), LINEAR TRANSFORMATION

Parabolic Geometry

see EUCLIDEAN GEOMETRY

Parabolic Horn Cyclide



A PARABOLIC CYCLIDE formed by inversion of a HORN TORUS when the inversion sphere is tangent to the TORUS.

see also CYCLIDE, PARABOLIC RING CYCLIDE, PARABOLIC SPINDLE CYCLIDE

Parabolic Partial Differential Equation

A PARTIAL DIFFERENTIAL EQUATION of second-order, i.e., one of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0, \tag{1}$$

is called parabolic if the MATRIX

$$\mathsf{Z} \equiv \begin{bmatrix} A & B \\ B & C \end{bmatrix} \tag{2}$$

satisfies $\det(\mathsf{Z}) = 0$. The HEAT CONDUCTION EQUATION and other diffusion equations are examples. Initial-boundary conditions are used to give

$$u(x,t) = g(x,t) \quad \text{for } x \in \partial\Omega, t > 0 \tag{3}$$
$$u(x,0) = v(x) \quad \text{for } x \in \Omega, \tag{4}$$

where

$$u_{xx} = f(u_x, u_y, u, x, y) \tag{5}$$

holds in Ω.

see also ELLIPTIC PARTIAL DIFFERENTIAL EQUATION, HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION, PARTIAL DIFFERENTIAL EQUATION

Parabolic Point

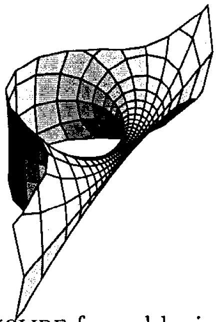
A point $\mathbf{p}$ on a REGULAR SURFACE $M \in \mathbb{R}^3$ is said to be parabolic if the GAUSSIAN CURVATURE $K(\mathbf{p}) = 0$ but $S(\mathbf{p}) \neq 0$ (where S is the SHAPE OPERATOR), or equivalently, exactly one of the PRINCIPAL CURVATURES κ_1 and κ_2 is 0.

see also ANTICLASTIC, ELLIPTIC POINT, GAUSSIAN CURVATURE, HYPERBOLIC POINT, PLANAR POINT, SYNCLASTIC

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 280, 1993.

Parabolic Ring Cyclide



A PARABOLIC CYCLIDE formed by inversion of a RING TORUS when the inversion sphere is tangent to the TORUS.

see also CYCLIDE, PARABOLIC HORN CYCLIDE, PARABOLIC SPINDLE CYCLIDE

Parabolic Rotation

The MAP

$$x' = x + 1 \tag{1}$$
$$y' = 2x + y + 1, \tag{2}$$

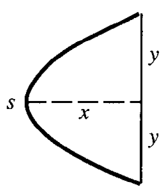
which leaves the PARABOLA

$$x'^2 - y' = (x+1)^2 - (2x+y+1) = x^2 - y \tag{3}$$

invariant.

see also PARABOLA, ROTATION

Parabolic Segment



The ARC LENGTH of the parabolic segment shown above is given by

$$s = \sqrt{4x^2+y^2} + \frac{y^2}{2x}\ln\left(\frac{2x+\sqrt{4x^2+y^2}}{y}\right). \tag{1}$$

The AREA contained between the curves

$$y = x^2 \tag{2}$$
$$y = ax + b \tag{3}$$

can be found by eliminating y,

$$x^2 - ax - b = 0, \tag{4}$$

so the points of intersection are

$$x_\pm = \tfrac{1}{2}(a \pm \sqrt{a^2+4b}). \tag{5}$$

Therefore, for the AREA to be NONNEGATIVE, $a^2 + 4b > 0$, and

$$\begin{aligned} x_\pm &= \tfrac{1}{4}(a^2 \pm 2a\sqrt{a^2+b^2} + a^2 + 4b) \\ &= \tfrac{1}{4}(2a^2 + 4b \pm 2a\sqrt{a^2+4b}) \\ &= \tfrac{1}{2}(a^2 + 2b \pm a\sqrt{a^2+4b}), \end{aligned} \tag{6}$$

so the AREA is

$$\begin{aligned} A &= \int_{x_-}^{x_+} [(ax+b) - x^2]\,dx \\ &= \left[\tfrac{1}{2}ax^2 + bx - \tfrac{1}{3}x^3\right]_{(a-\sqrt{a^2+4b})/2}^{(a+\sqrt{a^2+4b})/2}. \end{aligned} \tag{7}$$

Now,

$$\begin{aligned} {x_+}^2 - {x_-}^2 &= \tfrac{1}{4}\Big[(a^2 + 2a\sqrt{a^2+4b} + a^2 + 4b) \\ &\quad - (a^2 - 2a\sqrt{a^2+4b} + a^2 + 4b)\Big] \\ &= \tfrac{1}{4}\left[4a\sqrt{a^2+4b}\right] = a\sqrt{a^2+4b} \end{aligned} \tag{8}$$

$$\begin{aligned} {x_+}^3 - {x_-}^3 &= (x_+ - x_-)({x_+}^2 + x_-x_+ + {x_-}^2) \\ &= \sqrt{a^2+4b}\Big\{\tfrac{1}{4}(a^2 + 2a\sqrt{a^2+4b} + a^2 + 4b) \\ &\quad + \tfrac{1}{4}[a^2 - (a^2+4b)] + \tfrac{1}{4}(a^2 - 2a\sqrt{a^2+4b} + a^2 + 4b)\Big\} \\ &= \tfrac{1}{4}\sqrt{a^2+4b}\,(4a^2+4b) = \sqrt{a^2+4b}\,(a^2+b). \end{aligned} \tag{9}$$

So

$$\begin{aligned} A &= \tfrac{1}{2}a^2\sqrt{a^2+4b} + b\sqrt{a^2+4b} = \tfrac{1}{3}(a^2+b)\sqrt{a^2+4b} \\ &= \sqrt{a^2+4b}\left[(\tfrac{1}{2} - \tfrac{1}{3})a^2 + b(1 - \tfrac{1}{3})\right] \\ &= (\tfrac{1}{6}a^2 + \tfrac{2}{3}b)\sqrt{a^2+4b} \\ &= \tfrac{1}{6}(a^2+4b)\sqrt{a^2+4b} = \tfrac{1}{6}(a^2+4b)^{3/2}. \end{aligned} \tag{10}$$

We now wish to find the maximum AREA of an inscribed TRIANGLE. This TRIANGLE will have two of its VERTICES at the intersections, and AREA

$$A_\Delta = \tfrac{1}{2}(x_-y_* - x_*y_- - x_+y_* + x_*y_+ + x_+y_- - x_-y_+). \tag{11}$$

But $y_* = {x_*}^2$, so

$$\begin{aligned} A_\Delta &= \tfrac{1}{2}(x_-{x_*}^2 - x_*y_- - x_+{x_*}^2 \\ &\quad + x_*y_* + x_+y_- - x_-y_+) \\ &= \tfrac{1}{2}[-{x_*}^2(x_+ - x_-) + x_*(y_+ - y_-) \\ &\quad + (x_+y_- - x_-y_+)]. \end{aligned} \tag{12}$$

The maximum AREA will occur when

$$\frac{\partial A_\Delta}{\partial x_*} = \tfrac{1}{2}[-2(x_+ - x_-)x_* + (y_+ - y_-)] = 0. \tag{13}$$

But

$$x_+ - x_- = \sqrt{a^2+4b} \quad (14)$$

$$y_+ - y_- = a\sqrt{a^2+4b}, \quad (15)$$

so

$$x_* = \frac{1}{2}\frac{y_+ - y_-}{x_+ - x_-} = \tfrac{1}{2}a \quad (16)$$

and

$$A_\Delta = \tfrac{1}{2}[-(\tfrac{1}{2}a)^2(x_+ - x_-) + (\tfrac{1}{2}a)(y_+ - y_-) + (x_+y_- - x_-y_+)]. \quad (17)$$

Working on the third term

$$\begin{aligned} x_+y_- &= \tfrac{1}{4}(a+\sqrt{a^2+4b})(a^2+2b-a\sqrt{a^2+4b}) \\ &= \tfrac{1}{4}\Big[a^3+2ab-a^2\sqrt{a^2+4b}+a^2\sqrt{a^2+4b} \\ &\quad +2b\sqrt{a^2+4b}-a(a^2+4b)\Big] \\ &= \tfrac{1}{4}[-2ab+2b\sqrt{a^2+4b}] \quad (18) \\ x_-y_+ &= \tfrac{1}{4}(a-\sqrt{a^2+4b})(a^2+2b+a\sqrt{a^2+4b}) \\ &= \tfrac{1}{4}\Big[a^3+2ab+a^2\sqrt{a^2+4b}-a^2\sqrt{a^2+4b} \\ &\quad -2b\sqrt{a^2+4b}-a(a^2+4b)\Big] \\ &= \tfrac{1}{4}[-2ab-2b\sqrt{a^2+4b}], \quad (19) \end{aligned}$$

so

$$x_+y_- - x_-y_+ = \tfrac{1}{4}(4b\sqrt{a^2+4b}) = b\sqrt{a^2+4b} \quad (20)$$

and

$$\begin{aligned} A_\Delta &= \tfrac{1}{2}(-\tfrac{1}{4}a^2\sqrt{a^2+4b}+\tfrac{1}{2}a^2\sqrt{a^2+4b}+b\sqrt{a^2+b^2}) \\ &= \tfrac{1}{2}\sqrt{a^2+4b}\left[(\tfrac{1}{2}-\tfrac{1}{4})a^2+b\right] = \tfrac{1}{2}\sqrt{a^2+4b}\,(\tfrac{1}{4}a^2+b) \\ &= \tfrac{1}{8}\sqrt{a^2+4b}\,(a^2+4b) = \tfrac{1}{8}(a^2+4b)^{3/2}, \quad (21) \end{aligned}$$

which gives the result known to Archimedes in the third century BC that

$$\frac{A}{A_\Delta} = \frac{\frac{1}{6}}{\frac{1}{8}} = \frac{4}{3}. \quad (22)$$

The AREA of the parabolic segment of height h opening upward along the y-AXIS is

$$A = 2\int_0^h \sqrt{y}\,dy = \tfrac{1}{3}h^{3/2}. \quad (23)$$

The weighted mean of y is

$$\langle y\rangle = 2\int_0^h y\sqrt{y}\,dy = 2\int_0^h y^{3/2}\,dy = \tfrac{4}{5}h^{5/2}. \quad (24)$$

The CENTROID is then given by

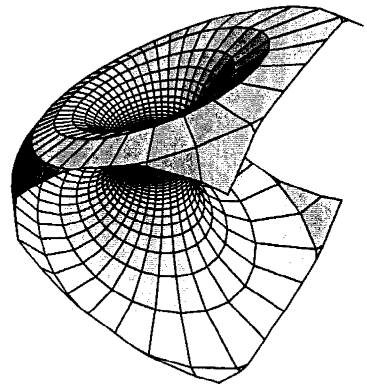
$$\bar{y} = \frac{\langle y\rangle}{A} = \tfrac{3}{5}h. \quad (25)$$

see also CENTROID (GEOMETRIC), PARABOLA, SEGMENT

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 125, 1987.

Parabolic Spindle Cyclide



A PARABOLIC CYCLIDE formed by inversion of a SPINDLE TORUS when the inversion sphere is tangent to the TORUS.

see also CYCLIDE, PARABOLIC HORN CYCLIDE, PARABOLIC RING CYCLIDE

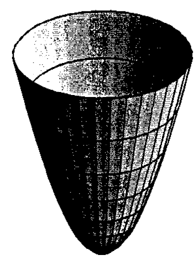
Parabolic Spiral

see FERMAT'S SPIRAL

Parabolic Umbilic Catastrophe

A CATASTROPHE which can occur for four control factors and two behavior axes.

Paraboloid



The SURFACE OF REVOLUTION of the PARABOLA. It is a QUADRATIC SURFACE which can be specified by the Cartesian equation

$$z = a(x^2+y^2), \quad (1)$$

or parametrically by

$$x(u,v) = \sqrt{u}\cos v \quad (2)$$

$$y(u,v) = \sqrt{u}\sin v \quad (3)$$

$$z(u,v) = u, \quad (4)$$

where $u \in [0,h]$, $v \in [0,2\pi)$, and h is the height.

The VOLUME of the paraboloid is

$$V = \pi\int_0^h z\,dz = \tfrac{1}{2}\pi h^2. \quad (5)$$

The weighted mean of z over the paraboloid is

$$\langle z\rangle = \pi\int_0^h z^2\,dz = \tfrac{1}{3}\pi h^3. \tag{6}$$

The CENTROID is then given by

$$\bar{z} = \frac{\langle z\rangle}{V} = \tfrac{2}{3}h \tag{7}$$

(Beyer 1987).

see also ELLIPTIC PARABOLOID, HYPERBOLIC PARABOLOID, PARABOLA

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 133, 1987.

Gray, A. "The Paraboloid." §11.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 221–222, 1993.

Paraboloid Geodesic

A GEODESIC on a PARABOLOID has differential parameters defined by

$$\begin{aligned} P &\equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ &= 1 + \frac{\cos^2 v}{4u} + \frac{\sin^2 v}{4u} = 1 + \frac{1}{4u} && (1) \\ Q &\equiv \frac{\partial^2 x}{\partial u\partial v} + \frac{\partial^2 y}{\partial u\partial v} + \frac{\partial^2 z}{\partial u\partial v} \\ &= 0 + u\cos^2 v + u\sin^2 v = u && (2) \\ R &\equiv 0 - \frac{\sin v}{2\sqrt{u}} + \frac{\cos v}{2\sqrt{u}} = \frac{1}{2\sqrt{u}}(\cos v - \sin v). && (3) \end{aligned}$$

The GEODESIC is then given by solving the EULER-LAGRANGE DIFFERENTIAL EQUATION

$$\frac{\frac{\partial P}{\partial v} + 2v'\frac{\partial Q}{\partial v} + v'^2\frac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + Rv'^2}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}}\right) = 0. \tag{4}$$

As given by Weinstock (1974), the solution simplifies to

$$\begin{aligned} &u - c^2 \\ &= u(1+4c^2)\sin^2\{v - 2c\ln[k(2\sqrt{u-c^2} + \sqrt{4u+1})]\}. \end{aligned} \tag{5}$$

see also GEODESIC

References

Weinstock, R. *Calculus of Variations, with Applications to Physics and Engineering.* New York: Dover, p. 45, 1974.

Paraboloidal Coordinates

see CONFOCAL PARABOLOIDAL COORDINATES

Paracompact Space

A paracompact space is a HAUSDORFF SPACE such that every open COVER has a LOCALLY FINITE open REFINEMENT. Paracompactness is a very common property that TOPOLOGICAL SPACES satisfy. Paracompactness is similar to the compactness property, but generalized for slightly "bigger" SPACES. All MANIFOLDS (e.g, second countable and Hausdorff) are paracompact.

see also HAUSDORFF SPACE, LOCALLY FINITE SPACE, MANIFOLD, TOPOLOGICAL SPACE

Paracycle

see ASTROID

Paradox

A statement which appears self-contradictory or contrary to expectations, also known as an ANTINOMY. Bertrand Russell classified known logical paradoxes into seven categories.

Ball and Coxeter (1987) give several examples of geometrical paradoxes.

see also ALIAS' PARADOX, ARISTOTLE'S WHEEL PARADOX, ARROW'S PARADOX, BANACH-TARSKI PARADOX, BARBER PARADOX, BERNOULLI'S PARADOX, BERRY PARADOX, BERTRAND'S PARADOX, CANTOR'S PARADOX, COASTLINE PARADOX, COIN PARADOX, ELEVATOR PARADOX, EPIMENIDES PARADOX, EUBULIDES PARADOX, GRELLING'S PARADOX, HAUSDORFF PARADOX, HEMPEL'S PARADOX, HETEROLOGICAL PARADOX, LEONARDO'S PARADOX, LIAR'S PARADOX, LOGICAL PARADOX, POTATO PARADOX, RICHARD'S PARADOX, RUSSELL'S PARADOX, SAINT PETERSBURG PARADOX, SIEGEL'S PARADOX, SIMPSON'S PARADOX, SKOLEM PARADOX, SMARANDACHE PARADOX, SOCRATES' PARADOX, SORITES PARADOX, THOMSON LAMP PARADOX, UNEXPECTED HANGING PARADOX, ZEEMAN'S PARADOX, ZENO'S PARADOXES

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 84–86, 1987.

Bunch, B. *Mathematical Fallacies and Paradoxes.* New York: Dover, 1982.

Carnap, R. *Introduction to Symbolic Logic and Its Applications.* New York: Dover, 1958.

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Kasner, E. and Newman, J. R. "Paradox Lost and Paradox Regained." In *Mathematics and the Imagination.* Redmond, WA: Tempus Books, pp. 193–222, 1989.

Northrop, E. P. *Riddles in Mathematics: A Book of Paradoxes.* Princeton, NJ: Van Nostrand, 1944.

O'Beirne, T. H. *Puzzles and Paradoxes.* New York: Oxford University Press, 1965.

Quine, W. V. "Paradox." *Sci. Amer.* **206**, 84–96, Apr. 1962.

Paradromic Rings

Rings produced by cutting a strip that has been given m half twists and been re-attached into n equal strips (Ball and Coxeter 1987, pp. 127–128).

see also MÖBIUS STRIP

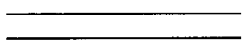
References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 127–128, 1987.

Paragyrate Diminished Rhombicosidodecahedron

see JOHNSON SOLID

Parallel



Two lines in 2-dimensional EUCLIDEAN SPACE are said to be parallel if they do not intersect. In 3-dimensional EUCLIDEAN SPACE, parallel lines not only fail to intersect, but also maintain a constant separation between points closest to each other on the two lines. (Lines in 3-space which are not parallel but do not intersect are called SKEW LINES.)

In a NON-EUCLIDEAN GEOMETRY, the concept of parallelism must be modified from its intuitive meaning. This is accomplished by changing the so-called PARALLEL POSTULATE. While this has counterintuitive results, the geometries so defined are still completely self-consistent.

see also ANTIPARALLEL, HYPERPARALLEL, LINE, NON-EUCLIDEAN GEOMETRY, PARALLEL CURVE, PARALLEL POSTULATE PERPENDICULAR, SKEW LINES

Parallel Axiom

see PARALLEL POSTULATE

Parallel Class

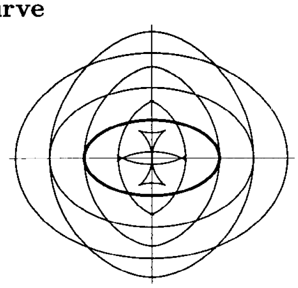
A set of blocks, also called a RESOLUTION CLASS, that partition the set V, where (V, B) is a balanced incomplete BLOCK DESIGN.

see also BLOCK DESIGN, RESOLVABLE

References

Abel, R. J. R. and Furino, S. C. "Resolvable and Near Resolvable Designs." §I.6 in *The CRC Handbook of Combinatorial Designs* (Ed. C. J. Colbourn and J. H. Dinitz). Boca Raton, FL: CRC Press, pp. 87–94, 1996.

Parallel Curve



The two branches of the parallel curve a distance k away from a parametrically represented curve $(f(t), g(t))$ are

$$x = f \pm \frac{kg'}{\sqrt{f'^2 + g'^2}}$$
$$y = g \mp \frac{kf'}{\sqrt{f'^2 + g'^2}}.$$

The above figure shows the curves parallel to the ELLIPSE.

References

Gray, A. "Parallel Curves." §5.7 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 95–97, 1993.

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 42–43, 1972.

Lee, X. "Parallel." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Parallel_dir/parallel.html`.

Yates, R. C. "Parallel Curves." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 155–159, 1952.

Parallel Postulate

Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line, no matter how far they are extended. This statement is equivalent to the fifth of EUCLID'S POSTULATES, which Euclid himself avoided using until proposition 29 in the *Elements.* For centuries, many mathematicians believed that this statement was not a true postulate, but rather a theorem which could be derived from the first four of EUCLID'S POSTULATES. (That part of geometry which could be derived using only postulates 1–4 came to be known as ABSOLUTE GEOMETRY.)

Over the years, many purported proofs of the parallel postulate were published. However, none were correct, including the 28 "proofs" G. S. Klügel analyzed in his dissertation of 1763 (Hofstadter 1989). In 1823, Janos Bolyai and Lobachevsky independently realized that entirely self-consistent "NON-EUCLIDEAN GEOMETRIES" could be created in which the parallel postulate *did not hold.* (Gauss had also discovered but suppressed the existence of non-Euclidean geometries.)

As stated above, the parallel postulate describes the type of geometry now known as PARABOLIC GEOMETRY. If, however, the phrase "exists one and only one straight line which passes" is replace by "exist no line which passes," or "exist at least two lines which pass," the postulate describes equally valid (though less intuitive) types of geometries known as ELLIPTIC and HYPERBOLIC GEOMETRIES, respectively.

The parallel postulate is equivalent to the EQUIDISTANCE POSTULATE, PLAYFAIR'S AXIOM, PROCLUS' AXIOM, TRIANGLE POSTULATE. There is also a single parallel axiom in HILBERT'S AXIOMS which is equivalent to Euclid's parallel postulate.

see also ABSOLUTE GEOMETRY, EUCLID'S AXIOMS, EUCLIDEAN GEOMETRY, HILBERT'S AXIOMS, NON-EUCLIDEAN GEOMETRY, PLAYFAIR'S AXIOM, TRIANGLE POSTULATE

References
Dixon, R. *Mathographics.* New York: Dover, p. 27, 1991.
Hilbert, D. *The Foundations of Geometry, 2nd ed.* Chicago, IL: Open Court, 1980.
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, pp. 88–92, 1989.
Iyanaga, S. and Kawada, Y. (Eds.). "Hilbert's System of Axioms." §163B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 544–545, 1980.

Parallel (Surface of Revolution)

A parallel of a SURFACE OF REVOLUTION is the intersection of the surface with a PLANE orthogonal to the axis of revolution.

see also MERIDIAN, SURFACE OF REVOLUTION

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 358, 1993.

Parallelepiped

In 3-D, a parallelepiped is a PRISM whose faces are all PARALLELOGRAMS. The volume of a 3-D parallelepiped is given by the SCALAR TRIPLE PRODUCT

$$\begin{aligned} V_{\text{parallelepiped}} &= |\mathbf{B}\cdot(\mathbf{B}\times\mathbf{C})| \\ &= |\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})| = |\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})|. \end{aligned}$$

In n-D, a parallelepiped is the POLYTOPE spanned by n VECTORS $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a VECTOR SPACE over the reals,

$$\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = t_1\mathbf{v}_1 + \ldots + t_n\mathbf{v}_n,$$

where $t_i \in [0,1]$ for $i = 1, \ldots, n$. In the usual interpretation, the VECTOR SPACE is taken as EUCLIDEAN SPACE, and the CONTENT of this parallelepiped is given by

$$\operatorname{abs}(\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)),$$

where the sign of the determinant is taken to be the "orientation" of the "oriented volume" of the parallelepiped.

see also PRISMATOID, RECTANGULAR PARALLELEPIPED, ZONOHEDRON

References
Phillips, A. W. and Fisher, I. *Elements of Geometry.* New York: Amer. Book Co., 1896.

Parallelism

see ANGLE OF PARALLELISM

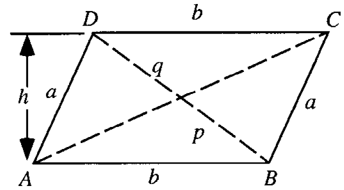
Parallelizable

A sphere $\mathbb{S}^n$ is parallelizable if there exist n cuts containing linearly independent tangent vectors. There exist only three parallelizable spheres: $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$ (Adams 1962, Le Lionnais 1983).

see also SPHERE

References
Adams, J. F. "On the Non-Existence of Elements of Hopf Invariant One." *Bull. Amer. Math. Soc.* **64**, 279–282, 1958.
Adams, J. F. "On the Non-Existence of Elements of Hopf Invariant One." *Ann. Math.* **72**, 20–104, 1960.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 49, 1983.

Parallelogram



A QUADRILATERAL with opposite sides parallel (and therefore opposite angles equal). A quadrilateral with equal sides is called a RHOMBUS, and a parallelogram whose ANGLES are all RIGHT ANGLES is called a RECTANGLE.

A parallelogram of base b and height h has AREA

$$A = bh = ab\sin A = ab\sin B. \tag{1}$$

The height of a parallelogram is

$$h = a\sin A = a\sin B, \tag{2}$$

and the DIAGONALS are

$$p = \sqrt{a^2 + b^2 - 2ab\cos A} \tag{3}$$
$$q = \sqrt{a^2 + b^2 - 2ab\cos B} \tag{4}$$
$$= \sqrt{a^2 + b^2 + 2ab\cos A} \tag{5}$$

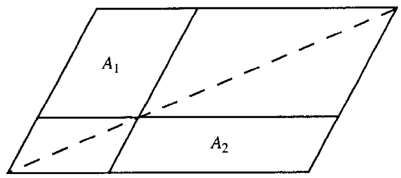
(Beyer 1987).

The AREA of the parallelogram with sides formed by the VECTORS (a, c) and (b, d) is

$$A = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = |ad - bc|. \tag{6}$$

Given a parallelogram P with area $A(P)$ and linear transformation T, the AREA of $T(P)$ is

$$A(T(P)) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} A(P). \tag{7}$$

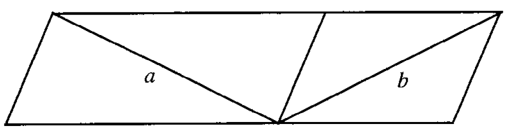


As shown by Euclid, if lines parallel to the sides are drawn through any point on a diagonal of a parallelogram, then the parallelograms not containing segments of that diagonal are equal in AREA (and conversely), so in the above figure, $A_1 = A_2$ (Johnson 1929).

see also DIAMOND, LOZENGE, PARALLELOGRAM ILLUSION, RECTANGLE, RHOMBUS, VARIGNON PARALLELOGRAM, WITTENBAUER'S PARALLELOGRAM

References
Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 123, 1987.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 61, 1929.

Parallelogram Illusion



The sides a and b have the same length, appearances to the contrary.

Parallelogram Law

Let $|\cdot|$ denote the NORM of a quantity. Then the quantities x and y satisfy the parallelogram law if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

If the NORM is defined as $|f| = \sqrt{\langle f|f\rangle}$ (the so-called L_2-NORM), then the law will always hold.

see also L_2-NORM, NORM

Parallelohedron

A special class of ZONOHEDRON. There are five parallelohedra with an infinity of equal and similarly situated replicas which are SPACE-FILLING POLYHEDRA: the CUBE, ELONGATED DODECAHEDRON, hexagonal PRISM, RHOMBIC DODECAHEDRON, and TRUNCATED OCTAHEDRON.

see also PARALLELOTOPE, SPACE-FILLING POLYHEDRON

References
Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, p. 29, 1973.

Parallelotope

Move a point Π_0 along a LINE for an initial point to a final point. It traces out a LINE SEGMENT Π_1. When Π_1 is translated from an initial position to a final position, it traces out a PARALLELOGRAM Π_2. When Π_2 is translated, it traces out a PARALLELEPIPED Π_3. The generalization of Π_n to n-D is then called a parallelotope. Π_n has 2^n vertices and

$$N_k = 2^{n-k}\binom{n}{k}$$

Π_ks, where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT and $k = 0$, 1, ..., n (Coxeter 1973). These are also the coefficients of $(2k+1)^n$.

see also HONEYCOMB, HYPERCUBE, ORTHOTOPE, PARALLELOHEDRON

References
Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 122–123, 1973.
Klee, V. and Wagon, S. *Old and New Unsolved Problems in Plane Geometry and Number Theory.* Washington, DC: Math. Assoc. Amer., 1991.
Zaks, J. "Neighborly Families of Congruent Convex Polytopes." *Amer. Math. Monthly* **94**, 151–155, 1987.

Paralogic Triangles

At the points where a line cuts the sides of a TRIANGLE $\Delta A_1A_2A_3$, perpendiculars to the sides are drawn, forming a TRIANGLE $\Delta B_1B_2B_3$ similar to the given TRIANGLE. The two triangles are also in perspective. One point of intersection of their CIRCUMCIRCLES is the SIMILITUDE CENTER, and the other is the PERSPECTIVE CENTER. The CIRCUMCIRCLES meet ORTHOGONALLY.

see also CIRCUMCIRCLE, ORTHOGONAL CIRCLES, PERSPECTIVE CENTER, SIMILITUDE CENTER

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 258–262, 1929.

Parameter

A parameter m used in ELLIPTIC INTEGRALS defined to be $m \equiv k^2$, where k is the MODULUS. An ELLIPTIC INTEGRAL is written $I(\phi|m)$ when the parameter is used. The complementary parameter is defined by

$$m' \equiv 1 - m, \tag{1}$$

where m is the parameter. Let q be the NOME, k the MODULUS, and $m \equiv k^2$ the PARAMETER. Then

$$q(m) = e^{-\pi K'(m)/K(m)} \tag{2}$$

where $K(m)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND. Then the inverse of $q(m)$ is given by

$$m(q) = \frac{\vartheta_2{}^4(q)}{\vartheta_3{}^4(q)}, \tag{3}$$

where ϑ_i is a THETA FUNCTION.

see also AMPLITUDE, CHARACTERISTIC (ELLIPTIC INTEGRAL), ELLIPTIC INTEGRAL, ELLIPTIC INTEGRAL OF THE FIRST KIND, MODULAR ANGLE, MODULUS (ELLIPTIC INTEGRAL), NOME, PARAMETER, THETA FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 590, 1972.

Parameter (Quadric)

The number θ in the QUADRIC

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1$$

is called the parameter.

see also QUADRIC

Parameterization

The specification of a curve, surface, etc., by means of one or more variables which are allowed to take on values in a given specified range.

see also ISOTHERMAL PARAMETERIZATION, REGULAR PARAMETERIZATION, SURFACE PARAMETERIZATION

Parametric Latitude

An AUXILIARY LATITUDE also called the REDUCED LATITUDE and denoted η or θ. It gives the LATITUDE on a SPHERE of RADIUS a for which the parallel has the same radius as the parallel of geodetic latitude ϕ and the ELLIPSOID through a given point. It is given by

$$\eta=\tan^{-1}(\sqrt{1-e^2}\tan\phi).$$

In series form,

$$\eta=\phi-e_1\sin(2\phi)+\tfrac{1}{2}{e_1}^2\sin(4\phi)-\tfrac{1}{3}{e_1}^3\sin(6\phi)+\ldots,$$

where

$$e_1\equiv\frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}.$$

see also AUXILIARY LATITUDE, ELLIPSOID, LATITUDE, SPHERE

References

Adams, O. S. "Latitude Developments Connected with Geodesy and Cartography with Tables, Including a Table for Lambert Equal-Area Meridional Projections." Spec. Pub. No. 67. U. S. Coast and Geodetic Survey, 1921.
Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, p. 18, 1987.

Parametric Test

A STATISTICAL TEST in which assumptions are made about the underlying distribution of observed data.

Pareto Distribution

The distribution

$$P(x)=\left(\frac{x}{b}\right)^{a+2}.$$

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 252, 1993.

Parity

The parity of a number n is the sum of the bits in BINARY representation (mod 2). The parities of the first few integers (starting with 0) are 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, ... (Sloane's A010060) summarized in the following table.

N	Binary	Parity	N	Binary	Parity
1	1	1	11	1011	1
2	10	1	12	1100	0
3	11	0	13	1101	1
4	100	1	14	1110	1
5	101	0	15	1111	0
6	110	0	16	10000	1
7	111	1	17	10001	0
8	1000	1	18	10010	0
9	1001	0	19	10011	1
10	1010	0	20	10100	0

The constant generated by the sequence of parity digits is called the THUE-MORSE CONSTANT.

see also BINARY, THUE-MORSE CONSTANT

References

Sloane, N. J. A. Sequence A010060 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Parity Constant

see THUE-MORSE CONSTANT

Parking Constant

see RÉNYI'S PARKING CONSTANTS

Parodi's Theorem

The EIGENVALUES λ satisfying $P(\lambda)=0$, where $P(\lambda)$ is the CHARACTERISTIC POLYNOMIAL, lie in the unions of the DISKS

$$|z|\leq 1$$

$$|z+b_1|\leq\sum_{j=1}^n|b_j|.$$

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1119, 1979.

Parry Circle

The CIRCLE passing through the ISODYNAMIC POINTS and the CENTROID of a TRIANGLE (Kimberling 1998, pp. 227–228).

see also CENTROID (TRIANGLE), ISODYNAMIC POINTS, PARRY POINT

References

Kimberling, C. "Triangle Centers and Central Triangles." *Congr. Numer.* **129**, 1–295, 1998.

Parry Point

The intersection of the PARRY CIRCLE and the CIRCUMCIRCLE of a TRIANGLE. The TRILINEAR COORDINATES of the Parry point are

$$\frac{a}{2a^2-b^2-c^2}:\frac{b}{2b^2-c^2-a^2}:\frac{c}{2c^2-a^2-b^2}$$

(Kimberling 1998, pp. 227–228).

see also PARRY CIRCLE

References

Kimberling, C. "Parry Point." `http://www.evansville.edu/~ck6/tcenters/recent/parry.html`.

Kimberling, C. "Triangle Centers and Central Triangles." *Congr. Numer.* **129**, 1–295, 1998.

Parseval's Integral

The POISSON INTEGRAL with $n=0$.

$$J_0(z)=\frac{1}{[\Gamma(n+\frac{1}{2})]^2}\int_0^\pi \cos(z\cos\theta)\,d\theta,$$

where $J_0(z)$ is a BESSEL FUNCTION OF THE FIRST KIND and $\Gamma(x)$ is a GAMMA FUNCTION.

Parseval's Relation

Let $F(\nu)$ and $G(\nu)$ be the FOURIER TRANSFORMS of $f(t)$ and $g(t)$, respectively. Then

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(t)g^*(t)\,dt\\
&=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}F(\nu)e^{-2\pi i\nu t}\,d\nu\int_{-\infty}^{\infty}G^*(\nu')e^{2\pi i\nu' t}\,d\nu'\right]d\nu'\\
&=\int_{-\infty}^{\infty}F(\nu)\int_{-\infty}^{\infty}G^*(\nu')\delta(\nu'-\nu)\,d\nu'\,d\nu\\
&=\int_{-\infty}^{\infty}F(\nu)G^*(\nu)\,d\nu.
\end{aligned}$$

see also FOURIER TRANSFORM, PARSEVAL'S THEOREM

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 425, 1985.

Parseval's Theorem

Let $E(t)$ be a continuous function and $E(t)$ and E_ν be FOURIER TRANSFORM pairs so that

$$E(t)\equiv\int_{-\infty}^{\infty}E_\nu e^{-2\pi i\nu t}\,d\nu \tag{1}$$

$$E^*(t)\equiv\int_{-\infty}^{\infty}{E_{\nu'}}^* e^{2\pi i\nu' t}\,d\nu'. \tag{2}$$

Then

$$\begin{aligned}
&\int_{-\infty}^{\infty}|E(t)|^2\,dt=\int_{-\infty}^{\infty}E(t)E^*(t)\,dt\\
&=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}E_\nu e^{-2\pi i\nu t}\,d\nu\int_{-\infty}^{\infty}{E_{\nu'}}^*e^{2\pi i\nu' t}\,d\nu'\right]dt\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_\nu {E_{\nu'}}^* e^{2\pi it(\nu'-\nu)}\,d\nu\,d\nu'\,dt\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_\nu {E_{\nu'}}^* e^{2\pi it(\nu'-\nu)}\,dt\,d\nu\,d\nu'\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\nu'-\nu)E_\nu {E_{\nu'}}^*\,d\nu\,d\nu'\\
&=\int_{-\infty}^{\infty}E_\nu {E_\nu}^*\,d\nu=\int_{-\infty}^{\infty}|E_\nu|^2\,d\nu.
\end{aligned}\tag{3}$$

where $\delta(x-x_0)$ is the DELTA FUNCTION.

For finite FOURIER TRANSFORM pairs h_k and H_n,

$$\sum_{k=0}^{N-1}|h_k|^2=\frac{1}{N}\sum_{n=0}^{N-1}|H_n|^2. \tag{4}$$

If a function has a FOURIER SERIES given by

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx), \tag{5}$$

then BESSEL'S INEQUALITY becomes an equality known as Parseval's theorem. From (5),

$$\begin{aligned}
[f(x)]^2&=\tfrac{1}{4}{a_0}^2+a_0\sum_{n=1}^{\infty}[a_n\cos(nx)+b_n\sin(nx)]\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_na_m\cos(nx)\cos(mx)\\
&+a_nb_m\cos(nx)\sin(mx)+a_mb_n\sin(nx)\cos(mx)\\
&+b_nb_m\sin(nx)\sin(mx)].
\end{aligned}\tag{6}$$

Integrating

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = \tfrac{1}{4}{a_0}^2 \int_{-\pi}^{\pi} dx$$
$$+a_0 \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]\,dx$$
$$+ \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [a_n a_m \cos(nx)\cos(mx)$$
$$+a_n b_m \cos(nx)\sin(mx) + a_m b_n \sin(nx)\cos(mx)$$
$$+b_n b_m \sin(nx)\sin(mx)]\,dx = \tfrac{1}{4}{a_0}^2(2\pi) + 0$$
$$+\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [a_n a_m \pi \delta_{nm} + 0 + 0 + b_n b_m \pi \delta_{nm}], \tag{7}$$

so

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2\,dx = \tfrac{1}{2}{a_0}^2 + \sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2). \tag{8}$$

For a generalized FOURIER SERIES with a COMPLETE BASIS $\{\phi_i\}_{i=1}^{\infty}$, an analogous relationship holds. For a COMPLEX FOURIER SERIES,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |a_n|^2. \tag{9}$$

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1101, 1979.

Part Metric

A METRIC defined by

$$d(z,w) = \sup \left[\left| \frac{\ln u(z)}{u(w)} \right| : u \in H^+ \right],$$

where H^+ denotes the POSITIVE HARMONIC FUNCTIONS on a DOMAIN. The part metric is invariant under CONFORMAL MAPS for any DOMAIN.

References

Bear, H. S. "Part Metric and Hyperbolic Metric." *Amer. Math. Monthly* **98**, 109–123, 1991.

Partial Derivative

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation.

$$\frac{\partial f}{\partial x_m} \equiv \lim_{h \to 0} \frac{f(x_1, \ldots, x_m + h, \ldots, x_n) - f(x_1, \ldots, x_m, \ldots, x_n)}{h}. \tag{1}$$

The above partial derivative is sometimes denoted f_{x_m} for brevity. For a "nice" 2-D function $f(x,y)$ (i.e., one for which f, f_x, f_y, f_{xy}, f_{yx} exist and are continuous in a NEIGHBORHOOD (a,b)), then $f_{xy}(a,b) = f_{yx}(a,b)$. Partial derivatives involving more than one variable are called MIXED PARTIAL DERIVATIVES.

For nice functions, mixed partial derivatives must be equal regardless of the order in which the differentiation is performed so, for example,

$$f_{xy} = f_{yx} \tag{2}$$

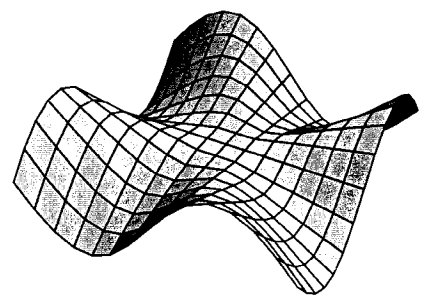
$$f_{xxy} = f_{xyx} = f_{yxx}. \tag{3}$$

For an EXACT DIFFERENTIAL,

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy, \tag{4}$$

so

$$\left(\frac{\partial y}{\partial x}\right)_f = -\frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x}. \tag{5}$$



If the continuity requirement for MIXED PARTIALS is dropped, it is possible to construct functions for which MIXED PARTIALS are *not* equal. An example is the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) = 0 \\ 0 & \text{for } (x,y) = 0, \end{cases} \tag{6}$$

which has $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ (Wagon 1991). This function is depicted above and by Fischer (1986).

Abramowitz and Stegun (1972) give FINITE DIFFERENCE versions for partial derivatives.

see also ABLOWITZ-RAMANI-SEGUR CONJECTURE, DERIVATIVE, MIXED PARTIAL DERIVATIVE, MONKEY SADDLE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 883–885, 1972.

Fischer, G. (Ed.). Plate 121 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 118, 1986.

Thomas, G. B. and Finney, R. L. §16.8 in *Calculus and Analytic Geometry, 9th ed.*0201531747 Reading, MA: Addison-Wesley, 1996.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 83–85, 1991.

Partial Differential Equation

A partial differential equation (PDE) is an equation involving functions and their PARTIAL DERIVATIVES; for example, the WAVE EQUATION

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}. \quad (1)$$

In general, partial differential equations are much more difficult to solve analytically than are ORDINARY DIFFERENTIAL EQUATIONS. They may sometimes be solved using a BÄCKLUND TRANSFORMATION, CHARACTERISTIC, GREEN'S FUNCTION, INTEGRAL TRANSFORM, LAX PAIR, SEPARATION OF VARIABLES, or—when all else fails (which it frequently does)—numerical methods.

Fortunately, partial differential equations of second-order are often amenable to analytical solution. Such PDEs are of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0. \quad (2)$$

Second-order PDEs are then classified according to the properties of the MATRIX

$$\mathsf{Z} \equiv \begin{bmatrix} A & B \\ B & C \end{bmatrix} \quad (3)$$

as ELLIPTIC, HYPERBOLIC, or PARABOLIC.

If Z is a POSITIVE DEFINITE MATRIX, i.e., $\det(\mathsf{Z}) > 0$, the PDE is said to be ELLIPTIC. LAPLACE'S EQUATION and POISSON'S EQUATION are examples. Boundary conditions are used to give the constraint $u(x,y) = g(x,y)$ on $\partial\Omega$, where

$$u_{xx} + u_{yy} = f(u_x, u_y, u, x, y) \quad (4)$$

holds in Ω.

If $\det(\mathsf{Z}) < 0$, the PDE is said to be HYPERBOLIC. The WAVE EQUATION is an example of a hyperbolic partial differential equation. Initial-boundary conditions are used to give

$$u(x,y,t) = g(x,y,t) \quad \text{for } x \in \partial\Omega, t > 0 \quad (5)$$

$$u(x,y,0) = v_0(x,y) \quad \text{in } \Omega \quad (6)$$

$$u_t(x,y,0) = v_1(x,y) \quad \text{in } \Omega, \quad (7)$$

where

$$u_{xy} = f(u_x, u_t, x, y) \quad (8)$$

holds in Ω.

If $\det(\mathsf{Z}) = 0$, the PDE is said to be parabolic. The HEAT CONDUCTION EQUATION equation and other diffusion equations are examples. Initial-boundary conditions are used to give

$$u(x,t) = g(x,t) \quad \text{for } x \in \partial\Omega, t > 0 \quad (9)$$

$$u(x,0) = v(x) \quad \text{for } x \in \Omega, \quad (10)$$

where

$$u_{xx} = f(u_x, u_y, u, x, y) \quad (11)$$

holds in Ω.

see also BÄCKLUND TRANSFORMATION, BOUNDARY CONDITIONS, CHARACTERISTIC (PARTIAL DIFFERENTIAL EQUATION), ELLIPTIC PARTIAL DIFFERENTIAL EQUATION, GREEN'S FUNCTION, HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION, INTEGRAL TRANSFORM, JOHNSON'S EQUATION, LAX PAIR, MONGE-AMPÈRE DIFFERENTIAL EQUATION, PARABOLIC PARTIAL DIFFERENTIAL EQUATION, SEPARATION OF VARIABLES

References

Arfken, G. "Partial Differential Equations of Theoretical Physics." §8.1 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 437–440, 1985.

Bateman, H. *Partial Differential Equations of Mathematical Physics.* New York: Dover, 1944.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Partial Differential Equations." Ch. 19 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 818–880, 1992.

Sobolev, S. L. *Partial Differential Equations of Mathematical Physics.* New York: Dover, 1989.

Sommerfeld, A. *Partial Differential Equations in Physics.* New York: Academic Press, 1964.

Webster, A. G. *Partial Differential Equations of Mathematical Physics, 2nd corr. ed.* New York: Dover, 1955.

Partial Fraction Decomposition

A RATIONAL FUNCTION $P(x)/Q(x)$ can be rewritten using what is known as partial fraction decomposition. This procedure often allows integration to be performed on each term separately by inspection. For each factor of $Q(x)$ the form $(ax+b)^m$, introduce terms

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_m}{(ax+b)^m}. \quad (1)$$

For each factor of the form $(ax^2+bx+c)^m$, introduce terms

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}. \quad (2)$$

Then write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \ldots + \frac{A_2x+B_2}{ax^2+bx+c} + \ldots \quad (3)$$

and solve for the A_is and B_is.

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 13–15, 1987.

Partial Latin Square

In a normal $n \times n$ LATIN SQUARE, the entries in each row and column are chosen from a "global" set of n objects. Like a Latin square, a partial Latin square has no two rows or columns which contain the same two symbols. However, in a partial Latin square, each cell is assigned one of its own set of n possible "local" (and distinct) symbols, chosen from an overall set of more than three distinct symbols, and these symbols may vary from location to location. For example, given the possible symbols $\{1, 2, \ldots, 6\}$ which must be arranged as

$$\begin{matrix} \{1,2,3\} & \{1,3,4\} & \{2,5,6\} \\ \{2,3,5\} & \{1,2,3\} & \{4,5,6\} \\ \{4,3,6\} & \{3,5,6\} & \{2,3,5\}, \end{matrix}$$

the 3×3 partial Latin square

$$\begin{matrix} 1 & 3 & 2 \\ 2 & 4 & 5 \\ 6 & 5 & 3 \end{matrix}$$

can be constructed.

see also DINITZ PROBLEM, LATIN SQUARE

References
Cipra, B. "Quite Easily Done." In *What's Happening in the Mathematical Sciences* **2**, pp. 41–46, 1994.

Partial Order

A RELATION "$\leq$" is a partial order on a SET S if it has:

1. Reflexivity: $a \leq a$ for all $a \in S$.
2. Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$.
3. Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$.

For a partial order, the size of the longest CHAIN (ANTICHAIN) is called the LENGTH (WIDTH). A partially ordered set is also called a POSET.

see also ANTICHAIN, CHAIN, FENCE POSET, IDEAL (PARTIAL ORDER), LENGTH (PARTIAL ORDER), LINEAR EXTENSION, PARTIALLY ORDERED SET, TOTAL ORDER, WIDTH (PARTIAL ORDER)

References
Ruskey, F. "Information on Linear Extension." `http://sue.csc.uvic.ca/~cos/inf/pose/LinearExt.html`.

Partial Quotient

If the SIMPLE CONTINUED FRACTION of a REAL NUMBER x is given by

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}},$$

then the quantities a_i are called partial quotients.

see also CONTINUED FRACTION, CONVERGENT, SIMPLE CONTINUED FRACTION

Partially Ordered Set

A partially ordered set (or POSET) is a SET taken together with a PARTIAL ORDER on it. Formally, a partially ordered set is defined as an ordered pair $P = (X, \leq)$, where X is called the GROUND SET of P and $\leq$ is the PARTIAL ORDER of P.

see also CIRCLE ORDER, COVER RELATION, DOMINANCE, GROUND SET, HASSE DIAGRAM, INTERVAL ORDER, ISOMORPHIC POSETS, PARTIAL ORDER, POSET DIMENSION, REALIZER, RELATION

References
Dushnik, B. and Miller, E. W. "Partially Ordered Sets." *Amer. J. Math.* **63**, 600–610, 1941.
Fishburn, P. C. *Interval Orders and Interval Sets: A Study of Partially Ordered Sets.* New York: Wiley, 1985.
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Particularly Well-Behaved Functions

Functions which have DERIVATIVES of all orders at all points and which, together with their DERIVATIVES, fall off at least as rapidly as $|x|^{-n}$ as $|x| \to \infty$, no matter how large n is.

see also REGULAR SEQUENCE

Partisan Game

A GAME for which each player has a different set of moves in any position. Every position in an IMPARTIAL GAME has a NIM-VALUE.

Partition

A partition is a way of writing an INTEGER n as a sum of POSITIVE INTEGERS without regard to order, possibly subject to one or more additional constraints. Particular types of partition functions include the PARTITION FUNCTION P, giving the number of partitions of a number without regard to order, and PARTITION FUNCTION Q, giving the number of ways of writing the INTEGER n as a sum of POSITIVE INTEGERS without regard to order with the constraint that all INTEGERS in each sum are distinct.

see also AMENABLE NUMBER, DURFEE SQUARE, ELDER'S THEOREM, FERRERS DIAGRAM, GRAPHICAL PARTITION, PARTITION FUNCTION P, Partition Function Q, PERFECT PARTITION, PLANE PARTITION, SET PARTITION, SOLID PARTITION, STANLEY'S THEOREM

References
Andrews, G. E. *The Theory of Partitions.* Cambridge, England: Cambridge University Press, 1998.
Dickson, L. E. "Partitions." Ch. 3 in *History of the Theory of Numbers, Vol. 2: Diophantine Analysis.* New York: Chelsea, pp. 101–164, 1952.

Partition Function P

$P(n)$ gives the number of ways of writing the INTEGER n as a sum of POSITIVE INTEGERS without regard to order. For example, since 4 can be written

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1, \end{aligned} \tag{1}$$

so $P(4) = 5$. $P(n)$ satisfies

$$P(n) \leq \tfrac{1}{2}[P(n+1) + P(n-1)] \tag{2}$$

(Honsberger 1991). The values of $P(n)$ for $n = 1$, 2, ..., are 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, ... (Sloane's A000041). The following table gives the value of $P(n)$ for selected small n.

n	$P(n)$
50	204226
100	190569292
200	3972999029388
300	9253082936723602
400	6727090051741041926
500	2300165032574323995027
600	458004788008144308553622
700	60378285202834474611028659
800	5733052172321422504456911979
900	415873681190459054784114365430
1000	24061467864032622473692149727991

n for which $P(n)$ is PRIME are 2, 3, 4, 5, 6, 13, 36, 77, 132, 157, 168, 186, ... (Sloane's A046063). Numbers which cannot be written as a PRODUCT of $P(n)$ are 13, 17, 19, 23, 26, 29, 31, 34, 37, 38, 39, ... (Sloane's A046064), corresponding to numbers of nonisomorphic ABELIAN GROUPS which are not possible for any group order.

When explicitly listing the partitions of a number n, the simplest form is the so-called *natural representation* which simply gives the sequence of numbers in the representation (e.g., (2, 1, 1) for the number $4 = 2+1+1$). The *multiplicity representation* instead gives the number of times each number occurs together with that number (e.g., (2, 1), (1, 2) for $4 = 2 \cdot 1 + 1 \cdot 2$). The FERRERS DIAGRAM is a pictorial representation of a partition.

Euler invented a GENERATING FUNCTION which gives rise to a POWER SERIES in $P(n)$,

$$\begin{aligned} P(n) = \sum_{m=1}^{\infty} (-1)^{m+1} [P(n - \tfrac{1}{2}m(3m-1)) \\ + P(n - \tfrac{1}{2}m(3m+1))]. \end{aligned} \tag{3}$$

A RECURRENCE RELATION is

$$P(n) = \frac{1}{n} \sum_{m=0}^{n-1} \sigma(n-m) P(m), \tag{4}$$

where $\sigma(n)$ is the DIVISOR FUNCTION (Berndt 1994, p. 108). Euler also showed that, for

$$f(x) \equiv \prod_{m=1}^{\infty} (1 - x^m) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2} \tag{5}$$

$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} + \ldots, \tag{6}$$

where the exponents are generalized PENTAGONAL NUMBERS 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, ... (Sloane's A001318) and the sign of the kth term (counting 0 as the 0th term) is $(-1)^{\lfloor (k+1)/2 \rfloor}$ (with $\lfloor x \rfloor$ the FLOOR FUNCTION), the partition numbers $P(n)$ are given by the GENERATING FUNCTION

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} P(n) x^n. \tag{7}$$

MacMahon obtained the beautiful RECURRENCE RELATION

$$\begin{aligned} P(n) - P(n-1) - P(n-2) + P(n-5) + P(n-7) \\ - P(n-12) - P(n-15) + \ldots = 0, \end{aligned} \tag{8}$$

where the sum is over generalized PENTAGONAL NUMBERS $\leq n$ and the sign of the kth term is $(-1)^{\lfloor (k+1)/2 \rfloor}$, as above.

In 1916–1917, Hardy and Ramanujan used the CIRCLE METHOD and elliptic MODULAR FUNCTIONS to obtain the approximate solution

$$P(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}. \tag{9}$$

Rademacher (1937) subsequently obtained an exact series solution which yields the Hardy-Ramanujan FORMULA (9) as the first term:

$$P(n) = \sum_{q=1}^{\infty} L_q(n) \psi_q(n), \tag{10}$$

where

$$K = \pi\sqrt{\frac{2}{3}} \tag{11}$$

$$L_q(n) = \sum_p \omega_{p,q} e^{-2np\pi i/q} \tag{12}$$

$$\omega_{p,q} = e^{\pi i s_{p,q}} \tag{13}$$

$$s_{p,q} = \frac{1}{q} \sum_{\mu=1}^{q-1} \mu \left(\frac{\mu p}{q} - \left\lfloor \frac{\mu p}{q} \right\rfloor - \frac{1}{2} \right) \tag{14}$$

$$\lambda_n = \sqrt{n - \tfrac{1}{24}} \tag{15}$$

$$\psi_q(n) = \frac{\sqrt{q}}{\pi\sqrt{2}} \left\{ \frac{d}{dm} \left[\frac{\sinh\left(\frac{K\lambda_m}{q}\right)}{\lambda_m} \right] \right\}_{m=n}, \tag{16}$$

$\lfloor x \rfloor$ is the FLOOR FUNCTION, and p runs through the INTEGERS less than and RELATIVELY PRIME to q (when $q = 1$, $p = 0$). The remainder after Q terms is

$$R(Q) < CQ^{-1/2} + D\sqrt{\frac{Q}{n}}\sinh\left(\frac{K\sqrt{n}}{Q}\right), \tag{17}$$

where C and D are fixed constants.

With $f(x)$ as defined above, Ramanujan also showed that

$$5\frac{f^5(x^5)}{f^6(x)} = \sum_{m=0}^{\infty} P(5m+4)x^m. \tag{18}$$

Ramanujan also found numerous CONGRUENCES such as

$$P(5m+4) \equiv 0 \pmod{5} \tag{19}$$

$$P(7m+5) \equiv 0 \pmod{7} \tag{20}$$

$$P(11m+6) \equiv 0 \pmod{11}. \tag{21}$$

RAMANUJAN'S IDENTITY gives the first of these.

Let $P_O(n)$ be the number of partitions of n containing ODD numbers only and $P_D(n)$ be the number of partitions of n without duplication, then

$$P_O(n) = P_D(n) = \prod_{k=1,3,\ldots}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \ldots)$$

$$= \prod_{k=1}^{\infty}(1+x^k) = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + \ldots, \tag{22}$$

as discovered by Euler (Honsberger 1985). The first few values of $P_O = P_D$ are 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, ... (Sloane's A000009).

Let $P_E(n)$ be the number of partitions of EVEN numbers only, and let $P_{EO}(n)$ ($P_{DO}(n)$) be the number of partitions in which the parts are all EVEN (ODD) and all different. The first few values of $P_{DO}(n)$ are 1, 1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, ... (Sloane's A000700). Some additional GENERATING FUNCTIONS are given by Honsberger (1985, pp. 241–242)

$$\sum_{n=1}^{\infty} P_{\text{no even part repeated}}(n)x^n$$

$$= \prod_{k=1}(1 - x^{2k-1})^{-1}(1 + x^{2k}) \tag{23}$$

$$\sum_{n=1}^{\infty} P_{\text{no part occurs more than 3 times}}(n)x^n$$

$$= \prod_{k=1}(1 + x^k + x^{2k} + x^{3k}) \tag{24}$$

$$\sum_{n=1}^{\infty} P_{\text{no part divisible by 4}}(n)x^n = \prod_{k=1}\frac{1 - x^{4k}}{1 - x^k} \tag{25}$$

$$\sum_{n=1}^{\infty} P_{\text{no part occurs more than } d \text{ times}}(n)x^n$$

$$= \prod_{k=1}\sum_{i=0}^{d} x^{ik} = \prod_{k=1}\frac{1 - x^{(d+1)k}}{1 - x^k} \tag{26}$$

$$\sum_{n=1}^{\infty} P_{\text{every part occurs 2, 3, or 5 times}}(n)x^n$$

$$= \prod_{k=1}(1 + x^{2k} + x^{3k} + x^{5k})$$

$$= \prod_{k=1}(1 + x^{2k})(1 + x^{3k}) = \prod_{k=1}\frac{1 - x^{4k}}{1 - x^{2k}}\frac{1 - x^{6k}}{1 - x^{3k}} \tag{27}$$

$$\sum_{n=1}^{\infty} P_{\text{no part occurs exactly once}}(n)x^n$$

$$= (1 + x^{2k} + x^{3k} + \ldots) = \prod_k \frac{1 + x^{6k}}{(1 - x^{2k})(1 - x^{3k})}. \tag{28}$$

Some additional interesting theorems following from these (Honsberger 1985, pp. 64–68 and 143–146) are:

1. The number of partitions of n in which no EVEN part is repeated is the same as the number of partitions of n in which no part occurs more than three times and also the same as the number of partitions in which no part is divisible by four.
2. The number of partitions of n in which no part occurs more often than d times is the same as the number of partitions in which no term is a multiple of $d + 1$.
3. The number of partitions of n in which each part appears either 2, 3, or 5 times is the same as the number of partitions in which each part is CONGRUENT mod 12 to either 2, 3, 6, 9, or 10.
4. The number of partitions of n in which no part appears exactly once is the same as the number of partitions of n in which no part is CONGRUENT to 1 or 5 mod 6.
5. The number of partitions in which the parts are all EVEN and different is equal to the absolute difference of the number of partitions with ODD and EVEN parts.

$P(n,k)$, also written $P_k(n)$, is the number of ways of writing n as a sum of k terms, and can be computed from the RECURRENCE RELATION

$$P(n,k) = P(n-1,k-1) + P(n-k,k) \tag{29}$$

(Ruskey). The number of partitions of n with largest part k is the same as $P(n,k)$.

The function $P(n,k)$ can be given explicitly for the first few values of k,

$$P(n,2) = \left\lfloor \tfrac{1}{2}n \right\rfloor \tag{30}$$

$$P(n,3) = [\tfrac{1}{12}n^2], \tag{31}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and $[x]$ is the NINT function (Honsberger 1985, pp. 40–45).

see also ALCUIN'S SEQUENCE, ELDER'S THEOREM, EULER'S PENTAGONAL NUMBER THEOREM, FERRERS DIAGRAM, PARTITION FUNCTION Q, PENTAGONAL NUMBER, $r_k(n)$, ROGERS-RAMANUJAN IDENTITIES, STANLEY'S THEOREM

References

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Rademacher, H. "On the Partition Function $p(n)$." *Proc. London Math. Soc.* **43**, 241–254, 1937.

Ruskey, F. "Information of Numerical Partitions." `http://sue.csc.uvic.ca/~cos/inf/nump/NumPartition.html`.

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Partition Function Q

$Q(n)$ gives the number of ways of writing the INTEGER n as a sum of POSITIVE INTEGERS without regard to order with the constraint that all INTEGERS in each sum are distinct. The values for $n = 1, 2, \ldots$ are 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, ... (Sloane's A000009). The GENERATING FUNCTION for $Q(n)$ is

$$\prod_{m=1}^{\infty}(1+x^m) = \frac{1}{\prod_{m=0}^{\infty}(1-x^{2m+1})}$$
$$= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + \ldots.$$

The values of n for which $Q(n)$ is PRIME are 3, 4, 5, 7, 22, 70, 100, 495, 1247, 2072, ... (Sloane's A046065), with no others for $n \leq 15,000$.

The number of PARTITIONS of n with $\leq k$ summands is denoted $q(n,k)$ or $q_k(n)$. Therefore, $q_n(n) = P(n)$ and

$$q_k(n) = q_{k-1}(n) + q_k(n-k).$$

see also PARTITION FUNCTION P

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Partitions into Distinct Parts." §24.2.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 823–824, 1972.

Sloane, N. J. A. Sequences A046065 and A000009/M0281 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Party Problem

Also known as the MAXIMUM CLIQUE PROBLEM. Find the minimum number of guests that must be invited so that at least m will know each other or at least n will not know each other. The solutions are known as RAMSEY NUMBERS.

see also CLIQUE, RAMSEY NUMBER

Parzen Apodization Function

An APODIZATION FUNCTION similar to the BARTLETT FUNCTION.

see also APODIZATION FUNCTION, BARTLETT FUNCTION

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 547, 1992.

Pascal Distribution

see NEGATIVE BINOMIAL DISTRIBUTION

Pascal's Formula

Each subsequent row of PASCAL'S TRIANGLE is obtained by adding the two entries diagonally above. This follows immediately from the BINOMIAL COEFFICIENT identity

$$\begin{aligned}\binom{n}{r} &\equiv \frac{n!}{(n-r)!r!} = \frac{(n-1)!n}{(n-r)!r!} \\ &= \frac{(n-1)!(n-r)}{(n-r)!r!} + \frac{(n-1)!r}{(n-r)!r!} \\ &= \frac{(n-1)!}{(n-r-1)!r!} + \frac{(n-1)!}{(n-r)!(r-1)!} \\ &= \binom{n-1}{r} + \binom{n-1}{r-1}.\end{aligned}$$

see also BINOMIAL COEFFICIENT, PASCAL'S TRIANGLE

Pascal's Hexagrammum Mysticum

see PASCAL'S THEOREM

Pascal's Limaçon

see LIMAÇON

Pascal Line

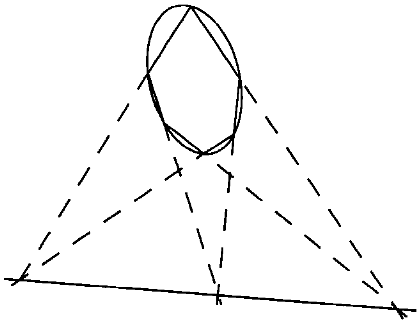
The line containing the three points of the intersection of the three pairs of opposite sides of two TRIANGLES.

see also PASCAL'S THEOREM

Pascal's Rule

see PASCAL'S FORMULA

Pascal's Theorem



The dual of BRIANCHON'S THEOREM. It states that, given a (not necessarily REGULAR, or even CONVEX) HEXAGON inscribed in a CONIC SECTION, the three pairs of the continuations of opposite sides meet on a straight LINE, called the PASCAL LINE. There are 6! (6! means 6 FACTORIAL, where $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$) possible ways of taking all VERTICES in any order, but among these are six equivalent CYCLIC PERMUTATIONS and two possible orderings, so the total number of different hexagons (not all simple) is

$$\frac{6!}{2 \cdot 6} = \frac{720}{12} = 60.$$

There are therefore a total of 60 PASCAL LINES created by connecting VERTICES in any order. These intersect three by three in 20 STEINER POINTS.

see also BRAIKENRIDGE-MACLAURIN CONSTRUCTION, BRIANCHON'S THEOREM, CAYLEY-BACHARACH THEOREM, CONIC SECTION, DUALITY PRINCIPLE, HEXAGON, PAPPUS'S HEXAGON THEOREM, PASCAL LINE, STEINER POINTS

References

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 73–76, 1967.

Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 105–106, 1990.

Pappas, T. "The Mystic Hexagram." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 118, 1989.

Pascal's Triangle

A TRIANGLE of numbers arranged in staggered rows such that

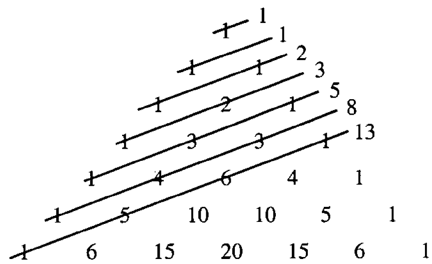
$$a_{nr} \equiv \frac{n!}{r!(n-r)!} \equiv \binom{n}{r}, \qquad (1)$$

where $\binom{n}{r}$ is a BINOMIAL COEFFICIENT. The triangle was studied by B. Pascal, although it had been described centuries earlier by Chinese mathematician Yanghui (about 500 years earlier, in fact) and the Arabian poet-mathematician Omar Khayyám. It is therefore known as the YANGHUI TRIANGLE in China. Starting with $n = 0$, the TRIANGLE is

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

(Sloane's A007318). PASCAL'S FORMULA shows that each subsequent row is obtained by adding the two entries diagonally above,

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n-1}{r} + \binom{n-1}{r-1}. \qquad (2)$$



In addition, the "SHALLOW DIAGONALS" of Pascal's triangle sum to FIBONACCI NUMBERS,

$$\sum_{k=1}^{n} \binom{k}{n-k} = \frac{(-1)^n {}_3F_2(1,2,1-n;\frac{1}{2}(3-n),2-\frac{1}{2}n;-\frac{1}{4})}{\pi(2-3n+n^2)}$$
$$= F_{n+1}, \qquad (3)$$

where ${}_3F_2(a,b,c;d,e;z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION.

Pascal's triangle contains the FIGURATE NUMBERS along its diagonals. It can be shown that

$$\sum_{i=1}^{n} a_{ij} = \frac{n+1}{j+1} a_{nj} = a_{(n+1),(j+1)} \qquad (4)$$

and

$$\binom{m+1}{1} \sum k^m + \binom{m+1}{2} \sum k^{m-1}$$
$$+ \ldots + \binom{m+1}{m} \sum k = (n+1)[(n+1)^m - 1]. \qquad (5)$$

The "shallow diagonals" sum to the FIBONACCI SEQUENCE, i.e.,

$$\begin{aligned} 1 &= 1 \\ 1 &= 1 \\ 2 &= 1+1 \\ 3 &= 2+1 \\ 5 &= 1+3+1 \\ 8 &= 3+4+1. \end{aligned} \tag{6}$$

In addition,

$$\sum_{j=1}^{i} a_{ij} = 2^i - 1. \tag{7}$$

It is also true that the first number after the 1 in each row divides all other numbers in that row IFF it is a PRIME. If P_n is the number of ODD terms in the first n rows of the Pascal triangle, then

$$0.812\ldots < P_n n^{-\ln 2/\ln 3} < 1 \tag{8}$$

(Harborth 1976, Le Lionnais 1983).

The BINOMIAL COEFFICIENT $\binom{m}{n}$ mod 2 can be computed using the XOR operation n XOR m, making Pascal's triangle mod 2 very easy to construct. Pascal's triangle is unexpectedly connected with the construction of regular POLYGONS and with the SIERPIŃSKI SIEVE.

see also BELL TRIANGLE, BINOMIAL COEFFICIENT, BINOMIAL THEOREM, BRIANCHON'S THEOREM, CATALAN'S TRIANGLE, CLARK'S TRIANGLE, EULER'S TRIANGLE, FIBONACCI NUMBER, FIGURATE NUMBER TRIANGLE, LEIBNIZ HARMONIC TRIANGLE, NUMBER TRIANGLE, PASCAL'S FORMULA, POLYGON, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, SIERPIŃSKI SIEVE, TRINOMIAL TRIANGLE

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Pascal's Wager

"God is or He is not... Let us weigh the gain and the loss in choosing... 'God is.' If you gain, you gain all, if you lose, you lose nothing. Wager, then, unhesitatingly, that He is."

Pasch's Axiom

In the plane, if a line intersects one side of a TRIANGLE and misses the three VERTICES, then it must intersect one of the other two sides. This is a special case of the generalized MENELAUS' THEOREM with $n = 3$.

see also HELLY'S THEOREM, MENELAUS' THEOREM, PASCH'S THEOREM

Pasch's Theorem

A theorem stated in 1882 which cannot be derived from EUCLID'S POSTULATES. Given points a, b, c, and d on a LINE, if it is known that the points are ordered as (a, b, c) and (b, c, d), it is also true that (a, b, d).

see also EUCLID'S POSTULATES, LINE, PASCH'S AXIOM

Pass Equivalent

Two KNOTS are pass equivalent if there exists a sequence of pass moves taking one to the other. Every KNOT is either pass equivalent to the UNKNOT or TREFOIL KNOT. These two knots are not pass equivalent to each other, but the ENANTIOMERS of the TREFOIL KNOT are pass equivalent. A KNOT has ARF INVARIANT 0 if the KNOT is pass equivalent to the UNKNOT and 1 if it is pass equivalent to the TREFOIL KNOT.

see also ARF INVARIANT, KNOT, PASS MOVE, TREFOIL KNOT, UNKNOT

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 223–228, 1994.

Pass Move

A change in a knot projection such that a pair of oppositely oriented strands are passed through another pair of oppositely oriented strands.

see also PASS EQUIVALENT

Patch

A patch (also called a LOCAL SURFACE) is a differentiable mapping $\mathbf{x} : U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^2$. More generally, if A is any SUBSET of $\mathbb{R}^2$, then a map $\mathbf{x} : A \to \mathbb{R}^n$ is a patch provided that $\mathbf{x}$ can be extended to a differentiable map from U into $\mathbb{R}^n$, where U is an open set containing A. Here, $\mathbf{x}(U)$ (or more generally, $\mathbf{x}(A)$) is called the TRACE of $\mathbf{x}$.

see also GAUSS MAP, INJECTIVE PATCH, MONGE PATCH, REGULAR PATCH, TRACE (MAP)

References

Gray, A. "Patches in $\mathbb{R}^3$." §10.2 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 183–184 and 192–193, 1993.

Path

A path γ is a continuous mapping $\gamma : [a, b] \mapsto \mathbb{C}$, where $\gamma(a)$ is the initial point and $\gamma(b)$ is the final point. It is often written parametrically as $\sigma(t)$.

Path Graph

The path P_n is a TREE with two nodes of valency 1, and the other $n-2$ nodes of valency 2. Path graphs P_n are always GRACEFUL for $n > 4$.

see also CHAIN (GRAPH), GRACEFUL GRAPH, HAMILTONIAN PATH, TREE

Path Integral

Let γ be a PATH given parametrically by $\sigma(t)$. Let s denote ARC LENGTH from the initial point. Then

$$\begin{aligned}\int_\gamma f(s)\,ds &= \int_\gamma f(\sigma(t))\,|\sigma'(t)|\,dt\\ &= \int_\gamma f(x(t), y(t), z(t))\,|\sigma'(t)|\,dt.\end{aligned}$$

see also LINE INTEGRAL

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Evaluation of Functions by Path Integration." §5.14 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 201–204, 1992.

Pathwise-Connected

A TOPOLOGICAL SPACE X is pathwise-connected IFF for every two points $x, y \in X$, there is a CONTINUOUS FUNCTION f from [0,1] to X such that $f(0) = x$ and $f(1) = y$. Roughly speaking, a SPACE X is pathwise-connected if, for every two points in X, there is a path connecting them. For LOCALLY PATHWISE-CONNECTED SPACES (which include most "interesting spaces" such as MANIFOLDS and CW-COMPLEXES), being CONNECTED and being pathwise-connected are equivalent, although there are connected spaces which are not pathwise connected. Pathwise-connected spaces are also called 0-connected.

see also CONNECTED SPACE, CW-COMPLEX, LOCALLY PATHWISE-CONNECTED SPACE, TOPOLOGICAL SPACE

Patriarchal Cross

see GAULLIST CROSS

Pauli Matrices

Matrices which arise in Pauli's treatment of spin in quantum mechanics. They are defined by

$$\sigma_1 = \sigma_x \equiv \mathsf{P}_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1)$$

$$\sigma_2 = \sigma_y \equiv \mathsf{P}_2 \equiv \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad (2)$$

$$\sigma_3 = \sigma_z \equiv \mathsf{P}_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3)$$

The Pauli matrices plus the 2×2 IDENTITY MATRIX I form a complete set, so any 2×2 matrix A can be expressed as

$$\mathsf{A} = c_0\mathsf{I} + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3. \quad (4)$$

The associated matrices

$$\sigma_+ \equiv 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (5)$$

$$\sigma_- \equiv 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (6)$$

$$\sigma^2 \equiv 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (7)$$

can also be defined. The Pauli spin matrices satisfy the identities

$$\sigma_i\sigma_j = \mathsf{I}\delta_{ij} + \epsilon_{ijk}i\sigma_k \quad (8)$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\sigma_{ij} \quad (9)$$

$$\sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \sqrt{{p_x}^2 + {p_y}^2 + {p_z}^2}. \quad (10)$$

see also DIRAC MATRICES, QUATERNION

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 211–212, 1985.

Goldstein, H. "The Cayley-Klein Parameters and Related Quantities." *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, p. 156, 1980.

Pauli Spin Matrices

see PAULI MATRICES

Payoff Matrix

A $m \times n$ MATRIX which gives the possible outcome of a two-person ZERO-SUM GAME when player A has m possible strategies and player B n strategies. The analysis of the MATRIX in order to determine optimal strategies is the aim of GAME THEORY. The so-called "augmented" payoff matrix is defined as follows:

$$\mathsf{G} = \begin{array}{c} \begin{array}{cccccccc} P_0 & P_1 & P_2 & \cdots & P_n & P_{n+1} & P_{n+2} & \cdots & P_{n+m} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -1 & a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ -1 & a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array}.$$

see also GAME THEORY, ZERO-SUM GAME

Peacock's Tail

One name for the figure used by Euclid to prove the PYTHAGOREAN THEOREM.

see also BRIDE'S CHAIR, WINDMILL

Peano Arithmetic

The theory of NATURAL NUMBERS defined by the five PEANO'S AXIOMS. Any universal statement which is undecidable in Peano arithmetic is necessarily TRUE. Undecidable statements may be either TRUE or FALSE. Paris and Harrington (1977) gave the first "natural" example of a statement which is true for the integers but unprovable in Peano arithmetic (Spencer 1983).

see also KREISEL CONJECTURE, NATURAL INDEPENDENCE PHENOMENON, NUMBER THEORY, PEANO'S AXIOMS

References
Kirby, L. and Paris, J. "Accessible Independence Results for Peano Arithmetic." *Bull. London Math. Soc.* **14**, 285–293, 1982.
Paris, J. and Harrington, L. "A Mathematical Incompleteness in Peano Arithmetic." In *Handbook of Mathematical Logic* (Ed. J. Barwise). Amsterdam, Netherlands: North-Holland, pp. 1133–1142, 1977.
Spencer, J. "Large Numbers and Unprovable Theorems." *Amer. Math. Monthly* **90**, 669–675, 1983.

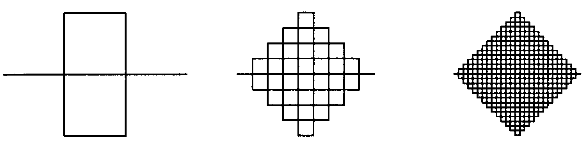
Peano's Axioms

1. Zero is a number.
2. If a is a number, the successor of a is a number.
3. ZERO is not the successor of a number.
4. Two numbers of which the successors are equal are themselves equal.
5. (INDUCTION AXIOM.) If a set S of numbers contains ZERO and also the successor of every number in S, then every number is in S.

Peano's axioms are the basis for the version of NUMBER THEORY known as PEANO ARITHMETIC.

see also INDUCTION AXIOM, PEANO ARITHMETIC

Peano Curve

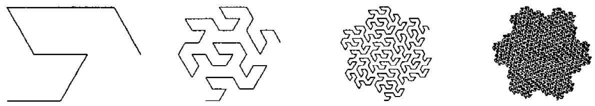


A FRACTAL curve which can be written as a LINDENMAYER SYSTEM.

see also DRAGÓN CURVE, HILBERT CURVE, LINDENMAYER SYSTEM, SIERPIŃSKI CURVE

References
Dickau, R. M. "Two-Dimensional L-Systems." `http://forum.swarthmore.edu/advanced/robertd/lsys2d.html`.
Hilbert, D. "Über die stetige Abbildung einer Linie auf ein Flachenstück." *Math. Ann.* **38**, 459–460, 1891.
Peano, G. "Sur une courbe, qui remplit une aire plane." *Math. Ann.* **36**, 157–160, 1890.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 207, 1991.

Peano-Gosper Curve

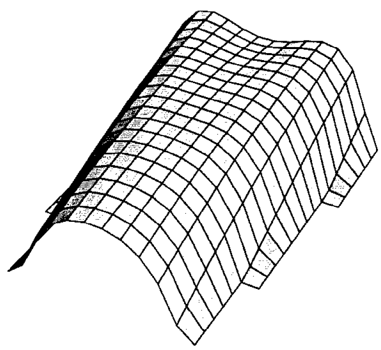


A PLANE-FILLING CURVE originally called a FLOWSNAKE by R. W. Gosper and M. Gardner. Mandelbrot (1977) subsequently coined the name Peano-Gosper curve. The GOSPER ISLAND bounds the space that the Peano-Gosper curve fills.

see also DRAGON CURVE, EXTERIOR SNOWFLAKE, GOSPER ISLAND, HILBERT CURVE, KOCH SNOWFLAKE, PEANO CURVE, SIERPIŃSKI ARROWHEAD CURVE, SIERPIŃSKI CURVE

References
Dickau, R. M. "Two-Dimensional L-Systems." `http://forum.swarthmore.edu/advanced/robertd/lsys2d.html`.
Mandelbrot, B. B. *Fractals: Form, Chance, & Dimension.* San Francisco, CA: W. H. Freeman, 1977.

Peano Surface



The function

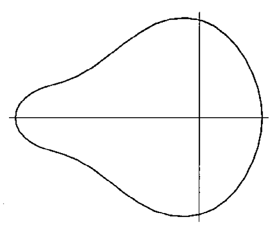
$$f(x,y) = (2x^2 - y)(y - x^2)$$

which does *not* have a LOCAL MAXIMUM at (0, 0), despite criteria commonly touted in the second half of the 1800s which indicated the contrary.

see also LOCAL MAXIMUM

References
Fischer, G. (Ed.). Plate 122 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 119, 1986.
Leitere, J. "Functions." §7.1.2 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 70–71, 1986.

Pear Curve

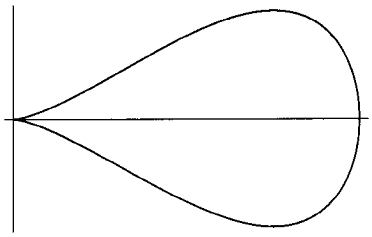


The LEMNISCATE L_3 in the iteration towards the MANDELBROT SET. In CARTESIAN COORDINATES with a constant r, the equation is given by

$$r^2 = (x^2+y^2)(1+2x+5x^2+6x^3+6x^4+4x^5+x^6-3y^2 -2xy^2+8x^2y^2+8x^3y^2+3x^4y^2+2y^4+4xy^4 +3x^2y^4+y^6).$$

see also PEAR-SHAPED CURVE

Pear-Shaped Curve



A curve given by the Cartesian equation

$$b^2y^2 = x^3(a-x).$$

see also PEAR CURVE, TEARDROP CURVE

References
MacTutor History of Mathematics Archive. "Pear-Shaped Cubic." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Pearshaped.html.

Pearson's Correlation

see CORRELATION COEFFICIENT

Pearson-Cunningham Function

see CUNNINGHAM FUNCTION

Pearson's Function

$$I\left(\frac{{\chi_s}^2}{\sqrt{2(k-1)}}, \frac{k-3}{2}\right) \equiv \frac{\Gamma\left(\frac{1}{2}{\chi_s}^2, \frac{k-1}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)},$$

where $\Gamma(x)$ is the GAMMA FUNCTION.

see also CHI-SQUARED TEST, GAMMA FUNCTION

Pearson Kurtosis

Let μ_4 be the fourth MOMENT of a DISTRIBUTION and σ its VARIANCE. Then the Pearson kurtosis is defined by

$$\beta_2 \equiv \frac{\mu_4}{\sigma^4}.$$

see also FISHER KURTOSIS, KURTOSIS

Pearson Mode Skewness

Given a DISTRIBUTION with measured MEAN, MODE, and STANDARD DEVIATION s, the Pearson mode skewness is

$$\frac{\text{mean} - \text{mode}}{s}.$$

see also MEAN, MODE, PEARSON SKEWNESS, PEARSON'S SKEWNESS COEFFICIENTS, SKEWNESS

Pearson Skewness

Let a DISTRIBUTION have third MOMENT μ_3 and STANDARD DEVIATION σ, then the Pearson skewness is defined by

$$\beta_1 = \left(\frac{\mu_3}{\sigma^3}\right)^2.$$

see also FISHER SKEWNESS, PEARSON'S SKEWNESS COEFFICIENTS, SKEWNESS

Pearson's Skewness Coefficients

Given a DISTRIBUTION with measured MEAN, MEDIAN, MODE, and STANDARD DEVIATION s, Pearson's first skewness coefficient is

$$\frac{3[\text{mean}] - [\text{mode}]}{s},$$

and the second coefficient is

$$\frac{3[\text{mean}] - [\text{median}]}{s}.$$

see also FISHER SKEWNESS, PEARSON SKEWNESS, SKEWNESS

Pearson System

Generalizes the differential equation for the GAUSSIAN DISTRIBUTION

$$\frac{dy}{dx} = \frac{y(m-x)}{a} \tag{1}$$

to

$$\frac{dy}{dx} = \frac{y(m-x)}{a+bx+cx^2}. \tag{2}$$

Let c_1, c_2 be the roots of $a+bx+cx^2$. Then the possible types of curves are

0. $b=c=0$, $a>0$. E.g., NORMAL DISTRIBUTION.
I. $b^2/4ac<0$, $c_1 \leq x \leq c_2$. E.g., BETA DISTRIBUTION.
II. $b^2/4ac=0$, $c<0$, $-c_1 \leq x \leq c_1$ where $c_1 \equiv \sqrt{-c/a}$.
III. $b^2/4ac=\infty$, $c=0$, $c_1 \leq x < \infty$ where $c_1 \equiv -a/b$. E.g., GAMMA DISTRIBUTION. This case is intermediate to cases I and VI.
IV. $0 < b^2/4ac < 1$, $-\infty < x < \infty$.
V. $b^2/4ac = 1$, $c_1 \leq x < \infty$ where $c_1 \equiv -b/2a$. Intermediate to cases IV and VI.

VII. $b^2/4ac > 1$, $c_1 \leq x < \infty$ where c_1 is the larger root. E.g., BETA PRIME DISTRIBUTION.

VII. $b^2/4ac = 0$, $c > 0$, $-\infty < x < \infty$. E.g., STUDENT'S t-DISTRIBUTION.

Classes IX–XII are discussed in Pearson (1916). See also Craig (in Kenney and Keeping 1951). If a Pearson curve possesses a MODE, it will be at $x = m$. Let $y(x) = 0$ at c_1 and c_2, where these may be $-\infty$ or ∞. If yx^{r+2} also vanishes at c_1, c_2, then the rth MOMENT and $(r+1)$th MOMENTS exist.

$$\int_{c_1}^{c_2} \frac{dy}{dx}(ax^r + bx^{r+1} + cx^{r+2})\,dx = \int_{c_1}^{c_2} y(mx^r - x^{r+1})\,dx \tag{3}$$

giving

$$[y(ax^r + bx^{r+1} + cx^{r+2})]_{c_1}^{c_2} - \int_{c_1}^{c_2} y[arx^{r-1} + b(r+1)x^r + c(r+2)x^{r+1}]\,dx = \int_{c_1}^{c_2} y(mx^r - x^{r+1})\,dx \tag{4}$$

$$0 - \int_{c_1}^{c_2} y[arx^{r-1} + b(r+1)x^r + c(r+2)x^{r+1}]\,dx = \int_{c_1}^{c_2} y(mx^r - x^{r+1})\,dx \tag{5}$$

also,

$$\nu_r = \int_{c_1}^{c_2} yx^r\,dx, \tag{6}$$

so

$$ar\nu_{r-1} + b(r+1)\nu_r + c(r+2)\nu_{r+1} = -m\nu_r + \nu_{r+1}. \tag{7}$$

For $r = 0$,

$$b + 2c\nu_1 = -m + \nu_1, \tag{8}$$

so

$$\nu_1 = \frac{m+b}{1-2c}. \tag{9}$$

For $r = 1$,

$$a + 2b\nu_1 + 3c\nu_2 = -m\nu_1 + \nu_2, \tag{10}$$

so

$$\nu_2 = \frac{a + (m+2b)\nu_1}{1-3c}. \tag{11}$$

Now let $t \equiv (x - \nu_1)/\sigma$. Then

$$\nu_1 = 0 \tag{12}$$

$$\nu_2 = \mu_2 = 1 \tag{13}$$

$$\alpha_r = \mu_r = \nu_r. \tag{14}$$

Hence $b = -m$, and $a = 1 - c$ so

$$(1-3c)r\alpha_{r-1} - mr\alpha_r + [c(r+2) - 1]\alpha_{r+1} = 0. \tag{15}$$

For $r = 2$,

$$2m + (1-4c)\alpha_3 = 0. \tag{16}$$

For $r = 3$,

$$3(1-3c) - 3m\alpha_3 - (1-5c)\alpha_4 = 0. \tag{17}$$

So the SKEWNESS and KURTOSIS are

$$\gamma_1 = \alpha_3 = \frac{2m}{4c-1} \tag{18}$$

$$\gamma_2 = \alpha_4 - 3 = \frac{6(m^2 - 4c^2 + c)}{(4c-1)(5c-1)}. \tag{19}$$

So the parameters a, b, and c can be written

$$a = 1 - 3c \tag{20}$$

$$b = -m = \frac{\gamma_1}{2(1+2\delta)} \tag{21}$$

$$c = \frac{\delta}{2(1+2\delta)}, \tag{22}$$

where

$$\delta \equiv \frac{2\gamma_2 - 3\gamma_1{}^2}{\gamma_2 + 6}. \tag{23}$$

References

Craig, C. C. "A New Exposition and Chart for the Pearson System of Frequency Curves." *Ann. Math. Stat.* **7**, 16–28, 1936.

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, p. 107, 1951.

Pearson, K. "Second Supplement to a Memoir on Skew Variation." *Phil. Trans. A* **216**, 429–457, 1916.

Pearson Type III Distribution

A skewed distribution which is similar to the BINOMIAL DISTRIBUTION when $p \neq q$ (Abramowitz and Stegun 1972, p. 930).

$$y = k(t+A)^{A^2-1}e^{-At}, \tag{1}$$

for $t \in [0, \infty)$ where

$$A \equiv 2/\gamma \tag{2}$$

$$K \equiv \frac{A^{A^2}e^{-A^2}}{\Gamma(A^2)}, \tag{3}$$

$\Gamma(z)$ is the GAMMA FUNCTION, and t is a standardized variate. Another form is

$$P(x) = \frac{1}{\beta\Gamma(p)}\left(\frac{x-\alpha}{\beta}\right)^{p-1} \exp\left(\frac{x-\alpha}{\beta}\right). \tag{4}$$

For this distribution, the CHARACTERISTIC FUNCTION is

$$\phi(t) = e^{i\alpha t}(1 - i\beta t)^{-p}, \tag{5}$$

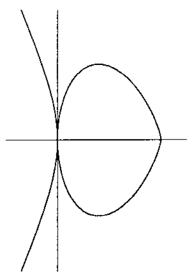
and the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \alpha + p\beta \tag{6}$$
$$\sigma^2 = p\beta^2 \tag{7}$$
$$\gamma_1 = \frac{2}{\sqrt{p}} \tag{8}$$
$$\gamma_2 = \frac{6}{p}. \tag{9}$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, 1972.

Pearls of Sluze



$$y^m = kx^n(a-x)^b.$$

The curves with integral n, p, and m were studied by de Sluze between 1657 and 1698. The name "Pearls of Sluze" was given to these curves by Blaise Pascal (MacTutor Archive).

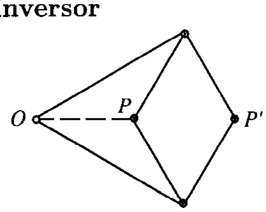
References

MacTutor History of Mathematics Archive. "Pearls of Sluze." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Pearls.html`.

Peaucellier Cell

see PEAUCELLIER INVERSOR

Peaucellier Inversor



A LINKAGE with six rods which draws the inverse of a given curve. When a pencil is placed at P, the inverse is drawn at P' (or vice versa). If a seventh rod (dashed) is added (with an additional pivot), P is kept on a circle and the locus traced out by P' is a straight line. It therefore converts circular motion to linear motion without sliding, and was discovered in 1864. Another LINKAGE which performs this feat using hinged squares had been published by Sarrus in 1853 but ignored. Coxeter (1969, p. 428) shows that

$$OP \times OP' = OA^2 - PA^2.$$

see also HART'S INVERSOR, LINKAGE

References

Bogomolny, A. "Peaucellier Linkage." `http://www.cut-the-knot.com/pythagoras/invert.html`.
Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods.* Oxford, England: Oxford University Press, p. 156, 1978.
Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, pp. 82–83, 1969.
Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 46–48, 1990.
Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 121–126, 1957.
Smith, D. E. *A Source Book in Mathematics.* New York: Dover, p. 324, 1994.

Peaucellier's Linkage

see PEAUCELLIER INVERSOR

Pedal

The pedal of a curve with respect to a point P is the locus of the foot of the PERPENDICULAR from P to the TANGENT to the curve. When a CLOSED CURVE rolls on a straight line, the AREA between the line and ROULETTE after a complete revolution by any point on the curve is twice the AREA of the pedal (taken with respect to the generating point) of the rolling curve.

Pedal Circle

The pedal CIRCLE of a point P in a TRIANGLE is the CIRCLE through the feet of the perpendiculars from P to the sides of the TRIANGLE (the CIRCUMCIRCLE about the PEDAL TRIANGLE). When P is on a side of the TRIANGLE, the line between the two perpendiculars is called the PEDAL LINE. Given four points, no three of which are COLLINEAR, then the four PEDAL CIRCLES of each point for the TRIANGLE formed by the other three have a common point through which the NINE-POINT CIRCLES of the four TRIANGLES pass. The radius of the pedal circle of a point P is

$$r = \frac{\overline{A_1P} \cdot \overline{A_2P} \cdot \overline{A_3P}}{2(R^2 - \overline{OP}^2)}$$

(Johnson 1929, p. 141).

see also MIQUEL POINT, NINE-POINT CIRCLE, PEDAL TRIANGLE

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Pedal Coordinates

The pedal coordinates of a point P with respect to the curve C and the PEDAL POINT O are the radial distant r from O to P and the PERPENDICULAR distance p from O to the line L tangent to C at P.

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 2–3, 1972.

Yates, R. C. "Pedal Equations." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 166–169, 1952.

Pedal Curve

Given a curve C, the pedal curve of C with respect to a fixed point O (the PEDAL POINT) is the locus of the point P of intersection of the PERPENDICULAR from O to a TANGENT to C. The parametric equations for a curve $(f(t), g(t))$ relative to the PEDAL POINT (x_0, y_0) are

$$x = \frac{x_0 f'^2 + f g'^2 + (y_0 - g) f' g'}{f'^2 + g'^2}$$

$$y = \frac{g f'^2 + y_0 g'^2 + (x_0 - f) f' g'}{f'^2 + g'2^2}.$$

Curve	Pole	Pedal
astroid	center	quadrifolium
cardioid	cusp	Cayley's sextic
central conic	focus	circle
circle	any point	limaçon
circle	on circumference	cardioid
circle involute	center of circle	Archimedean spiral
cissoid of Diocles	focus	cardioid
deltoid	center	trifolium
deltoid	cusp	simple folium
deltoid	on the curve	unsymmetrical double folium
deltoid	vertex	double folium
epicycloid	center	rose
hypocycloid	center	rose
line	any point	point
logarithmic spiral	pole	logarithmic spiral
parabola	focus	line
parabola	foot of directrix	right strophoid
parabola	on directrix	strophoid
parabola	refl. of focus by dir.	Maclaurin trisectrix
parabola	vertex	cissoid of Diocles
sinusoidal spiral	pole	sinusoidal spiral
Tschirnhausen cubic	focus of pedal	parabola

see also NEGATIVE PEDAL CURVE

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 46–49 and 204, 1972.

Lee, X. "Pedal." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Pedal_dir/pedal.html`.

Lockwood, E. H. "Pedal Curves." Ch. 18 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 152–155, 1967.

Yates, R. C. "Pedal Curves." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 160–165, 1952.

Pedal Line

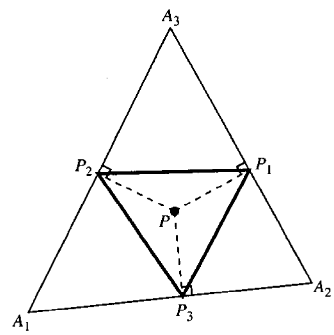
Mark a point P on a side of a TRIANGLE and draw the perpendiculars from the point to the two other sides. The line between the feet of these two perpendiculars is called the pedal line.

see also PEDAL TRIANGLE, SIMSON LINE

Pedal Point

The fixed point with respect to which a PEDAL CURVE is drawn.

Pedal Triangle



Given a point P, the pedal triangle of P is the TRIANGLE whose VERTICES are the feet of the perpendiculars from P to the side lines. The pedal triangle of a TRIANGLE with TRILINEAR COORDINATES $\alpha : \beta : \gamma$ and angles A, B, and C has VERTICES with TRILINEAR COORDINATES

$$0 : \beta + \alpha \cos C : \gamma + \alpha \cos B \tag{1}$$

$$\alpha + \beta \cos C : 0 : \gamma + \beta \cos A \tag{2}$$

$$\alpha + \gamma \cos B : \beta + \gamma \cos A : 0. \tag{3}$$

The third pedal triangle is similar to the original one. This theorem can be generalized to: the nth pedal n-gon of any n-gon is similar to the original one. It is also true that

$$P_2 P_3 = A_1 P \sin \alpha_1 \tag{4}$$

(Johnson 1929, pp. 135–136). The AREA A of the pedal triangle of a point P is proportional to the POWER of P with respect to the CIRCUMCIRCLE,

$$A = \tfrac{1}{2}(R^2 - \overline{OP}^2) \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 = \frac{R^2 - \overline{OP}^2}{4R^2} \Delta \tag{5}$$

(Johnson 1929, pp. 139–141).

see also ANTIPEDAL TRIANGLE, FAGNANO'S PROBLEM, PEDAL CIRCLE, PEDAL LINE, SCHWARZ'S TRIANGLE PROBLEM

References

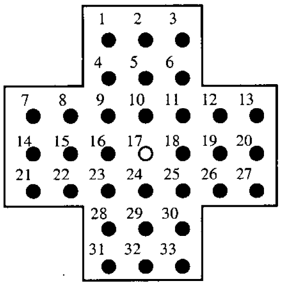
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 22–26, 1967.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Peg Knot

see CLOVE HITCH

Peg Solitaire



A game played on a cross-shaped board with 33 holes. All holes but the middle one are initially filled with pegs. The goal is to remove all pegs but one by jumping pegs from one side of an occupied peg hole to an empty space, removing the peg which was jumped over. Strategies and symmetries are discussed in Beeler *et al.* (1972, Item 75). A triangular version called HI-Q also exists (Beeler *et al.* 1972, Item 76). Kraitchik (1942) considers a board with one additional hole placed at the vertices of the central right angles.

see also HI-Q

References

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Gardner, M. "Peg Solitaire." Ch. 11 in *The Unexpected Hanging and Other Mathematical Diversions.* New York: Simon and Schuster, pp. 122–135 and 250–251, 1969.

Kraitchik, M. "Peg Solitaire." §12.19 in *Mathematical Recreations.* New York: W. W. Norton, pp. 297–298, 1942.

Peg Top

see PIRIFORM

Peirce's Theorem

The only linear associative algebra in which the coordinates are REAL NUMBERS and products vanish only if one factor is zero are the FIELD of REAL NUMBERS, the FIELD of COMPLEX NUMBERS, and the algebra of QUATERNIONS with REAL COEFFICIENTS.

see also WEIERSTRAß'S THEOREM

Pell Equation

A special case of the quadratic DIOPHANTINE EQUATION having the form

$$x^2 - Dy^2 = 1, \tag{1}$$

where D is a nonsquare NATURAL NUMBER. Dörrie (1965) defines the equation as

$$x^2 - Dy^2 = 4 \tag{2}$$

and calls it the FERMAT DIFFERENCE EQUATION. The general Pell equation was solved by the Indian mathematician Bhaskara.

Pell equations, as well as the analogous equation with a minus sign on the right, can be solved by finding the CONTINUED FRACTION $[a_1, a_2, \ldots]$ for $\sqrt{D}$. (The trivial solution $x = 1$, $y = 0$ is ignored in all subsequent discussion.) Let p_n/q_n denote the nth CONVERGENT $[a_1, a_2, \ldots, a_n]$, then we are looking for a convergent which obeys the identity

$$p_n{}^2 - Dq_n{}^2 = (-1)^n, \tag{3}$$

which turns out to always be possible since the CONTINUED FRACTION of a QUADRATIC SURD always becomes periodic at some term a_{r+1}, where $a_{r+1} = 2a_1$, i.e.,

$$\sqrt{D} = [a_1, \overline{a_2, \ldots, a_r, 2a_1}]. \tag{4}$$

Writing $n = rk$ gives

$$p_{rk}{}^2 - Dq_{rk}{}^2 = (-1)^{rk}, \tag{5}$$

for k anPOSITIVE INTEGER. If r is ODD, solutions to

$$x^2 - Dy^2 = \pm 1 \tag{6}$$

can be obtained if k is chosen to be EVEN or ODD, but if r is EVEN, there are no values of k which can make the exponent ODD.

If r is EVEN, then $(-1)^r$ is POSITIVE and the solution in terms of smallest INTEGERS is $x = p_r$ and $y = q_r$, where p_r/q_r is the rth CONVERGENT. If r is ODD, then $(-1)^r$ is NEGATIVE, but we can take $k = 2$ in this case, to obtain

$$p_{2r}{}^2 - Dq_{2r}{}^2 = 1, \tag{7}$$

so the solution in smallest INTEGERS is $x = p_{2r}$, $y = q_{2r}$. Summarizing,

$$(x, y) = \begin{cases} (p_r, q_r) & \text{for } r \text{ even} \\ (p_{2r}, p_{2r}) & \text{for } r \text{ odd.} \end{cases} \tag{8}$$

Given one solution $(x, y) = (p, q)$ (which can be found as above), a whole family of solutions can be found by taking each side to the nth POWER,

$$x^2 - Dy^2 = (p^2 - Dq^2)^n = 1. \tag{9}$$

Factoring gives

$$(x + \sqrt{D}\, y)(x - \sqrt{D}\, y) = (p + \sqrt{D}\, q)^n (p - \sqrt{D}\, q)^n \tag{10}$$

and

$$x + \sqrt{D}\, y = (p + \sqrt{D}\, q)^n \tag{11}$$

$$x - \sqrt{D}\, y = (p - \sqrt{D}\, q)^n, \tag{12}$$

which gives the family of solutions

$$x = \frac{(p + q\sqrt{D})^n + (p - q\sqrt{D})^n}{2} \tag{13}$$

$$y = \frac{(p + q\sqrt{D})^n - (p - q\sqrt{D})^n}{2\sqrt{D}}. \tag{14}$$

These solutions also hold for

$$x^2 - Dy^2 = -1, \tag{15}$$

except that n can take on only ODD values.

The following table gives the smallest integer solutions (x, y) to the Pell equation with constant $D \leq 102$ (Beiler 1966, p. 254). SQUARE $D = d^2$ are not included, since they would result in an equation of the form

$$x^2 - d^2y^2 = x^2 - (dy)^2 = x^2 - y'^2 = 1, \tag{16}$$

which has no solutions (since the difference of two SQUARES cannot be 1).

D	x	y	D	x	y
2	3	2	54	485	66
3	2	1	55	89	12
5	9	4	56	15	2
6	5	2	57	151	20
7	8	3	58	19603	2574
8	3	1	59	530	69
10	19	6	60	31	4
11	10	3	61	1766319049	226153980
12	7	2	62	63	8
13	649	180	63	8	1
14	15	4	65	129	16
15	4	1	66	65	8
17	33	8	67	48842	5967
18	17	4	68	33	4
19	170	39	69	7775	936
20	9	2	70	251	30
21	55	12	71	3480	413
22	197	42	72	17	2
23	24	5	73	2281249	267000
24	5	1	74	3699	430
26	51	10	75	26	3
27	26	5	76	57799	6630
28	127	24	77	351	40
29	9801	1820	78	53	6
30	11	2	79	80	9
31	1520	273	80	9	1
32	17	3	82	163	18
33	23	4	83	82	9
34	35	6	84	55	6
35	6	1	85	285769	30996
37	73	12	86	10405	1122
38	37	6	87	28	3
39	25	4	88	197	21
40	19	3	89	500001	53000
41	2049	320	90	19	2
42	13	2	91	1574	165
43	3482	531	92	1151	120
44	199	30	93	12151	1260
45	161	24	94	2143295	221064
46	24335	3588	95	39	4
47	48	7	96	49	5
48	7	1	97	62809633	6377352
50	99	14	98	99	10
51	50	7	99	10	1
52	649	90	101	201	20
53	66249	9100	102	101	10

The first few minimal values of x and y for nonsquare D are 3, 2, 9, 5, 8, 3, 19, 10, 7, 649, ... (Sloane's A033313) and 2, 1, 4, 2, 3, 1, 6, 3, 2, 180, ... (Sloane's A033317), respectively. The values of D having $x = 2, 3, \ldots$ are 3, 2, 15, 6, 35, 12, 7, 5, 11, 30, ... (Sloane's A033314) and the values of D having $y = 1, 2, \ldots$ are 3, 2, 7, 5, 23, 10, 47, 17, 79, 26, ... (Sloane's A033318). Values of the incrementally largest minimal x are 3, 9, 19, 649, 9801, 24335, 66249, ... (Sloane's A033315) which occur at $D = 2$, 5, 10, 13, 29, 46, 53, 61, 109, 181, ... (Sloane's A033316). Values of the incrementally largest minimal

y are 2, 4, 6, 180, 1820, 3588, 9100, 226153980, ... (Sloane's A033319), which occur at $D = 2$, 5, 10, 13, 29, 46, 53, 61, ... (Sloane's A033320).

see also DIOPHANTINE EQUATION, DIOPHANTINE EQUATION—QUADRATIC, LAGRANGE NUMBER (DIOPHANTINE EQUATION)

References
Beiler, A. H. "The Pellian." Ch. 22 in *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains.* New York: Dover, pp. 248–268, 1966.
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Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, 1965.
Lagarias, J. C. "On the Computational Complexity of Determining the Solvability or Unsolvability of the Equation $X^2 - DY^2 = -1$." *Trans. Amer. Math. Soc.* **260**, 485–508, 1980.
Smarandache, F. "Un metodo de resolucion de la ecuacion diofantica." *Gaz. Math.* **1**, 151–157, 1988.
Smarandache, F. " Method to Solve the Diophantine Equation $ax^2 - by^2 + c = 0$." In *Collected Papers, Vol. 1.* Lupton, AZ: Erhus University Press, 1996.
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Whitford, E. E. *Pell Equation.* New York: Columbia University Press, 1912.

Pell-Lucas Number

see PELL NUMBER

Pell-Lucas Polynomial

see PELL POLYNOMIAL

Pell Number

The numbers obtained by the U_ns in the LUCAS SEQUENCE with $P = 2$ and $Q = -1$. They and the Pell-Lucas numbers (the V_ns in the LUCAS SEQUENCE) satisfy the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}. \tag{1}$$

Using P_i to denote a Pell number and Q_i to denote a Pell-Lucas number,

$$P_{m+n} = P_m P_{n+1} + P_{m-1} P_n \tag{2}$$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}, \tag{3}$$

$$P_{2^t m} = P_m (2Q_m)(2Q_{2m})(2Q_{4m}) \cdots (2Q_{2^{t-1}m}) \tag{4}$$

$$Q_m{}^2 = 2P_m{}^2 + (-1)^m \tag{5}$$

$$Q_{2m} = 2Q_m{}^2 - (-1)^m. \tag{6}$$

The Pell numbers have $P_0 = 0$ and $P_1 = 1$ and are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, ... (Sloane's A000129). The Pell-Lucas numbers have $Q_0 = 2$ and $Q_1 = 2$ and are 2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, ... (Sloane's A002203).

The only TRIANGULAR Pell number is 1 (McDaniel 1996).

see also BRAHMAGUPTA POLYNOMIAL, PELL POLYNOMIAL

References
McDaniel, W. L. "Triangular Numbers in the Pell Sequence." *Fib. Quart.* **34**, 105–107, 1996.
Sloane, N. J. A. Sequences A000129/M1413 and A002203/M0360 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Pell Polynomial

The Pell polynomials $P(x)$ and Lucas-Pell polynomials $Q(x)$ are generated by a LUCAS POLYNOMIAL SEQUENCE using generator $(2x, 1)$. This gives recursive equations for $P(x)$ from $P_0(x) = P_1(x) = 1$ and

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x). \tag{1}$$

The first few are

$$\begin{aligned} P_1 &= 1 \\ P_2 &= 2x \\ P_3 &= 4x^2 - 1 \\ P_4 &= 8x^3 - 4x \\ P_5 &= 16x^4 - 12x^2 + 1. \end{aligned}$$

The Pell-Lucas numbers are defined recursively by $q_0(x) = 1$, $q_1(x) = x$ and

$$q_{n+2}(x) = 2xq_{n+1}(x) + q_n(x), \tag{2}$$

together with

$$Q_n(x) \equiv 2q_n(x). \tag{3}$$

The first few are

$$\begin{aligned} Q_1 &= 2x \\ Q_2 &= 4x^2 - 2 \\ Q_3 &= 8x^3 - 6x \\ Q_4 &= 16x^4 - 16x^2 + 2 \\ Q_5 &= 32x^5 - 40x^3 + 10x. \end{aligned}$$

see also LUCAS POLYNOMIAL SEQUENCE

References
Horadam, A. F. and Mahon, J. M. "Pell and Pell-Lucas Polynomials." *Fib. Quart.* **23**, 7–20, 1985.
Mahon, J. M. M. A. (Honors) thesis, The University of New England. Armidale, Australia, 1984.
Sloane, N. J. A. Sequence A000129/M1413 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Pell Sequence

see PELL NUMBER

Pencil

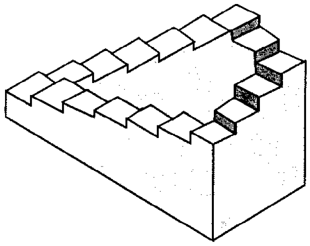
The set of all LINES through a point. Woods (1961), however, uses this term as a synonym for RANGE.

see also NEAR-PENCIL, PERSPECTIVITY, RANGE (LINE SEGMENT), SECTION (PENCIL), SHEAF (GEOMETRY)

References

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, pp. 8 and 11–12, 1961.

Penrose Stairway



An IMPOSSIBLE FIGURE (also called the SCHROEDER STAIRS) in which a stairway in the shape of a square appears to circulate indefinitely while still possessing normal steps. The Dutch artist M. C. Escher included Penrose stairways in many of his mind-bending illustrations.

see also IMPOSSIBLE FIGURE

References

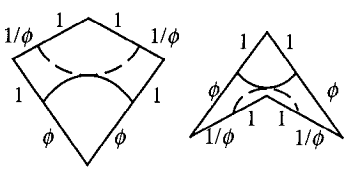
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 15, 1989.

Jablan, S. "Impossible Figures." `http://members.tripod.com/~modularity/impos.htm`.

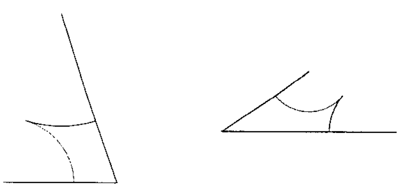
Pappas, T. "Optical Illusions and Computer Graphics." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 5, 1989.

Robinson, J. O. and Wilson, J. A. "The Impossible Colonnade and Other Variations of a Well-Known Figure." *Brit. J. Psych.* **64**, 363–365, 1973.

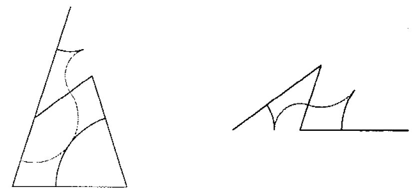
Penrose Tiles



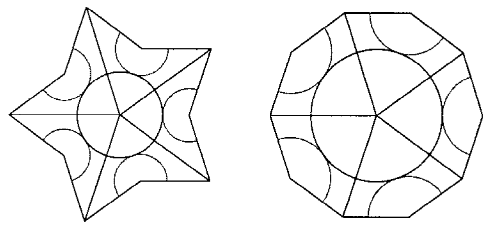
A pair of shapes which tile the plane only aperiodically (when the markings are constrained to match at borders). The two tiles, illustrated above, are called the "KITE" and "DART."



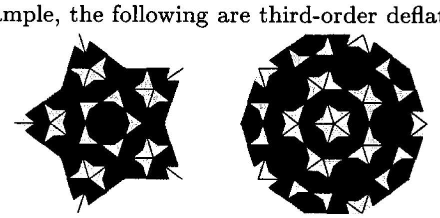
To see how the plane may be tiled aperiodically using the kite and dart, divide the kite into acute and obtuse tiles, shown above. Now define "deflation" and "inflation" operations. The deflation operator takes an acute TRIANGLE to the union of two ACUTE TRIANGLES and one OBTUSE, and the OBTUSE TRIANGLE goes to an ACUTE and an OBTUSE TRIANGLE. These operations are illustrated below.



When applied to a collection of tiles, the deflation operator leads to a more refined collection. The operators do not respect tile boundaries, but do respect the half tiles defined above. There are two ways to obtain aperiodic TILINGS with 5-fold symmetry about a single point. They are known an the "star" and "sun" configurations, and are show below.



Higher order versions can then be obtained by deflation. For example, the following are third-order deflations:



References

Gardner, M. Chs. 1–2 in *Penrose Tiles and Trapdoor Ciphers... and the Return of Dr. Matrix, reissue ed.* New York: W. H. Freeman, pp. 299–300, 1989.

Hurd, L. P. "PenroseTiles." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/2D/0206-772`.

Peterson, I. *The Mathematical Tourist: Snapshots of Modern Mathematics.* New York: W. H. Freeman, pp. 86–95, 1988.

Wagon, S. "Penrose Tiles." §4.3 in *Mathematica in Action.* New York: W. H. Freeman, pp. 108–117, 1991.

Penrose Triangle

see TRIBAR

Penrose Tribar

see TRIBAR

Pentabolo

A 5-POLYABOLO.

Pentacle

see PENTAGRAM

Pentacontagon

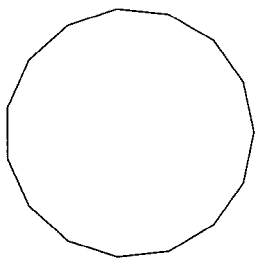
A 50-sided POLYGON.

Pentad

A group of five elements.

see also MONAD, PAIR, QUADRUPLET, QUINTUPLET, TETRAD, TRIAD, TRIPLET, TWINS

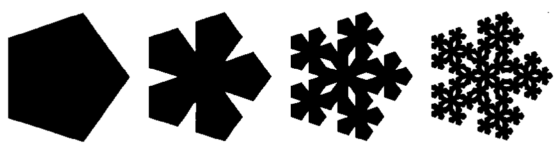
Pentadecagon

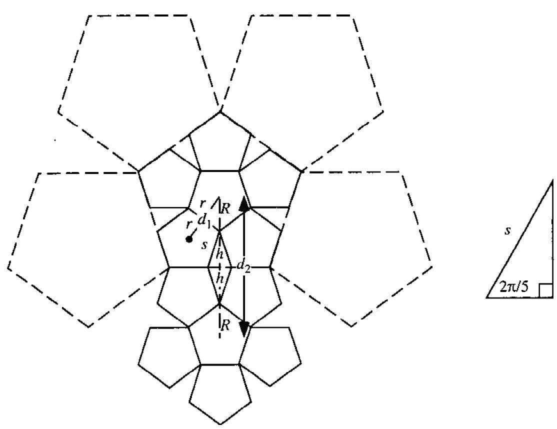


A 15-sided POLYGON, sometimes also called the PENTAKAIDECAGON.

see also POLYGON, REGULAR POLYGON, TRIGONOMETRY VALUES—$\pi/15$

Pentaflake





A FRACTAL with 5-fold symmetry. As illustrated above, five PENTAGONS can be arranged around an identical PENTAGON to form the first iteration of the pentaflake. This cluster of six pentagons has the shape of a pentagon with five triangular wedges removed. This construction was first noticed by Albrecht Dürer (Dixon 1991).

For a pentagon of side length 1, the first ring of pentagons has centers at RADIUS

$$d_1 = 2r = \tfrac{1}{2}(1+\sqrt{5})R = \phi R, \tag{1}$$

where ϕ is the GOLDEN RATIO. The INRADIUS r and CIRCUMRADIUS R are related by

$$r = R\cos(\tfrac{1}{5}\pi) = \tfrac{1}{4}(\sqrt{5}+1)R, \tag{2}$$

and these are related to the side length s by

$$s = 2\sqrt{R^2 - r^2} = \tfrac{1}{2}R\sqrt{10 - 2\sqrt{5}}. \tag{3}$$

The height h is

$$h = s\sin(\tfrac{2}{5}\pi) = \tfrac{1}{4}s\sqrt{10+2\sqrt{5}} = \tfrac{1}{2}\sqrt{5}\,R, \tag{4}$$

giving a RADIUS of the second ring as

$$d_2 = 2(R+h) = (2+\sqrt{5})R = \phi^3 R. \tag{5}$$

Continuing, the nth pentagon ring is located at

$$d_n = \phi^{2n-1}. \tag{6}$$

Now, the length of the side of the first pentagon compound is given by

$$s_2 = 2\sqrt{(2r+R)^2 - (h+R)^2} = R\sqrt{5+2\sqrt{5}}, \tag{7}$$

so the ratio of side lengths of the original pentagon to that of the compound is

$$\frac{s_2}{s} = \frac{R\sqrt{5+2\sqrt{5}}}{\frac{1}{2}R\sqrt{10-2\sqrt{5}}} = 1+\phi. \tag{8}$$

We can now calculate the dimension of the pentaflake fractal. Let N_n be the number of black pentagons and L_n the length of side of a pentagon after the n iteration,

$$N_n = 6^n \tag{9}$$
$$L_n = (1+\phi)^{-n}. \tag{10}$$

The CAPACITY DIMENSION is therefore

$$d_{\text{cap}} = -\lim_{n\to\infty}\frac{\ln N_n}{\ln L_n} = \frac{\ln 6}{\ln(1+\phi)} = \frac{\ln 2 + \ln 3}{\ln(1+\phi)} = 1.861715\ldots. \tag{11}$$

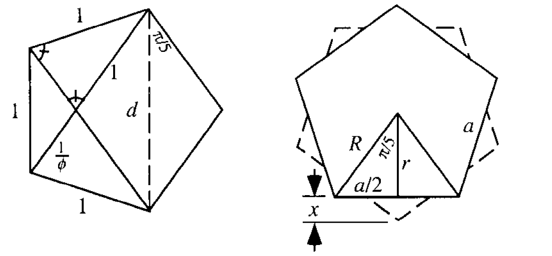
see also PENTAGON

References

Dixon, R. *Mathographics.* New York: Dover, pp. 186–188, 1991.

Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Pentagon



The regular convex 5-gon is called the pentagon. By SIMILAR TRIANGLES in the figure on the left,

$$\frac{d}{1} = \frac{1}{\frac{1}{\phi}} = \phi, \tag{1}$$

where d is the diagonal distance. But the dashed vertical line connecting two nonadjacent VERTICES is the same length as the diagonal one, so

$$\phi = 1 + \frac{1}{\phi} \tag{2}$$

$$\phi^2 - \phi - 1 \tag{3}$$

$$\phi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}. \tag{4}$$

This number is the GOLDEN RATIO. The coordinates of the VERTICES relative to the center of the pentagon with unit sides, starting at the right VERTEX and moving clockwise, are $(\cos(\frac{1}{5}n\pi), \sin(\frac{1}{5}n\pi))$ for $n = 0, 1, \ldots, 4$, or

$$(1, 0), (c_1, s_1), (c_2, s_2), (c_2, -s_2), (c_1, -s_1), \tag{5}$$

where

$$c_1 = \cos\left(\frac{\pi}{5}\right) = \tfrac{1}{4}(\sqrt{5}+1) \tag{6}$$

$$c_2 = \cos\left(\frac{2\pi}{5}\right) = \tfrac{1}{4}(\sqrt{5}-1) \tag{7}$$

$$s_1 = \sin\left(\frac{\pi}{5}\right) = \tfrac{1}{4}\sqrt{10-2\sqrt{5}} \tag{8}$$

$$s_2 = \sin\left(\frac{2\pi}{5}\right) = \tfrac{1}{4}\sqrt{10+2\sqrt{5}}. \tag{9}$$

For a regular POLYGON, the CIRCUMRADIUS, INRADIUS, SAGITTA, and AREA are given by

$$R_n = \tfrac{1}{2}a\csc\left(\frac{\pi}{n}\right) \tag{10}$$

$$r_n = \tfrac{1}{2}a\cot\left(\frac{\pi}{n}\right) \tag{11}$$

$$x_n = R_n - r_n = \tfrac{1}{2}a\tan\left(\frac{\pi}{2n}\right) \tag{12}$$

$$A_n = \tfrac{1}{4}na^2\cot\left(\frac{\pi}{n}\right). \tag{13}$$

Plugging in $n = 5$ gives

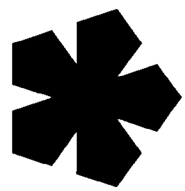
$$R = \tfrac{1}{2}a\csc(\tfrac{1}{5}\pi) = \tfrac{1}{10}a\sqrt{50+10\sqrt{5}} \tag{14}$$

$$r = \tfrac{1}{2}a\cot(\tfrac{1}{5}\pi) = \tfrac{1}{10}a\sqrt{25+10\sqrt{5}} \tag{15}$$

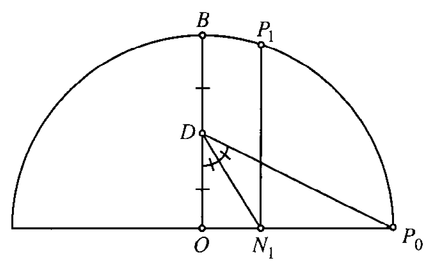
$$x = \tfrac{1}{10}a\sqrt{25-10\sqrt{5}} \tag{16}$$

$$A = \tfrac{5}{4}a^2\sqrt{25+10\sqrt{5}}. \tag{17}$$

Five pentagons can be arranged around an identical pentagon to form the first iteration of the "PENTAFLAKE," which itself has the shape of a pentagon with five triangular wedges removed. For a pentagon of side length 1, the first ring of pentagons has centers at radius ϕ, the second ring at ϕ^3, and the nth at ϕ^{2n-1}.



In proposition IV.11, Euclid showed how to inscribe a regular pentagon in a CIRCLE. Ptolemy also gave a RULER and COMPASS construction for the pentagon in his epoch-making work *The Almagest.* While Ptolemy's construction has a SIMPLICITY of 16, a GEOMETRIC CONSTRUCTION using CARLYLE CIRCLES can be made with GEOMETROGRAPHY symbol $2S_1 + S_2 + 8C_1 + 0C_2 + 4C_3$, which has SIMPLICITY 15 (De Temple 1991).



Pentagon

The following elegant construction for the pentagon is due to Richmond (1893). Given a point, a CIRCLE may be constructed of any desired RADIUS, and a DIAMETER drawn through the center. Call the center O, and the right end of the DIAMETER P_0. The DIAMETER PERPENDICULAR to the original DIAMETER may be constructed by finding the PERPENDICULAR BISECTOR. Call the upper endpoint of this PERPENDICULAR DIAMETER B. For the pentagon, find the MIDPOINT of OB and call it D. Draw DP_0, and BISECT $\angle ODP_0$, calling the intersection point with OP_0 N_1. Draw N_1P_1 PARALLEL to OB, and the first two points of the pentagon are P_0 and P_1 (Coxeter 1969).

Madachy (1979) illustrates how to construct a pentagon by folding and knotting a strip of paper.

see also CYCLIC PENTAGON, DECAGON, DISSECTION, FIVE DISKS PROBLEM, HOME PLATE, PENTAFLAKE, PENTAGRAM, POLYGON, TRIGONOMETRY VALUES—$\pi/5$

References

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De Temple, D. W. "Carlyle Circles and the Lemoine Simplicity of Polygonal Constructions." *Amer. Math. Monthly* **98**, 97–108, 1991.

Dixon, R. *Mathographics.* New York: Dover, p. 17, 1991.

Dudeney, H. E. *Amusements in Mathematics.* New York: Dover, p. 38, 1970.

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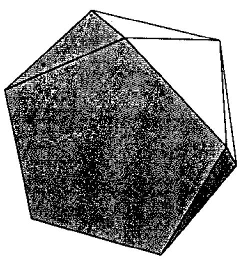
Pappas, T. "The Pentagon, the Pentagram & the Golden Triangle." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 188–189, 1989.

Richmond, H. W. "A Construction for a Regular Polygon of Seventeen Sides." *Quart. J. Pure Appl. Math.* **26**, 206–207, 1893.

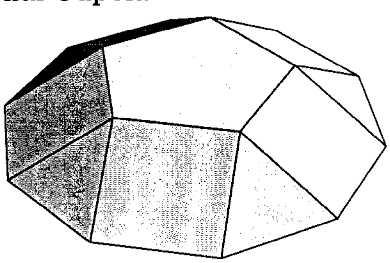
Wantzel, P. L. "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas." *J. Math. pures appliq.* **1**, 366–372, 1836.

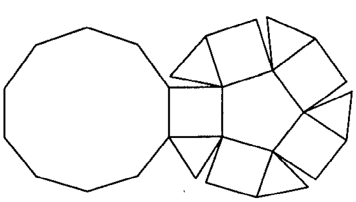
Pentagonal Antiprism



An ANTIPRISM and UNIFORM POLYHEDRON U_{77} whose DUAL POLYHEDRON is the PENTAGONAL DELTAHEDRON.

Pentagonal Cupola





JOHNSON SOLID J_5. The bottom 10 VERTICES are

$$\left(\pm\frac{(1+\sqrt{5})\sqrt{5+\sqrt{5}}}{4\sqrt{2}}, \pm\frac{1}{2}, 0\right),$$
$$\left(\pm\frac{(1+\sqrt{5})\sqrt{5-\sqrt{5}}}{4\sqrt{2}}, \pm\frac{3+\sqrt{5}}{2}, 0\right),$$
$$(0, \pm\tfrac{1}{2}(1+\sqrt{5}), 0)$$

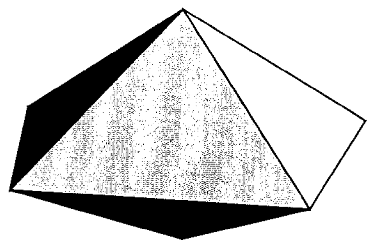
and the top five VERTICES are

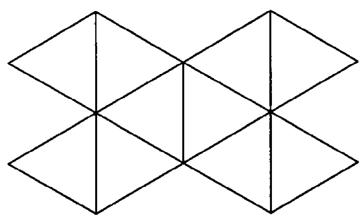
$$\left(\frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}}, 0, \frac{\sqrt{5-\sqrt{5}}}{\sqrt{10}}\right),$$
$$\left(\frac{(\sqrt{5}-1)\sqrt{5+\sqrt{5}}}{4\sqrt{10}}, \pm\tfrac{1}{4}(1+\sqrt{5}), \frac{\sqrt{5-\sqrt{5}}}{\sqrt{10}}\right),$$
$$\left(\frac{-(\sqrt{5}+1)\sqrt{5+\sqrt{5}}}{4\sqrt{10}}, \pm\frac{1}{2}, \frac{\sqrt{5-\sqrt{5}}}{\sqrt{10}}\right).$$

Pentagonal Deltahedron

A DELTAHEDRON which is the DUAL POLYHEDRON of the PENTAGONAL ANTIPRISM.

Pentagonal Dipyramid





The pentagonal dipyramid is one of the convex DELTAHEDRA, and JOHNSON SOLID J_{13}. It is also the DUAL POLYHEDRON of the PENTAGONAL PRISM. The distance between two adjacent VERTICES on the base of the PENTAGON is

$$\begin{aligned} {d_{12}}^2 &= [1-\cos(\tfrac{2}{5}\pi)]^2 + \sin^2(\tfrac{2}{5}\pi) \\ &= [1-\tfrac{1}{4}(\sqrt{5}-1)]^2 + \left[\frac{(1+\sqrt{5})\sqrt{5-\sqrt{5}}}{4\sqrt{2}}\right]^2 \\ &= \tfrac{1}{2}(5-\sqrt{5}), \end{aligned} \tag{1}$$

and the distance between the apex and one of the base points is

$$d_{1h}{}^2 = (0-1)^2 + (0-0)^2 + (h-0)^2 = 1 + h^2. \quad (2)$$

But

$$d_{12}{}^2 = d_{12}{}^2 \quad (3)$$

$$\tfrac{1}{2}(5-\sqrt{5}) = 1 + h^2 \quad (4)$$

$$h^2 = \tfrac{1}{2}(3-\sqrt{5}), \quad (5)$$

and

$$h = \sqrt{\frac{3-\sqrt{5}}{2}}. \quad (6)$$

This root is of the form $\sqrt{a+b\sqrt{c}}$, so applying SQUARE ROOT simplification gives

$$h = \tfrac{1}{2}(\sqrt{5}-1) \equiv \phi - 1, \quad (7)$$

where ϕ is the GOLDEN MEAN.

see also DELTAHEDRON, DIPYRAMID, GOLDEN MEAN, ICOSAHEDRON, JOHNSON SOLID, TRIANGULAR DIPYRAMID

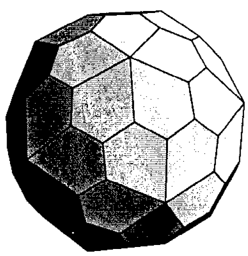
Pentagonal Gyrobicupola

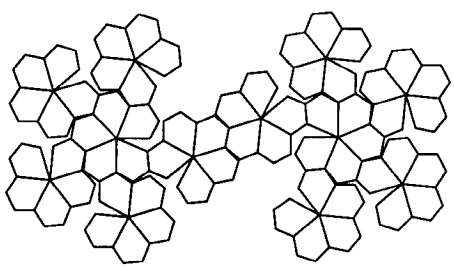
see JOHNSON SOLID

Pentagonal Gyrocupolarotunda

see JOHNSON SOLID

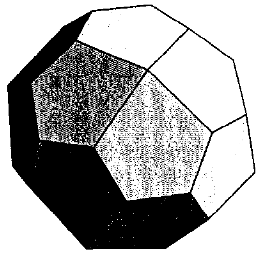
Pentagonal Hexecontahedron

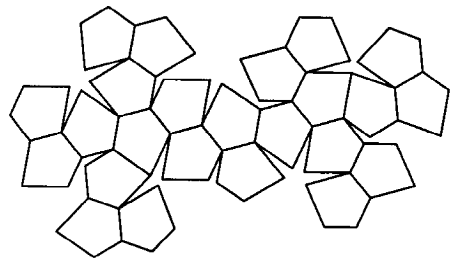




The DUAL POLYHEDRON of the SNUB DODECAHEDRON.

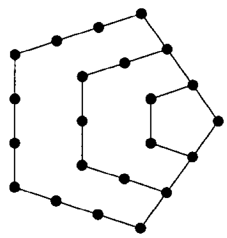
Pentagonal Icositetrahedron





The DUAL POLYHEDRON of the SNUB CUBE.

Pentagonal Number



A POLYGONAL NUMBER of the form $n(3n-1)/2$. The first few are 1, 5, 12, 22, 35, 51, 70, ... (Sloane's A000326). The GENERATING FUNCTION for the pentagonal numbers is

$$\frac{x(2x+1)}{(1-x)^3} = x + 5x^2 + 12x^3 + 22x^4 + \ldots.$$

Every pentagonal number is 1/3 of a TRIANGULAR NUMBER.

The so-called generalized pentagonal numbers are given by $n(3n-1)/2$ with $n = 0, \pm 1, \pm 2, \ldots$, the first few of which are 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, ... (Sloane's A001318).

see also EULER'S PENTAGONAL NUMBER THEOREM, PARTITION FUNCTION P, POLYGONAL NUMBER, TRIANGULAR NUMBER

References

Guy, R. K. "Sums of Squares." §C20 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 136–138, 1994.

Pappas, T. "Triangular, Square & Pentagonal Numbers." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 214, 1989.

Sloane, N. J. A. Sequences A000326/M3818 and A001318/M1336 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Pentagonal Orthobicupola

see JOHNSON SOLID

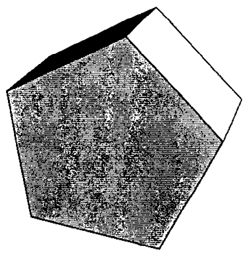
Pentagonal Orthobirotunda

see JOHNSON SOLID

Pentagonal Orthocupolarontunda

see JOHNSON SOLID

Pentagonal Prism



A PRISM and UNIFORM POLYHEDRON U_{76} whose DUAL POLYHEDRON is the PENTAGONAL DIPYRAMID.

see also PENTAGRAMMIC PRISM

Pentagonal Pyramid

see JOHNSON SOLID

Pentagonal Pyramidal Number

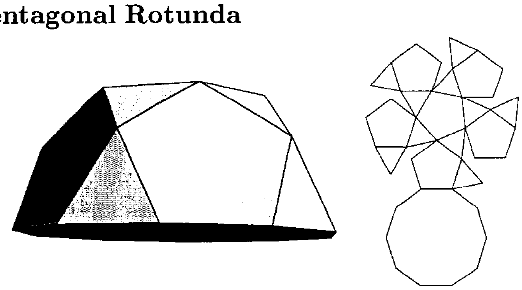
A PYRAMIDAL NUMBER of the form $n^2(n+1)/2$. The first few are 1, 6, 18, 40, 75, ... (Sloane's A002411). The GENERATING FUNCTION for the pentagonal pyramidal numbers is

$$\frac{x(2x+1)}{(x-1)^4} = x + 6x^2 + 18x^3 + 40x^4 + \ldots.$$

References

Sloane, N. J. A. Sequence A002411/M4116 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Pentagonal Rotunda



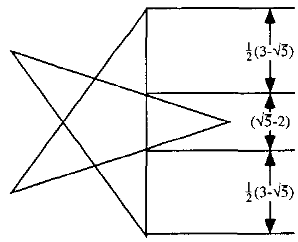
Half of an ICOSIDODECAHEDRON, denoted R_5. It has 10 triangular and five pentagonal faces separating a PENTAGONAL ceiling and a DODECAHEDRAL floor. It is JOHNSON SOLID J_6, and the only true ROTUNDA.

see also ICOSIDODECAHEDRON, JOHNSON SOLID, ROTUNDA

Pentagonal Tiling

see TILING

Pentagram



The STAR POLYGON $\{\frac{5}{2}\}$, also called the PENTACLE, PENTALPHA, or PENTANGLE.

see also DISSECTION, HEXAGRAM, HOEHN'S THEOREM, PENTAGON, STAR FIGURE, STAR OF LAKSHMI

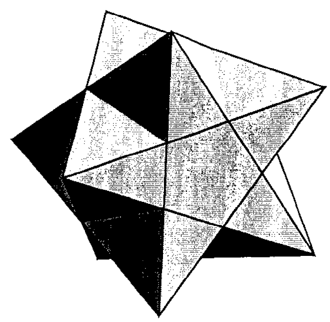
References

Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 122–125, 1990.

Pappas, T. "The Pentagon, the Pentagram & the Golden Triangle." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 188–189, 1989.

Schwartzman, S. *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English.* Washington, DC: Math. Assoc. Amer., 1994.

Pentagrammic Antiprism

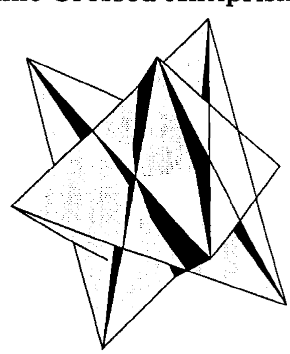


An ANTIPRISM and UNIFORM POLYHEDRON U_{79} whose DUAL POLYHEDRON is the PENTAGRAMMIC DELTAHEDRON.

Pentagrammic Concave Deltahedron

The DUAL POLYHEDRON of the PENTAGRAMMIC CROSSED ANTIPRISM.

Pentagrammic Crossed Antiprism



An ANTIPRISM and UNIFORM POLYHEDRON U_{80} whose DUAL POLYHEDRON is the PENTAGRAMMIC CONCAVE DELTAHEDRON.

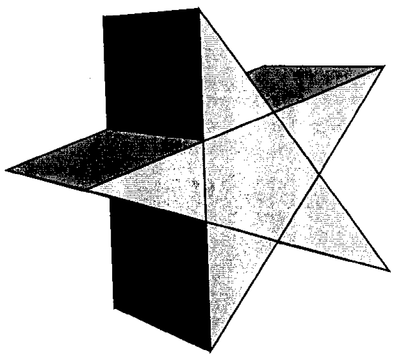
Pentagrammic Deltahedron

The DUAL POLYHEDRON of the PENTAGRAMMIC ANTIPRISM.

Pentagrammic Dipyramid

The DUAL POLYHEDRON of the PENTAGRAMMIC PRISM.

Pentagrammic Prism



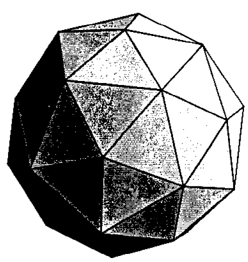
A PRISM and UNIFORM POLYHEDRON U_{78} whose DUAL POLYHEDRON is the PENTAGRAMMIC DIPYRAMID.

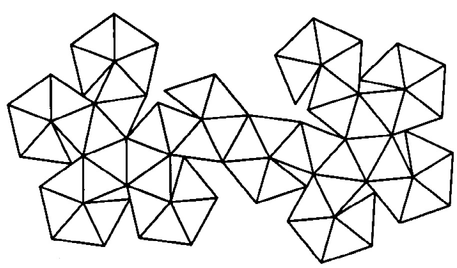
see also PENTAGONAL PRISM

Pentakaidecagon

see PENTADECAGON

Pentakis Dodecahedron





The DUAL POLYHEDRON of the TRUNCATED ICOSAHEDRON.

see also ARCHIMEDEAN SOLID, DUAL POLYHEDRON, TRUNCATED ICOSAHEDRON

Pentalpha

see PENTAGRAM

Pentangle

see PENTAGRAM

Pentatope

The simplest regular figure in 4-D.

Pentatope Number

A FIGURATE NUMBER which is given by

$$Ptop_n = \tfrac{1}{4} Te_n(n+3) = \tfrac{1}{24} n(n+1)(n+2)(n+3),$$

where Te_n is the nth TETRAHEDRAL NUMBER. The first few pentatope numbers are 1, 5, 15, 35, 70, 126, ... (Sloane's A000332). The GENERATING FUNCTION for the pentatope numbers is

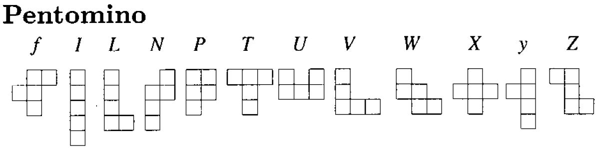
$$\frac{x}{(1-x)^5} = x + 5x^2 + 15x^3 + 35x^4 + \ldots.$$

see also FIGURATE NUMBER, TETRAHEDRAL NUMBER

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 55–57, 1996.

Pentomino



The twelve 5-POLYOMINOES illustrated above and known by the letters of the alphabet they most closely resemble: $f, I, L, N, P, T, U, V, W, X, y, Z$ (Gardner 1960).

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 110–111, 1987.

Dudeney, H. E. "The Broken Chessboard." Problem 74 in *The Canterbury Puzzles and Other Curious Problems, 7th ed.* London: Thomas Nelson and Sons, pp. 119–120, 1949.

Gardner, M. "Mathematical Games: More About the Shapes that Can Be Made with Complex Dominoes." *Sci. Amer.* **203**, 186–198, Nov. 1960.

Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, pp. 80–86, 1975.

Lei, A. "Pentominoes." `http://www.cs.ust.hk/~philipl/omino/pento.html`.

Ruskey, F. "Information on Pentomino Puzzles." `http://sue.csc.uvic.ca/~cos/inf/misc/PentInfo.html`.

Pépin's Test

A test for the PRIMALITY of FERMAT NUMBERS $F_n = 2^{2^n} + 1$, with $n \geq 2$ and $k \geq 2$. Then the two following conditions are equivalent:

1. F_n is PRIME and $k/F_n = -1$.
2. $k^{(F_n-1)/2} \equiv -1 \pmod{F_n}$.

k is usually taken as 3 as a first test.

see also FERMAT NUMBER, PÉPIN'S THEOREM

References

Ribenboim, P. *The Little Book of Big Primes.* New York: Springer-Verlag, p. 62, 1991.

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 119–120, 1993.

Pépin's Theorem

The FERMAT NUMBER F_n is PRIME IFF

$$3^{2^{2^n-1}} \equiv -1 \pmod{F_n}.$$

see also FERMAT NUMBER, PÉPIN'S TEST, SELFRIDGE-HURWITZ RESIDUE

Percent

The use of percentages is a way of expressing RATIOS in terms of whole numbers. Given a RATIO or FRACTION, it is converted to a percentage by multiplying by 100 and appending a "percentage sign" %. For example, if an investment grows from a number $P = 13.00$ to a number $A = 22.50$, then A is $22.50/13.00 = 1.7308$ times as much as P, or 173.08%, and the investment has grown by 73.08%.

see also PERCENTAGE ERROR, PERMIL

Percentage Error

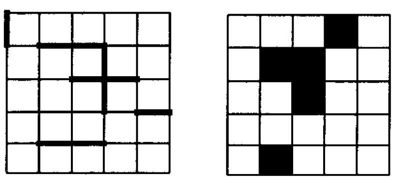
The percentage error is 100% times the RELATIVE ERROR.

see also ABSOLUTE ERROR, ERROR PROPAGATION, PERCENT, RELATIVE ERROR

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

Percolation Theory



bond percolation *site percolation*

Percolation theory deals with fluid flow (or any other similar process) in random media. If the medium is a set of regular LATTICE POINTS, then there are two types of percolation. A SITE PERCOLATION considers the lattice vertices as the relevant entities; a BOND PERCOLATION considers the lattice edges as the relevant entities.

see also BOND PERCOLATION, CAYLEY TREE, CLUSTER, CLUSTER PERIMETER, LATTICE ANIMAL, PERCOLATION THRESHOLD, POLYOMINO, s-CLUSTER, s-RUN, SITE PERCOLATION

References

Deutscher, G.; Zallen, R.; and Adler, J. (Eds.). *Percolation Structures and Processes.* Bristol: Adam Hilger, 1983.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/rndprc/rndprc.html`.

Grimmett, G. *Percolation.* New York: Springer-Verlag, 1989.

Kesten, H. *Percolation Theory for Mathematicians.* Boston, MA: Birkhäuser, 1982.

Stauffer, D. and Aharony, A. *Introduction to Percolation Theory, 2nd ed.* London: Taylor & Francis, 1992.

Percolation Threshold

The critical fraction of lattice points which must be filled to create a continuous path of nearest neighbors from one side to another. The following table is from Stauffer and Aharony (1992, p. 17).

Lattice	Site	Bond
Cubic (Body-Centered)	0.246	0.1803
Cubic (Face-Centered)	0.198	0.119
Cubic (Simple)	0.3116	0.2488
Diamond	0.43	0.388
Honeycomb	0.6962	0.65271
4-Hypercubic	0.197	0.1601
5-Hypercubic	0.141	0.1182
6-Hypercubic	0.107	0.0942
7-Hypercubic	0.089	0.0787
Square	0.592746	0.50000
Triangular	0.50000	0.34729

The square bond value is 1/2 exactly, as is the triangular site. $p_c = 2\sin(\pi/18)$ for the triangular bond and $p_c = 1 - 2\sin(\pi/18)$ for the honeycomb bond. An exact answer for the square site percolation threshold is not known.

see also PERCOLATION THEORY

References

Essam, J. W.; Gaunt, D. S.; and Guttmann, A. J. "Percolation Theory at the Critical Dimension." *J. Phys. A* **11**, 1983–1990, 1978.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/rndprc/rndprc.html`.

Kesten, H. *Percolation Theory for Mathematicians.* Boston, MA: Birkhäuser, 1982.

Stauffer, D. and Aharony, A. *Introduction to Percolation Theory, 2nd ed.* London: Taylor & Francis, 1992.

Perfect Box

see EULER BRICK

Perfect Cubic

A perfect cubic POLYNOMIAL can be factored into a linear and a quadratic term,

$$(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$$

$$(a^3 + b^3) = (a + b)(a^2 - ab + b^2).$$

see also CUBIC EQUATION, PERFECT SQUARE, POLYNOMIAL

Perfect Cuboid

see EULER BRICK

Perfect Difference Set

A SET of RESIDUES $\{a_1, a_2, \ldots, a_{k+1}\}$ (mod n) such that every NONZERO RESIDUE can be uniquely expressed in the form $a_i - a_j$. Examples include $\{1, 2, 4\}$ (mod 7) and $\{1, 2, 5, 7\}$ (mod 13). A NECESSARY condition for a difference set to exist is that n be of the form $k^2 + k + 1$. A SUFFICIENT condition is that k be a PRIME POWER. Perfect sets can be used in the construction of PERFECT RULERS.

see also PERFECT RULER

References

Guy, R. K. "Modular Difference Sets and Error Correcting Codes." §C10 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 118–121, 1994.

Perfect Digital Invariant

see NARCISSISTIC NUMBER

Perfect Information

A class of GAME in which players move alternately and each player is completely informed of previous moves. FINITE, ZERO-SUM, two-player GAMES with perfect information (including checkers and chess) have a SADDLE POINT, and therefore one or more optimal strategies. However, the optimal strategy may be so difficult to compute as to be effectively impossible to determine (as in the game of CHESS).

see also FINITE GAME, GAME, ZERO-SUM GAME

Perfect Magic Cube

A perfect magic cube is a MAGIC CUBE for which the cross-section diagonals, as well as the space diagonals, sum to the MAGIC CONSTANT.

see also MAGIC CUBE, SEMIPERFECT MAGIC CUBE

References

Gardner, M. "Magic Squares and Cubes." Ch. 17 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, pp. 213–225, 1988.

Perfect Number

Perfect numbers are INTEGERS n such that

$$n = s(n), \tag{1}$$

where $s(n)$ is the RESTRICTED DIVISOR FUNCTION (i.e., the SUM of PROPER DIVISORS of n), or equivalently

$$\sigma(n) = 2n, \tag{2}$$

where $\sigma(n)$ is the DIVISOR FUNCTION (i.e., the SUM of DIVISORS of n including n itself). The first few perfect numbers are 6, 28, 496, 8128, ... (Sloane's A000396). This follows from the fact that

$$\begin{aligned} 6 &= \sum 1, 2, 3 \\ 28 &= \sum 1, 2, 4, 7, 14 \\ 496 &= \sum 1, 2, 4, 8, 16, 31, 62, 124, 248, \end{aligned}$$

etc.

Perfect numbers are intimately connected with a class of numbers known as MERSENNE PRIMES. This can be demonstrated by considering a perfect number P of the form $P = q2^{p-1}$ where q is PRIME. Then

$$\sigma(P) = 2P, \tag{3}$$

and using

$$\sigma(q) = q + 1 \tag{4}$$

for q prime, and

$$\sigma(2^\alpha) = 2^{\alpha+1} - 1 \tag{5}$$

gives

$$\begin{aligned} \sigma(q2^{p-1}) &= \sigma(q)\sigma(2^{p-1}) = (q+1)(2^p - 1) \\ &= 2q2^{p-1} = q2^p \end{aligned} \tag{6}$$

$$q(2^p - 1) + 2^p - 1 = q2^p \tag{7}$$

$$q = 2^p - 1. \tag{8}$$

Therefore, if $M_p \equiv q = 2^p - 1$ is PRIME, then

$$P = \tfrac{1}{2}(M_p + 1)M_p = 2^{p-1}(2^p - 1) \tag{9}$$

is a perfect number, as was stated in Proposition IX.36 of Euclid's *Elements* (Dunham 1990). The first few perfect numbers are summarized in the following table.

#	p	P
1	2	6
2	3	28
3	5	496
4	7	8128
5	13	33550336
6	17	8589869056
7	19	137438691328
8	31	2305843008139952128

All EVEN perfect numbers are of this form, as was proved by Euler in a posthumous paper. The only even perfect number of the form x^3+1 is 28 (Mąkowski 1962).

It is not known if any ODD perfect numbers exist, although numbers up to 10^{300} have been checked (Brent *et al.* 1991, Guy 1994) without success, improving the result of Tuckerman (1973), who checked odd numbers up to 10^{36}. Euler showed that an ODD perfect number, if it exists, must be of the form

$$m = p^{4a+1}Q^2, \tag{10}$$

where p is an ODD PRIME RELATIVELY PRIME to Q. In 1887, Sylvester conjectured and in 1925, Gradshtein proved that any ODD perfect number must have at least six different prime aliquot factors (or eight if it is not divisible by 3; Ball and Coxeter 1987). Catalan (1888) proved that if an ODD perfect number is not divisible by 3, 5, or 7, it has at least 26 distinct prime aliquot factors. Stuyvaert (1896) proved that an ODD perfect number must be a sum of squares. All EVEN perfect numbers end in 16, 28, 36, 56, or 76 (Lucas 1891) and, with the exception of 6, have DIGITAL ROOT 1.

Every perfect number of the form $2^p(2^{p+1}-1)$ can be written

$$2^p(2^{p+1}-1) = \sum_{k=1}^{p/2}(2k-1)^3. \tag{11}$$

All perfect numbers are HEXAGONAL NUMBERS and therefore TRIANGULAR NUMBERS. It therefore follows that perfect numbers are always the sum of consecutive POSITIVE integers starting at 1, for example,

$$6 = \sum_{n=1}^{3} n \tag{12}$$

$$28 = \sum_{n=1}^{7} n \tag{13}$$

$$496 = \sum_{n=1}^{31} n \tag{14}$$

(Singh 1997). All EVEN perfect numbers $P > 6$ are of the form

$$P + 1 + 9T_n, \tag{15}$$

where T_n is a TRIANGULAR NUMBER

$$T_n = \tfrac{1}{2}n(n+1) \tag{16}$$

such that $n = 8j+2$ (Eaton 1995, 1996). The sum of reciprocals of all the divisors of a perfect number is 2, since

$$\underbrace{n + \ldots + c + b + a}_{n} = 2n \tag{17}$$

$$\frac{n}{a} + \frac{n}{b} + \ldots = 2n \tag{18}$$

$$\frac{1}{a} + \frac{1}{b} + \ldots = 2. \tag{19}$$

If $s(n) > n$, n is said to be an ABUNDANT NUMBER. If $s(n) < n$, n is said to be a DEFICIENT NUMBER. And if $s(n) = kn$ for a POSITIVE INTEGER $k > 1$, n is said to be a MULTIPERFECT NUMBER of order k.

see also ABUNDANT NUMBER, ALIQUOT SEQUENCE, AMICABLE NUMBERS, DEFICIENT NUMBER, DIVISOR FUNCTION, e-PERFECT NUMBER, HARMONIC NUMBER, HYPERPERFECT NUMBER, INFINARY PERFECT NUMBER, MERSENNE NUMBER, MERSENNE PRIME, MULTIPERFECT NUMBER, MULTIPLICATIVE PERFECT NUMBER, PLUPERFECT NUMBER, PSEUDOPERFECT NUMBER, QUASIPERFECT NUMBER, SEMIPERFECT NUMBER, SMITH NUMBER, SOCIABLE NUMBERS, SUBLIME NUMBER, SUPERPERFECT NUMBER, UNITARY PERFECT NUMBER, WEIRD NUMBER

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Perfect Partition

A PARTITION of n which can generate any number 1, 2, ..., n.

see also PARTITION

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Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 140–143, 1985.

Perfect Proportion

Since

$$\frac{2a}{a+b} = \frac{2ab}{(a+b)b}, \tag{1}$$

it follows that

$$\frac{a}{\frac{a+b}{2}} = \frac{\frac{2ab}{a+b}}{b}, \tag{2}$$

so

$$\frac{a}{A} = \frac{H}{b}, \tag{3}$$

where A and H are the ARITHMETIC MEAN and HARMONIC MEAN of a and b. This relationship was purportedly discovered by Pythagoras.

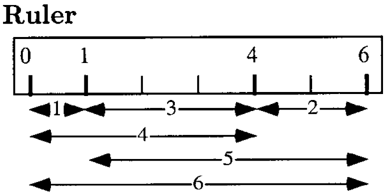
see also ARITHMETIC MEAN, HARMONIC MEAN

Perfect Rectangle

A RECTANGLE which cannot be built up of SQUARES all of different sizes is called an imperfect rectangle. A RECTANGLE which can be built up of SQUARES all of different sizes is called perfect.

order	perfect	imperfect
< 9	0	0
9	2	1
10	6	0
11	22	0
12	67	9
13	213	34
14	744	104
15	2609	282

Perfect Ruler



A type of RULER considered by Guy (1994) which has k distinct marks spaced such that the distances between marks can be used to measure all the distances 1, 2, 3, 4, ... up to some maximum distance $n > k$. Such a ruler can be constructed from a PERFECT DIFFERENCE SET by subtracting one from each element. For example, the PERFECT DIFFERENCE SET {1, 2, 5, 7} gives 0, 1, 4, 6, which can be used to measure $1-0=1$, $6-4=2$, $4-1=3$, $4-0=4$, $6-1=5$, $6-0=6$ (so we get 6 distances with only four marks).

see also PERFECT DIFFERENCE SET

References

Guy, R. K. "Modular Difference Sets and Error Correcting Codes." §C10 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 118–121, 1994.

Perfect Set

A SET P is called perfect if $P = P'$, where P' is the DERIVED SET of P.

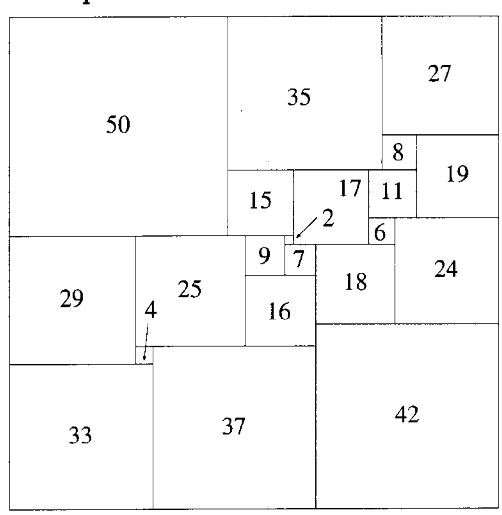
see also DERIVED SET, SET

Perfect Square

The term perfect square is used to refer to a SQUARE NUMBER, a PERFECT SQUARE DISSECTION, or a factorable quadratic polynomial of the form $a^2 - b^2 = (a-b)(a+b)$.

see also PERFECT SQUARE DISSECTION, QUADRATIC EQUATION, SQUARE NUMBER, SQUAREFREE

Perfect Square Dissection



A SQUARE which can be DISSECTED into a number of smaller SQUARES with no two equal is called a PERFECT SQUARE DISSECTION (or a SQUARED SQUARE). Square dissections in which the squares need not be different sizes are called MRS. PERKINS' QUILTS. If no subset of the SQUARES forms a RECTANGLE, then the perfect square is called "simple." Lusin claimed that perfect squares were impossible to construct, but this assertion was proved erroneous when a 55-SQUARE perfect square was published by R. Sprague in 1939 (Wells 1991).

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by

A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above. There is a simple notation (sometimes called Bouwkamp code) used to describe perfect squares. In this notation, brackets are used to group adjacent squares with flush tops, and then the groups are sequentially placed in the highest (and left-most) possible slots. For example, the 21-square illustrated above is denoted [50, 35, 27], [8, 19], [15, 17, 11], [6, 24], [29, 25, 9, 2], [7, 18], [16], [42], [4, 37], [33].

The number of simple perfect squares of order n for $n \geq 21$ are 1, 8, 12, 26, 160, 441, ... (Sloane's A006983). Duijvestijn's Table I gives a list of the 441 simple perfect squares of order 26, the smallest with side length 212 and the largest with side length 825. Skinner (1993) gives the smallest possible side length (and smallest order for each) as 110 (22), 112 (21), 120 (24), 139 (22), 140 (23), ... for simple perfect squared squares, and 175 (24), 235 (25), 288 (26), 324 (27), 325 (27), ... for compound perfect squared squares.

There are actually three simple perfect squares having side length 110. They are [60, 50], [23, 27], [24, 22, 14], [7, 16], [8, 6], [12, 15], [13], [2, 28], [26], [4, 21, 3], [18], [17] (order 22; discovered by A. J. W. Duijvestijn); [60, 50], [27, 23], [24, 22, 14], [4, 19], [8, 6], [3, 12, 16], [9], [2, 28], [26], [21], [1, 18], [17] (order 22; discovered by T. H. Willcocks); and [44, 29, 37], [21, 8], [13, 32], [28, 16], [15, 19], [12,4], [3, 1], [2, 14], [5], [10, 41], [38, 7], [31] (order 23; discovered by A. J. W. Duijvestijn).

D. Sleator has developed an efficient ALGORITHM for finding *non*-simple perfect squares using what he calls rectangle and "ell" grow sequences. This algorithm finds a slew of compound perfect squares of orders 24–32. Weisstein gives a partial list of known simple and compound perfect squares (where the number of simple perfect squares is exact for orders less than 27) as well as *Mathematica*® (Wolfram Research, Champaign, IL) algorithms for drawing them.

Order	# Simple	# Compound
21	1	0
22	8	0
23	12	0
24	26	1
25	160	1
26	441	2
27	?	2
28	?	4
29	?	2
30	?	3
31	?	2
32	?	2
38	1	0
69	1	0

see also MRS. PERKINS' QUILT

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Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 115–116, 1987.

Beiler, A. H. *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains.* New York: Dover, pp. 157–161, 1966.

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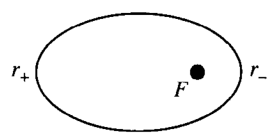
Periapsis



The smallest radial distance of an ELLIPSE as measured from a FOCUS. Taking $v = 0$ in the equation of an ELLIPSE

$$r = \frac{a(1-e^2)}{1+e\cos v}$$

gives the periapsis distance

$$r_- = a(1-e).$$

Periapsis for an orbit around the Earth is called perigee, and periapsis for an orbit around the Sun is called perihelion.

see also APOAPSIS, ECCENTRICITY, ELLIPSE, FOCUS

Perigon

An ANGLE of 2π radians $= 360°$ corresponding to the CENTRAL ANGLE of an entire CIRCLE.

Perimeter

The ARC LENGTH along the boundary of a closed 2-D region. The perimeter of a CIRCLE is called the CIRCUMFERENCE.

see also CIRCUMFERENCE, CLUSTER PERIMETER, SEMIPERIMETER

Perimeter Polynomial

A sum over all CLUSTER PERIMETERS.

Period Doubling

A characteristic of some systems making a transition to CHAOS. Doubling is followed by quadrupling, etc. An example of a map displaying period doubling is the LOGISTIC MAP.

see also CHAOS, LOGISTIC MAP

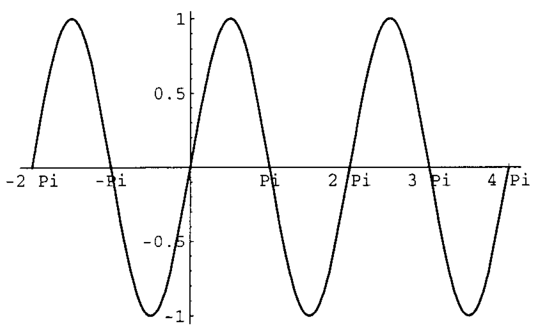
Period Three Theorem

Li and Yorke (1975) proved that *any* 1-D system which exhibits a regular CYCLE of period three will also display regular CYCLES of every other length as well as completely CHAOTIC CYCLES.

see also CHAOS, CYCLE (MAP)

References
Li, T. Y. and Yorke, J. A. "Period Three Implies Chaos." *Amer. Math. Monthly* **82**, 985–992, 1975.

Periodic Function



A FUNCTION $f(x)$ is said to be periodic with period p if $f(x) = f(x + np)$ for $n = 1, 2, \ldots$. For example, the SINE function $\sin x$ is periodic with period 2π (as well as with period -2π, 4π, 6π, etc.).

The CONSTANT FUNCTION $f(x) = 0$ is periodic with any period R for all NONZERO REAL NUMBERS R, so there is no concept analogous to the LEAST PERIOD of a PERIODIC POINT for functions.

see also PERIODIC POINT, PERIODIC SEQUENCE

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Spanier, J. and Oldham, K. B. "Periodic Functions." Ch. 36 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 343–349, 1987.

Periodic Point

A point x_0 is said to be a periodic point of a FUNCTION f of period n if $f^n(x_0) = x_0$, where $f_0(x) = x$ and $f^n(x)$ is defined recursively by $f^n(x) = f(f^{n-1}(x))$.

see also LEAST PERIOD, PERIODIC FUNCTION, PERIODIC SEQUENCE

Periodic Sequence

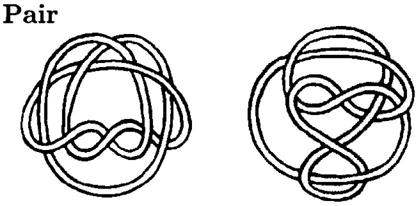
A SEQUENCE $\{a_i\}$ is said to be periodic with period p with if it satisfies $a_i = a_{i+np}$ for $n = 1, 2, \ldots$. For example, $\{1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots\}$ is a periodic sequence with LEAST PERIOD 2.

see also EVENTUALLY PERIODIC, PERIODIC FUNCTION, PERIODIC POINT

Perkins' Quilt

see MRS. PERKINS' QUILT

Perko Pair



The KNOTS 10_{161} and 10_{162} illustrated above. They are listed as separate knots in the pictorial enumeration of Rolfsen (1976, Appendix C), but were identified as identical by Perko (1974).

References
Perko, K. A. Jr. "On the Classification of Knots." *Proc. Amer. Math. Soc.* **45**, 262–266, 1974.
Rolfsen, D. "Table of Knots and Links." Appendix C in *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 280–287, 1976.

Permanence of Algebraic Form

All ELEMENTARY FUNCTIONS can be extended to the COMPLEX PLANE. Such definitions agree with the REAL definitions on the x-AXIS and constitute an ANALYTIC CONTINUATION.

see also ANALYTIC CONTINUATION, ELEMENTARY FUNCTION, PERMANENCE OF MATHEMATICAL RELATIONS PRINCIPLE

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 380, 1985.

Permanence of Mathematical Relations Principle

The metric properties discovered for a primitive figure remain applicable, without modifications other than changes of signs, to all correlative figures which can be considered to arise from the first.

This principle was formulated by Poncelet, and amounts to the statement that if an analytic identity in any finite number of variables holds for all real values of the variables, then it also holds by ANALYTIC CONTINUATION for all complex values (Bell 1945). This principle is also called PONCELET'S CONTINUITY PRINCIPLE.

see also ANALYTIC CONTINUATION, CONSERVATION OF NUMBER PRINCIPLE, DUALITY PRINCIPLE, PERMANENCE OF ALGEBRAIC FORM

References

Bell, E. T. *The Development of Mathematics, 2nd ed.* New York: McGraw-Hill, p. 340, 1945.

Permanent

An analog of a DETERMINANT where all the signs in the expansion by MINORS are taken as POSITIVE. The permanent of a MATRIX A is the coefficient of $x_1 \cdots x_n$ in

$$\prod_{i=1}^{n}(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)$$

(Vardi 1991). Another equation is the RYSER FORMULA

$$\operatorname{perm}(a_{ij}) = (-1)^n \sum_{s\subseteq\{1,\ldots,n\}} (-1)^{|s|} \prod_{i=1}^{n} \sum_{j\in s} a_{ij},$$

where the SUM is over all SUBSETS of $\{1, \ldots, n\}$, and $|s$ is the number of elements in s (Vardi 1991).

If M is a UNITARY MATRIX, then

$$|\operatorname{perm}(\mathsf{M})| \leq 1$$

(Minc 1978, p. 25; Vardi 1991).

see also DETERMINANT, FROBENIUS-KÖNIG THEOREM, IMMANANT, RYSER FORMULA, SCHUR MATRIX

References

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Permil

The use of percentages is a way of expressing RATIOS in terms of whole numbers. Given a RATIO or FRACTION, it is converted to a permil-age by multiplying by 1000 and appending a "mil sign" ‰. For example, if an investment grows from a number $P = 13.00$ to a number $A = 22.50$, then A is $22.50/13.00 = 1.7308$ times as much as P, or 1730.8 ‰.

see also PERCENT

Permutation

The rearrangement of elements in a set into a ONE-TO-ONE correspondence with itself, also called an ARRANGEMENT or ORDER. The number of ways of obtaining r *ordered* outcomes from a permutation of n elements is

$$_nP_r \equiv \frac{n!}{(n-r)!} = r!\binom{n}{r}, \tag{1}$$

where $n!$ is n FACTORIAL and $\binom{a}{b}$ is a BINOMIAL COEFFICIENT. The total number of permutations for n elements is given by $n!$.

A representation of a permutation as a product of CYCLES is unique (up to the ordering of the cycles). An example of a cyclic decomposition is ({1, 3, 4}, {2}), corresponding to the permutations $(1 \to 3, 3 \to 4, 4 \to 1)$ and $(2 \to 2)$, which combine to give {4, 2, 1, 3}.

Any permutation is also a product of TRANSPOSITIONS. Permutations are commonly denoted in LEXICOGRAPHIC or TRANSPOSITION ORDER. There is a correspondence between a PERMUTATION and a pair of YOUNG TABLEAUX known as the SCHENSTED CORRESPONDENCE.

The number of wrong permutations of n objects is $[n!/e]$ where $[x]$ is the NINT function. A permutation of n ordered objects in which no object is in its natural place is called a DERANGEMENT (or sometimes, a COMPLETE PERMUTATION) and the number of such permutations is given by the SUBFACTORIAL $!n$.

Using

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \tag{2}$$

with $x = y = 1$ gives

$$2^n = \sum_{r=0}^{n} \binom{n}{r}, \tag{3}$$

so the number of ways of choosing 0, 1, ..., or n at a time is 2^n.

The set of all permutations of a set of elements 1, ..., n can be obtained using the following recursive procedure

$$\begin{array}{ccc} & 1 & 2 \\ & / & \\ 2 & 1 & \end{array} \tag{4}$$

$$\begin{array}{ccccc} & 1 & & 2 & 3 \\ & & & / & \\ & 1 & 3 & 2 & \\ & / & & & \\ 3 & 1 & & 2 & \\ | & & & & \\ 3 & 2 & & 1 & \\ & \backslash & & & \\ & 2 & 3 & 1 & \\ & & & \backslash & \\ & 2 & & 1 & 3 \end{array} \tag{5}$$

Let the set of INTEGERS 1, 2, ..., n be permuted and the resulting sequence be divided into increasing RUNS. As n approaches INFINITY, the average length of the nth RUN is denoted L_n. The first few values are

$$L_1 = e - 1 = 1.7182818\ldots \tag{6}$$
$$L_2 = e^2 - 2e = 1.9524\ldots \tag{7}$$
$$L_3 = e^3 - 3e^2 + \tfrac{3}{2}e = 1.9957\ldots, \tag{8}$$

where e is the base of the NATURAL LOGARITHM (Knuth 1973, Le Lionnais 1983).

see also ALTERNATING PERMUTATION, BINOMIAL COEFFICIENT, CIRCULAR PERMUTATION, COMBINATION, COMPLETE PERMUTATION, DERANGEMENT, DISCORDANT PERMUTATION, EULERIAN NUMBER, LINEAR EXTENSION, PERMUTATION MATRIX, SUBFACTORIAL, TRANSPOSITION

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Kraitchik, M. "The Linear Permutations of n Different Things." §10.1 in *Mathematical Recreations.* New York: W. W. Norton, pp. 239–240, 1942.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, pp. 41–42, 1983.
Ruskey, F. "Information on Permutations." http://sue.csc.uvic.ca/~cos/inf/perm/PermInfo.html.
Sloane, N. J. A. Sequence A000142/M1675 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Permutation Group

A finite GROUP of substitutions of elements for each other. For instance, the order 4 permutation group {4, 2, 1, 3} would rearrange the elements $\{A, B, C, D\}$ in the order $\{D, B, A, C\}$. A SUBSTITUTION GROUP of two elements is called a TRANSPOSITION. Every SUBSTITUTION GROUP with > 2 elements can be written as a product of transpositions. For example,

$$(abc) = (ab)(ac)$$
$$(abcde) = (ab)(ac)(ad)(ae).$$

CONJUGACY CLASSES of elements which are interchanged are called CYCLES (in the above example, the CYCLES are {{1, 3, 4}, {2}}).

see also CAYLEY'S GROUP THEOREM, CYCLE (PERMUTATION), GROUP, SUBSTITUTION GROUP, TRANSPOSITION

Permutation Matrix

A MATRIX p_{ij} obtained by permuting the ith and jth rows of the IDENTITY MATRIX with $i < j$. Every row and column therefore contain precisely a single 1, and every permutation corresponds to a unique permutation matrix. The matrix is nonsingular, so the DETERMINANT is always NONZERO. It satisfies

$$\mathsf{p}_{ij}{}^2 = \mathsf{I},$$

where I is the IDENTITY MATRIX. Applying to another MATRIX, $\mathsf{p}_{ij}\mathsf{A}$ gives A with the ith and jth rows interchanged, and $\mathsf{A}\mathsf{p}_{ij}$ gives A with the ith and jth columns interchanged.

Interpreting the 1s in an $n \times n$ permutation matrix as ROOKS gives an allowable configuration of nonattacking ROOKS on an $n \times n$ CHESSBOARD.

see also ELEMENTARY MATRIX, IDENTITY, PERMUTATION, ROOK NUMBER

Permutation Pseudotensor

see PERMUTATION TENSOR

Permutation Symbol

A three-index object sometimes called the LEVI-CIVITA SYMBOL defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{for } i = j, j = k, \text{ or } k = i \\ +1 & \text{for } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{for } (i,j,k) \in \{(1,3,2),(3,2,1),(2,1,3)\}. \end{cases} \tag{1}$$

The permutation symbol satisfies

$$\delta_{ij}\epsilon_{ijk} = 0 \tag{2}$$
$$\epsilon_{ipq}\epsilon_{jpq} = 2\delta_{ij} \tag{3}$$
$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{4}$$
$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \tag{5}$$

where δ_{ij} is the KRONECKER DELTA. The symbol can be defined as the SCALAR TRIPLE PRODUCT of unit vectors in a right-handed coordinate system,

$$\epsilon_{ijk} \equiv \hat{\mathbf{x}}_i \cdot (\hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_k). \tag{6}$$

The symbol can also be interpreted as a TENSOR, in which case it is called the PERMUTATION TENSOR.

see also PERMUTATION TENSOR

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 132–133, 1985.

Permutation Tensor

A PSEUDOTENSOR which is ANTISYMMETRIC under the interchange of any two slots. Recalling the definition of the PERMUTATION SYMBOL in terms of a SCALAR TRIPLE PRODUCT of the Cartesian unit vectors,

$$\epsilon_{ijk} \equiv \hat{\mathbf{x}}_i \cdot (\hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_k) = [\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_j, \hat{\mathbf{x}}_k], \tag{1}$$

the pseudotensor is a generalization to an arbitrary BASIS defined by

$$\epsilon_{\alpha\beta\cdots\mu} = \sqrt{|g|}\,[\alpha, \beta, \ldots, \mu] \tag{2}$$

$$\epsilon^{\alpha\beta\cdots\mu} = \frac{[\alpha, \beta, \ldots, \mu]}{\sqrt{|g|}}, \tag{3}$$

where

$$[\alpha, \beta, \ldots, \mu] = \begin{cases} 1 & \text{the arguments are an even permutation} \\ -1 & \text{the arguments are an odd permutation} \\ 0 & \text{two or more arguments are equal,} \end{cases} \tag{4}$$

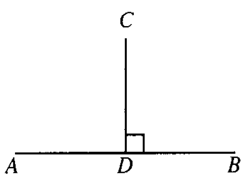
and $g \equiv \det(g_{\alpha\beta})$, where $g_{\alpha\beta}$ is the METRIC TENSOR. $\epsilon(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is NONZERO IFF the VECTORS are LINEARLY INDEPENDENT.

see also PERMUTATION SYMBOL, SCALAR TRIPLE PRODUCT

Peron Integral

see DENJOY INTEGRAL

Perpendicular



Two lines, vectors, planes, etc., are said to be perpendicular if they meet at a RIGHT ANGLE. In $\mathbb{R}^n$, two VECTORS $\mathbf{A}$ and $\mathbf{B}$ are PERPENDICULAR if their DOT PRODUCT

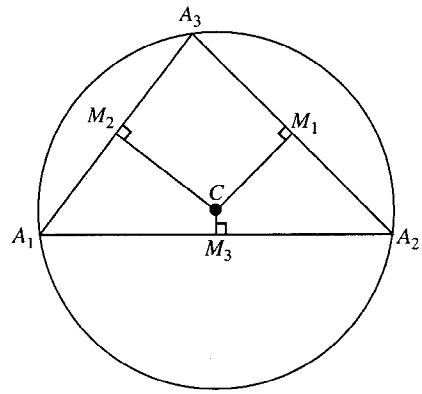
$$\mathbf{A} \cdot \mathbf{B} = 0.$$

In $\mathbb{R}^2$, a LINE with SLOPE $m_2 = -1/m_1$ is PERPENDICULAR to a LINE with SLOPE m_1. Perpendicular objects are sometimes said to be "orthogonal."

In the above figure, the LINE SEGMENT AB is perpendicular to the LINE SEGMENT CD. This relationship is commonly denoted with a small SQUARE at the vertex where perpendicular objects meet, as shown above.

see also ORTHOGONAL VECTORS, PARALLEL, PERPENDICULAR BISECTOR, PERPENDICULAR FOOT, RIGHT ANGLE

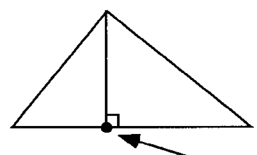
Perpendicular Bisector



The perpendicular bisectors of a TRIANGLE $\Delta A_1A_2A_3$ are lines passing through the MIDPOINT M_i of each side which are PERPENDICULAR to the given side. A TRIANGLE'S three perpendicular bisectors meet at a point C known as the CIRCUMCENTER (which is also the center of the TRIANGLE'S CIRCUMCIRCLE).

see also CIRCUMCENTER, MIDPOINT, PERPENDICULAR, PERPENDICULAR FOOT

Perpendicular Foot



The FOOT of the PERPENDICULAR is the point on the leg opposite a given vertex of a TRIANGLE at which the PERPENDICULAR passing through that vertex intersects the side. The length of the LINE SEGMENT front vertex to perpendicular foot is called the ALTITUDE of the TRIANGLE.

see also ALTITUDE, FOOT, PERPENDICULAR, PERPENDICULAR BISECTOR

Perrin Pseudoprime

If p is PRIME, then $p|P(p)$, where $P(p)$ is a member of the PERRIN SEQUENCE 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, ... (Sloane's A001608). A Perrin pseudoprime is a COMPOSITE NUMBER n such that $n|P(n)$. Several "unrestricted" Perrin pseudoprimes are known, the smallest of which are 271441, 904631, 16532714, 24658561, ... (Sloane's A013998).

Adams and Shanks (1982) discovered the smallest unrestricted Perrin pseudoprime after unsuccessful searches by Perrin (1899), Malo (1900), Escot (1901), and Jarden (1966). (Stewart's 1996 article stating no Perrin pseudoprimes were known was in error.)

Grantham (1996) generalized the definition of Perrin pseudoprime with parameters (r, s) to be an ODD COMPOSITE NUMBER n for which either

1. $(\Delta/n) = 1$ and n has an S-SIGNATURE, or
2. $(\Delta/n) = -1$ and n has a Q-SIGNATURE,

where (a/b) is the JACOBI SYMBOL. All the 55 Perrin pseudoprimes less than 50×10^9 have been computed by Kurtz *et al.* (1986). All have S-SIGNATURE, and form the sequence Sloane calls "restricted" Perrin pseudoprimes: 27664033, 46672291, 102690901, ... (Sloane's A018187).

see also PERRIN SEQUENCE, PSEUDOPRIME

References
Adams, W. W. "Characterizing Pseudoprimes for Third-Order Linear Recurrence Sequences." *Math Comput.* **48**, 1–15, 1987.
Adams, W. and Shanks, D. "Strong Primality Tests that Are Not Sufficient." *Math. Comput.* **39**, 255–300, 1982.
Bach, E. and Shallit, J. *Algorithmic Number Theory, Vol. 1: Efficient Algorithms.* Cambridge, MA: MIT Press, p. 305, 1996.
Escot, E.-B. "Solution to Item 1484." *L'Intermédiare des Math.* **8**, 63–64, 1901.
Grantham, J. "Frobenius Pseudoprimes." http://www.clark.net/pub/grantham/pseudo/pseudo.ps
Holzbaur, C. "Perrin Pseudoprimes." http://ftp.ai.univie.ac.at/perrin.html.
Jarden, D. *Recurring Sequences.* Jerusalem: Riveon Lematematika, 1966.
Kurtz, G. C.; Shanks, D.; and Williams, H. C. "Fast Primality Tests for Numbers Less than $50 \cdot 10^9$." *Math. Comput.* **46**, 691–701, 1986.
Malo, E. *L'Intermédiare des Math.* **7**, 281 and 312, 1900.
Perrin, R. "Item 1484." *L'Intermédiare des Math.* **6**, 76–77, 1899.
Ribenboim, P. *The New Book of Prime Number Records, 3rd ed.* New York: Springer-Verlag, p. 135, 1996.
Sloane, N. J. A. Sequences A013998, A018187, and A001608/M0429 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Stewart, I. "Tales of a Neglected Number." *Sci. Amer.* **274**, 102–103, June 1996.

Perrin Sequence

The INTEGER SEQUENCE defined by the recurrence

$$P(n) = P(n-2) + P(n-3) \tag{1}$$

with the initial conditions $P(0) = 3$, $P(1) = 0$, $P(2) = 2$. The first few terms are 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, ... (Sloane's A001608). $P(n)$ is the solution of a third-order linear homogeneous DIFFERENCE EQUATION having characteristic equation

$$x^3 - x - 1 = 0, \tag{2}$$

discriminant -23, and ROOTS

$$\alpha \approx 1.324717957 \tag{3}$$

$$\beta \approx -0.6623589786 + 0.5622795121i \tag{4}$$

$$\gamma \approx -0.6623589786 - 0.5622795121i. \tag{5}$$

The solution is then

$$A(n) = \alpha^n + \beta^n + \gamma^n, \tag{6}$$

where

$$A(n) \sim \alpha^n. \tag{7}$$

Perrin (1899) investigated the sequence and noticed that if n is PRIME, then $n|P(n)$. The first statement of this fact is attributed to É. Lucas in 1876 by Stewart (1996). Perrin also searched for but did not find any COMPOSITE NUMBER n in the sequence such that $n|P(n)$. Such numbers are now known as PERRIN PSEUDOPRIMES. Malo (1900), Escot (1901), and Jarden (1966) subsequently investigated the series and also found no PERRIN PSEUDOPRIMES. Adams and Shanks (1982) subsequently found that 271,441 is such a number.

see also PADOVAN SEQUENCE, PERRIN PSEUDOPRIME, SIGNATURE (RECURRENCE RELATION)

References
Adams, W. and Shanks, D. "Strong Primality Tests that Are Not Sufficient." *Math. Comput.* **39**, 255–300, 1982.
Escot, E.-B. "Solution to Item 1484." *L'Intermédiare des Math.* **8**, 63–64, 1901.
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Malo, E. *L'Intermédiare des Math.* **7**, 281 and 312, 1900.
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Sloane, N. J. A. Sequence A001608/M0429 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Perron-Frobenius Operator

An OPERATOR which describes the time evolution of densities in PHASE SPACE. The OPERATOR can be defined by

$$\rho_{n+1} = \tilde{L}\rho_n,$$

where ρ_n are the NATURAL DENSITIES after the nth iteration of a map f. This can be explicitly written as

$$\tilde{L}\rho(y) = \sum_{x \in f^{-1}(y)} \frac{\rho(x)}{|f'(x)|}.$$

References
Beck, C. and Schlögl, F. "Transfer Operator Methods." Ch. 17 in *Thermodynamics of Chaotic Systems.* Cambridge, England: Cambridge University Press, pp. 190–203, 1995.

Perron-Frobenius Theorem

If all elements a_{ij} of an IRREDUCIBLE MATRIX A are NONNEGATIVE, then $R = \min M_\lambda$ is an EIGENVALUE of A and all the EIGENVALUES of A lie on the DISK

$$|z| \leq R,$$

where, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a set of NONNEGATIVE numbers (which are not all zero),

$$M_\lambda = \inf\left\{\mu : \mu\lambda_i > \sum_{j=1}^n |a_{ij}|\lambda_j, 1 \leq i \leq n\right\}$$

and $R = \min M_\lambda$. Furthermore, if A has exactly p EIGENVALUES ($p \leq n$) on the CIRCLE $|z| = R$, then the set of all its EIGENVALUES is invariant under rotations by $2\pi/p$ about the ORIGIN.

see also WIELANDT'S THEOREM

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1121, 1979.

Perron's Theorem

If $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ is an arbitrary set of POSITIVE numbers, then all EIGENVALUES λ of the $n \times n$ MATRIX $\mathsf{A} = a_{ij}$ lie on the DISK $|z| \leq M_\mu$, where

$$M_\mu = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \frac{\mu_j}{\mu_i} |a_{ij}|.$$

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1121, 1979.

Persistence

see ADDITIVE PERSISTENCE, MULTIPLICATIVE PERSISTENCE, PERSISTENT NUMBER, PERSISTENT PROCESS

Persistent Number

An n-persistent number is a POSITIVE INTEGER k which contains the digits 0, 1, ..., 9, and for which $2k, \ldots, nk$ also share this property. No ∞-persistent numbers exist. However, the number $k = 1234567890$ is 2-persistent, since $2k = 2469135780$ but $3k = 3703703670$, and the number $k = 526315789473684210$ is 18-persistent. There exists at least one k-persistent number for each POSITIVE INTEGER k.

see also ADDITIVE PERSISTENCE, MULTIPLICATIVE PERSISTENCE

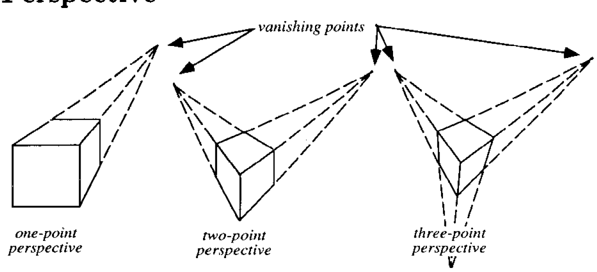
References
Honsberger, R. *More Mathematical Morsels.* Washington, DC: Math. Assoc. Amer., pp. 15–18, 1991.

Persistent Process

A FRACTAL PROCESS for which $H > 1/2$, so $r > 0$.

see also ANTIPERSISTENT PROCESS, FRACTAL PROCESS

Perspective



Perspective is the art and mathematics of realistically depicting 3-D objects in a 2-D plane. The study of the projection of objects in a plane is called PROJECTIVE GEOMETRY. The principles of perspective drawing were elucidated by the Florentine architect F. Brunelleschi (1377–1446). These rules are summarized by Dixon (1991):

1. The horizon appears as a line.
2. Straight lines in space appear as straight lines in the image.
3. Sets of PARALLEL lines meet at a VANISHING POINT.
4. Lines PARALLEL to the picture plane appear PARALLEL and therefore have no VANISHING POINT.

There is a graphical method for selecting vanishing points so that a CUBE or box appears to have the correct dimensions (Dixon 1991).

see also LEONARDO'S PARADOX, PERSPECTIVE AXIS, PERSPECTIVE CENTER, PERSPECTIVE COLLINEATION, PERSPECTIVE TRIANGLES, PERSPECTIVITY, PROJECTIVE GEOMETRY, VANISHING POINT, ZEEMAN'S PARADOX

References
de Vries, V. *Perspective.* New York: Dover, 1968.
Dixon, R. "Perspective Drawings." Ch. 3 in *Mathographics.* New York: Dover, pp. 79–88, 1991.
Parramon, J. M. *Perspective—How to Draw.* Barcelona, Spain: Parramon Editions, 1984.

Perspective Axis

The line joining the three collinear points of intersection of the extensions of corresponding sides in PERSPECTIVE TRIANGLES.

see also PERSPECTIVE CENTER, PERSPECTIVE TRIANGLES, SONDAT'S THEOREM

Perspective Center

The point at which the three LINES connecting the VERTICES of PERSPECTIVE TRIANGLES (from a point) CONCUR.

Perspective Collineation

A perspective collineation with center O and axis o is a COLLINEATION which leaves all lines through O and points of o invariant. Every perspective collineation is a PROJECTIVE COLLINEATION.

see also COLLINEATION, ELATION, HOMOLOGY (GEOMETRY), PROJECTIVE COLLINEATION

References
Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, pp. 247–248, 1969.

Perspective Triangles

Two TRIANGLES are perspective from a line if the extensions of their three pairs of corresponding sides meet in COLLINEAR points. The line joining these points is called the PERSPECTIVE AXIS. Two TRIANGLES are perspective from a point if their three pairs of corresponding VERTICES are joined by lines which meet in a point of CONCURRENCE. This point is called the PERSPECTIVE CENTER. DESARGUES' THEOREM guarantees that if two TRIANGLES are perspective from a point, they are perspective from a line.

see also DESARGUES' THEOREM, HOMOTHETIC TRIANGLES, PARALOGIC TRIANGLES, PERSPECTIVE AXIS, PERSPECTIVE CENTER

Perspectivity

A correspondence between two RANGES that are sections of one PENCIL by two distinct lines.

see also PENCIL, PROJECTIVITY, RANGE (LINE SEGMENT)

Pesin Theory

A theory of linear HYPERBOLIC MAPS in which the leading constants do depend on the variable x.

Peter-Weyl Theorem

Establishes completeness for a REPRESENTATION.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Peters Projection

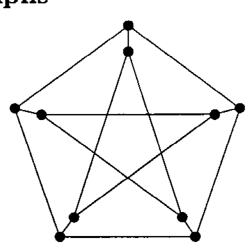
A CYLINDRICAL equal-area projection that shifts the standard parallels to 45° or 47°.

see also CYLINDRICAL PROJECTION

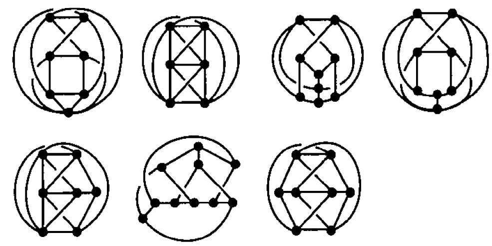
References

Dana, P. H. "Map Projections." `http://www.utexas.edu/depts/grg/gcraft/notes/mapproj/mapproj.html`.

Petersen Graphs



"The" Petersen graph is the GRAPH illustrated above possessing ten VERTICES all of whose nodes have DEGREE 3 (Saaty and Kainen 1986). The Petersen graph is the only smallest-girth graph which has no Tait coloring.



The seven graphs obtainable from the COMPLETE GRAPH K_6 by repeated triangle-Y exchanges are also called Petersen graphs, where the three EDGES forming the TRIANGLE are replaced by three EDGES and a new VERTEX that form a Y, and the reverse operation is also permitted. A GRAPH is intrinsically linked IFF it contains one of the seven Petersen graphs (Robertson *et al.* 1993).

see also HOFFMAN-SINGLETON GRAPH

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 221–222, 1994.

Robertson, N.; Seymour, P. D.; and Thomas, R. "Linkless Embeddings of Graphs in 3-Space." *Bull. Amer. Math. Soc.* **28**, 84–89, 1993.

Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, p. 102, 1986.

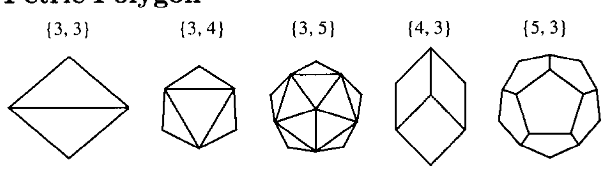
Petersen-Shoute Theorem

1. If ΔABC and $\Delta A'B'C'$ are two directly similar triangles, while $\Delta AA'A''$, $\Delta BB'B''$, and $\Delta CC'C''$ are three directly similar triangles, then $\Delta A''B''C''$ is directly similar to ΔABC.
2. When all the points P on AB are related by a SIMILARITY TRANSFORMATION to all the points P' on $A'B'$, the points dividing the segment PP' in a given ratio are distant and collinear, or else they coincide.

References

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 95–100, 1967.

Petrie Polygon



A skew POLYGON such that every two consecutive sides (but no three) belong to a face of a regular POLYHEDRON. Every finite POLYHEDRON can be orthogonally projected onto a plane in such a way that one Petrie polygon becomes a REGULAR POLYGON with the remainder of the projection interior to it. The Petrie polygon of the POLYHEDRON $\{p, q\}$ has h sides, where

$$\cos^2\left(\frac{\pi}{h}\right) = \cos^2\left(\frac{\pi}{p}\right) + \cos^2\left(\frac{\pi}{q}\right).$$

The Petrie polygons shown above correspond to the PLATONIC SOLIDS.

see also PLATONIC SOLID, REGULAR POLYGON

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 135, 1987.

Coxeter, H. S. M. "Petrie Polygons." §2.6 in *Regular Polytopes, 3rd ed.* New York: Dover, pp.24–25, 1973.

Petrov Notation

A TENSOR notation which considers the RIEMANN TENSOR $R_{\lambda\mu\nu\kappa}$ as a matrix $R_{(\lambda\mu)(\nu\kappa)}$ with indices $\lambda\mu$ and $\nu\kappa$.

References

Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: Wiley, p. 142, 1972.

Pfaffian Form

A 1-FORM

$$\omega = \sum_{i=1}^n a_i(x)\,dx_i$$

such that

$$\omega = 0.$$

References

Knuth, D. E. "Overlapping Pfaffians." *Electronic J. Combinatorics* **3**, No. 2, R5, 1–13, 1996. `http://www.combinatorics.org/Volume_3/volume3_2.html#R5`.

Phase

The angular position of a quantity. For example, the phase of a function $\cos(\omega t + \phi_0)$ as a function of time is

$$\phi(t) = \omega t + \phi_0.$$

The ARGUMENT of a COMPLEX NUMBER is sometimes also called the phase.

see also ARGUMENT (COMPLEX NUMBER), COMPLEX NUMBER, PHASOR, RETARDANCE

Phase Space

For a function or object with n DEGREES OF FREEDOM, the n-D SPACE which is accessible to the function or object is called its phase space.

see also WORLD LINE

Phase Transition

see RANDOM GRAPH

Phasor

The representation, beloved of engineers and physicists, of a COMPLEX NUMBER in terms of a COMPLEX exponential

$$x + iy = |z|e^{i\phi}, \tag{1}$$

where i (called j by engineers) is the IMAGINARY NUMBER and the MODULUS and ARGUMENT (also called PHASE) are

$$|z| = \sqrt{x^2 + y^2} \tag{2}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{3}$$

Here, ϕ is the counterclockwise ANGLE from the POSITIVE REAL axis. In the degenerate case when $x = 0$,

$$\phi = \begin{cases} -\frac{1}{2}\pi & \text{if } y < 0 \\ \text{undefined} & \text{if } y = 0 \\ \frac{1}{2}\pi & \text{if } y > 0. \end{cases} \tag{4}$$

It is trivially true that

$$\sum_i \Re[\psi_i] = \Re\left[\sum_i \psi_i\right]. \tag{5}$$

Now consider a SCALAR FUNCTION $\psi \equiv \psi_0 e^{i\phi}$. Then

$$\begin{aligned} I &\equiv [\Re(\psi)]^2 = [\tfrac{1}{2}(\psi + \psi^*)]^2 = \tfrac{1}{4}(\psi + \psi^*)^2 \\ &= \tfrac{1}{4}(\psi^2 + 2\psi\psi^* + \psi^{*2}). \end{aligned} \tag{6}$$

Look at the time averages of each term,

$$\left\langle \psi^2 \right\rangle = \left\langle {\psi_0}^2 e^{2i\phi} \right\rangle = {\psi_0}^2 \left\langle e^{2i\phi} \right\rangle = 0 \tag{7}$$

$$\langle \psi\psi^* \rangle = \left\langle \psi_0 e^{i\phi} \psi_0 e^{-i\phi} \right\rangle = {\psi_0}^2 = |\psi|^2 \tag{8}$$

$$\left\langle \psi^{*2} \right\rangle = \left\langle {\psi_0}^2 e^{-2i\phi} \right\rangle = {\psi_0}^2 \left\langle e^{-2i\phi} \right\rangle = 0. \tag{9}$$

Therefore,

$$\langle I \rangle = \tfrac{1}{2}|\psi|^2. \tag{10}$$

Consider now two scalar functions

$$\psi_1 \equiv \psi_{1,0} e^{i(kr_1 + \phi_1)} \tag{11}$$

$$\psi_2 \equiv \psi_{2,0} e^{i(kr_2 + \phi_2)}. \tag{12}$$

Then

$$\begin{aligned} I &\equiv [\Re(\psi_1) + \Re(\psi_2)]^2 = \tfrac{1}{4}[(\psi_1 + {\psi_1}^*) + (\psi_2 + {\psi_2}^*)]^2 \\ &= \tfrac{1}{4}[(\psi_1 + {\psi_1}^*)^2 + (\psi_2 + {\psi_2}^*)^2 \\ &\quad + 2(\psi_1\psi_2 + \psi_1{\psi_2}^* + {\psi_1}^*\psi_2 + {\psi_1}^*{\psi_2}^*)] \end{aligned} \tag{13}$$

$$\begin{aligned} \langle I \rangle &= \tfrac{1}{4}[2\psi_1{\psi_1}^* + 2\psi_2{\psi_2}^* + 2\psi_1{\psi_2}^* + 2{\psi_1}^*\psi_2] \\ &= \tfrac{1}{2}[\psi_1({\psi_1}^* + {\psi_2}^*) + \psi_2({\psi_1}^* + {\psi_2}^*)] \\ &= \tfrac{1}{2}(\psi_1 + \psi_2)({\psi_1}^* + {\psi_2}^*) = \tfrac{1}{2}|\psi_1 + \psi_2|^2. \end{aligned} \tag{14}$$

In general,

$$\langle I \rangle = \frac{1}{2}\left|\sum_{i=1}^n \psi_i\right|^2. \tag{15}$$

see also AFFIX, ARGUMENT (COMPLEX NUMBER), COMPLEX MULTIPLICATION, COMPLEX NUMBER, MODULUS (COMPLEX NUMBER), PHASE

Phi Curve

An ADJOINT CURVE which bears a special relation to the base curve.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 310, 1959.

Phi Number System

For every POSITIVE INTEGER n, there is a corresponding finite sequence of distinct INTEGERS $k_1, \ldots, k_m$ such that

$$n = \phi^{k_1} + \ldots + \phi^{k_m},$$

where ϕ is the GOLDEN MEAN.

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Phragmén-Lindêlöf Theorem

Let $f(z)$ be an ANALYTIC FUNCTION in an angular domain $W : |\arg z| < \alpha\pi/2$. Suppose there is a constant M such that for each $\epsilon > 0$, each finite boundary point has a NEIGHBORHOOD such that $|f(z)| < M + \epsilon$ on the intersection of D with this NEIGHBORHOOD, and that for some POSITIVE number $\beta > \alpha$ for sufficiently large $|z|$, the INEQUALITY $|f(z)| < \exp(|z|^{1/\beta})$ holds. Then $|f(z)| \leq M$ in D.

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Phyllotaxis

The beautiful arrangement of leaves in some plants, called phyllotaxis, obeys a number of subtle mathematical relationships. For instance, the florets in the head of a sunflower form two oppositely directed spirals: 55 of them clockwise and 34 counterclockwise. Surprisingly, these numbers are consecutive FIBONACCI NUMBERS. The ratios of alternate FIBONACCI NUMBERS are given by the convergents to ϕ^{-2}, where ϕ is the GOLDEN RATIO, and are said to measure the fraction of a turn between successive leaves on the stalk of a plant: 1/2 for elm and linden, 1/3 for beech and hazel, 2/5 for oak and apple, 3/8 for poplar and rose, 5/13 for willow and almond, etc. (Coxeter 1969, Ball and Coxeter 1987). A similar phenomenon occurs for DAISIES, pineapples, pinecones, cauliflowers, and so on.

Lilies, irises, and the trillium have three petals; columbines, buttercups, larkspur, and wild rose have five petals; delphiniums, bloodroot, and cosmos have eight petals; corn marigolds have 13 petals; asters have 21 petals; and daisies have 34, 55, or 84 petals—all FIBONACCI NUMBERS.

see also DAISY, FIBONACCI NUMBER, SPIRAL

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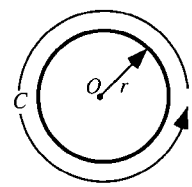
Pi



A REAL NUMBER denoted π which is defined as the ratio of a CIRCLE's CIRCUMFERENCE C to its DIAMETER $d = 2r$,

$$\pi \equiv \frac{C}{d} = \frac{C}{2r} \tag{1}$$

It is equal to

$$\pi = 3.141592653589793238462643383279502884197\ldots \tag{2}$$

(Sloane's A000796). π has recently (August 1997) been computed to a world record $51,539,600,000 \approx 3 \cdot 2^{34}$ DECIMAL DIGITS by Y. Kanada. This calculation was done using Borwein's fourth-order convergent algorithm and required 29 hours on a massively parallel 1024-processor Hitachi SR2201 supercomputer. It was checked in 37 hours using the BRENT-SALAMIN FORMULA on the same machine.

The SIMPLE CONTINUED FRACTION for π, which gives the "best" approximation of a given order, is [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...] (Sloane's A001203). The very large term 292 means that the CONVERGENT

$$[3,7,15,1]=[3,7,16]=\tfrac{355}{113}=3.14159292\ldots \tag{3}$$

is an extremely good approximation. The first few CONVERGENTS are 22/7, 333/106, 355/113, 103993/33102, 104348/33215, ... (Sloane's A002485 and A002486). The first occurrences of n in the CONTINUED FRACTION are 4, 9, 1, 30, 40, 32, 2, 44, 130, 100, ... (Sloane's A032523).

Gosper has computed 17,001,303 terms of π's CONTINUED FRACTION (Gosper 1977, Ball and Coxeter 1987), although the computer on which the numbers are stored may no longer be functional (Gosper, pers. comm., 1998). According to Gosper, a typical CONTINUED FRACTION term carries only slightly more significance than a decimal DIGIT. The sequence of increasing terms in the CONTINUED FRACTION is 3, 7, 15, 292, 436, 20776, ... (Sloane's A033089), occurring at positions 1, 2, 3, 5, 308, 432, ... (Sloane's A033090). In the first 26,491 terms of the CONTINUED FRACTION (counting 3 as the 0th), the only five-DIGIT terms are 20,776 (the 431st), 19,055 (15,543rd), and 19,308 (23,398th) (Beeler *et al.* 1972, Item 140). The first 6-DIGIT term is 528,210 (the 267,314th), and the first 8-DIGIT term is 12,996,958 (453,294th). The term having the largest known value is the whopping 9-DIGIT 87,878,3625 (the 11,504,931st term).

The SIMPLE CONTINUED FRACTION for π does not show any obvious patterns, but clear patterns *do* emerge in the beautiful non-simple CONTINUED FRACTIONS

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\ldots}}}} \tag{4}$$

(Brouckner), giving convergents 1, 3/2, 15/13, 105/76, 315/263, ... (Sloane's A025547 and A007509) and

$$\frac{\pi}{2}=1-\cfrac{1}{3-\cfrac{2\cdot 3}{1-\cfrac{1\cdot 2}{3-\cfrac{4\cdot 5}{1-\cfrac{3\cdot 4}{3-\cfrac{6\cdot 7}{1-\cfrac{5\cdot 6}{3-\ldots}}}}}}} \tag{5}$$

(Stern 1833), giving convergents 1, 2/3, 4/3, 16/15, 64/45, 128/105, ... (Sloane's A001901 and A046126).

π crops up in all sorts of unexpected places in mathematics besides CIRCLES and SPHERES. For example, it occurs in the normalization of the GAUSSIAN DISTRIBUTION, in the distribution of PRIMES, in the construction of numbers which are very close to INTEGERS (the RAMANUJAN CONSTANT), and in the probability that a pin dropped on a set of PARALLEL lines intersects a line (BUFFON'S NEEDLE PROBLEM). Pi also appears as the average ratio of the actual length and the direct distance between source and mouth in a meandering river (Støllum 1996, Singh 1997).

A brief history of NOTATION for pi is given by Castellanos (1988). π is sometimes known as LUDOLPH'S CONSTANT after Ludolph van Ceulen (1539–1610), a Dutch π calculator. The symbol π was first used by William Jones in 1706, and subsequently adopted by Euler. In *Measurement of a Circle,* Archimedes (ca. 225 BC) obtained the first rigorous approximation by INSCRIBING and CIRCUMSCRIBING $6\cdot 2^n$-gons on a CIRCLE using the ARCHIMEDES ALGORITHM. Using $n=4$ (a 96-gon), Archimedes obtained

$$3+\tfrac{10}{71}<\pi<3+\tfrac{1}{7} \tag{6}$$

(Shanks 1993, p. 140).

The Bible contains two references (I Kings 7:23 and Chronicles 4:2) which give a value of 3 for π. It should be mentioned, however, that both instances refer to a value obtained from physical measurements and, as such, are probably well within the bounds of experimental uncertainty. I Kings 7:23 states, "Also he made a molten sea of ten Cubits from brim to brim, round in compass, and five cubits in height thereof; and a line thirty cubits did compass it round about." This implies $\pi = C/d = 30/10 = 3$. The Babylonians gave an estimate of π as $3+1/8=3.125$. The Egyptians did better still, obtaining $2^8/3^4=3.1605\ldots$ in the Rhind papyrus, and 22/7 elsewhere. The Chinese geometers, however, did best of all, rigorously deriving π to 6 decimal places.

A method similar to Archimedes' can be used to estimate π by starting with an n-gon and then relating the AREA of subsequent $2n$-gons. Let β be the ANGLE from the center of one of the POLYGON's segments,

$$\beta=\tfrac{1}{4}(n-3)\pi. \tag{7}$$

Then

$$\pi=\frac{\frac{1}{2}n\sin(2\beta)}{\cos\beta\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2^2}\right)\cos\left(\frac{\beta}{2^3}\right)\cdots} \tag{8}$$

(Beckmann 1989, pp. 92–94). Viète (1593) was the first to give an exact expression for π by taking $n=4$ in the above expression, giving

$$\cos\beta=\sin\beta=\frac{1}{\sqrt{2}}=\tfrac{1}{2}\sqrt{2}, \tag{9}$$

which leads to an INFINITE PRODUCT of CONTINUED SQUARE ROOTS,

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}}\cdots \tag{10}$$

(Beckmann 1989, p. 95). However, this expression was not rigorously proved to converge until Rudio (1892). Another exact FORMULA is MACHIN'S FORMULA, which is

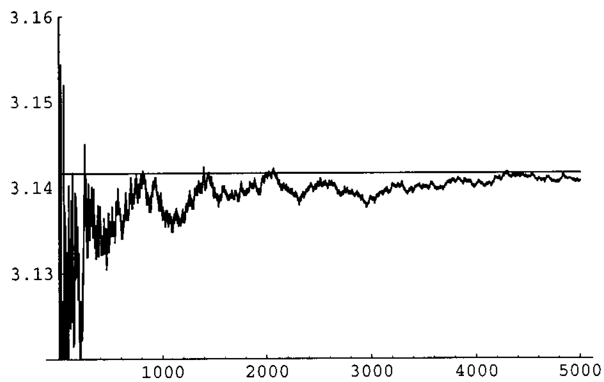
$$\frac{\pi}{4} = 4\tan^{-1}(\tfrac{1}{5}) - \tan^{-1}(\tfrac{1}{239}). \tag{11}$$

There are three other MACHIN-LIKE FORMULAS, as well as other FORMULAS with more terms. An interesting INFINITE PRODUCT formula due to Euler which relates π and the nth PRIME p_n is

$$\pi = \frac{2}{\prod_{i=n}^{\infty}\left[1+\frac{\sin(\frac{1}{2}\pi p_n)}{p_n}\right]} \tag{12}$$

$$= \frac{2}{\prod_{i=n}^{\infty}\left[1+\frac{(-1)^{(p_n-1)/2}}{p_n}\right]} \tag{13}$$

(Blatner 1997, p. 119), plotted below as a function of the number of terms in the product.



The AREA and CIRCUMFERENCE of the UNIT CIRCLE are given by

$$A = \pi = 4\int_0^1 \sqrt{1-x^2}\,dx \tag{14}$$

$$= \lim_{n\to\infty} \frac{4}{n^2}\sum_{k=0}^{n}\sqrt{n^2-k^2} \tag{15}$$

and

$$C = 2\pi = 4\int_0^1 \frac{dx}{\sqrt{1-x^2}} \tag{16}$$

$$= 4\int_0^1 \sqrt{1+\left(\frac{d}{dx}\sqrt{1-x^2}\right)^2}\,dx. \tag{17}$$

The SURFACE AREA and VOLUME of the unit SPHERE are

$$S = 4\pi \tag{18}$$

$$V = \tfrac{4}{3}\pi. \tag{19}$$

π is known to be IRRATIONAL (Lambert 1761, Legendre 1794) and even TRANSCENDENTAL (Lindemann 1882). Incidentally, Lindemann's proof of the transcendence of π also proved that the GEOMETRIC PROBLEM OF ANTIQUITY known as CIRCLE SQUARING is impossible. A simplified, but still difficult, version of Lindemann's proof is given by Klein (1955).

It is also known that π is not a LIOUVILLE NUMBER (Mahler 1953). In 1974, M. Mignotte showed that

$$\left|\pi - \frac{p}{q}\right| \leq q^{-20} \tag{20}$$

has only a finite number of solutions in INTEGERS (Le Lionnais 1983, p. 50). This result was subsequently improved by Chudnovsky and Chudnovsky (1984) who showed that

$$\left|\pi - \frac{p}{q}\right| > q^{-14.65}, \tag{21}$$

although it is likely that the exponent can be reduced to $2+\epsilon$, where ϵ is an infinitesimally small number (Borwein *et al.* 1989). It is *not* known if π is NORMAL (Wagon 1985), although the first 30 million DIGITS are very UNIFORMLY DISTRIBUTED (Bailey 1988). The following distribution is found for the first n DIGITS of $\pi-3$. It shows no statistically SIGNIFICANT departure from a UNIFORM DISTRIBUTION (technically, in the CHI-SQUARED TEST, it has a value of ${\chi_s}^2 = 5.60$ for the first 5×10^{10} terms).

digit	1×10^5	1×10^6	6×10^9	5×10^{10}
0	9,999	99,959	599,963,005	5,000,012,647
1	10,137	99,758	600,033,260	4,999,986,263
2	9,908	100,026	599,999,169	5,000,020,237
3	10,025	100,229	600,000,243	4,999,914,405
4	9,971	100,230	599,957,439	5,000,023,598
5	10,026	100,359	600,017,176	4,999,991,499
6	10,029	99,548	600,016,588	4,999,928,368
7	10,025	99,800	600,009,044	5,000,014,860
8	9,978	99,985	599,987,038	5,000,117,637
9	9,902	100,106	600,017,038	4,999,990,486

The digits of $1/\pi$ are also very uniformly distributed (${\chi_s}^2 = 7.04$), as shown in the following table.

digit	5×10^{10}
0	4,999,969,955
1	5,000,113,699
2	4,999,987,893
3	5,000,040,906
4	4,999,985,863
5	4,999,977,583
6	4,999,990,916
7	4,999,985,552
8	4,999,881,183
9	5,000,066,450

It is not known if $\pi + e$, π/e, or $\ln \pi$ are IRRATIONAL. However, it is known that they cannot satisfy any POLYNOMIAL equation of degree ≤ 8 with INTEGER COEFFICIENTS of average size 10^9 (Bailey 1988, Borwein *et al.* 1989).

π satisfies the INEQUALITY

$$\left(1+\frac{1}{\pi}\right)^{\pi+1} \approx 3.14097 < \pi. \tag{22}$$

Beginning with any POSITIVE INTEGER n, round up to the nearest multiple of $n-1$, then up to the nearest multiple of $n-2$, and so on, up to the nearest multiple of 1. Let $f(n)$ denote the result. Then the ratio

$$\lim_{n\to\infty} \frac{n^2}{f(n)} = \pi \tag{23}$$

(Brown). David (1957) credits this result to Jabotinski and Erdős and gives the more precise asymptotic result

$$f(n) = \frac{n^2}{\pi} + \mathcal{O}(n^{4/3}). \tag{24}$$

The first few numbers in the sequence $\{f(n)\}$ are 1, 2, 4, 6, 10, 12, 18, 22, 30, 34, ... (Sloane's A002491).

A particular case of the WALLIS FORMULA gives

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n-1)(2n+1)}\right] = \frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\cdots. \tag{25}$$

This formula can also be written

$$\lim_{n\to\infty}\frac{2^{4n}}{n\binom{2n}{n}^2} = \pi \lim_{n\to\infty}\frac{n[\Gamma(n)]^2}{[\Gamma(\frac{1}{2}+n)]^2} = \pi, \tag{26}$$

where $\binom{n}{k}$ denotes a BINOMIAL COEFFICIENT and $\Gamma(x)$ is the GAMMA FUNCTION (Knopp 1990). Euler obtained

$$\pi = \sqrt{6\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)}, \tag{27}$$

which follows from the special value of the RIEMANN ZETA FUNCTION $\zeta(2) = \pi^2/6$. Similar FORMULAS follow from $\zeta(2n)$ for all POSITIVE INTEGERS n. Gregory and Leibniz found

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots, \tag{28}$$

which is sometimes known as GREGORY'S FORMULA. The error after the nth term of this series in GREGORY'S FORMULA is larger than $(2n)^{-1}$ so this sum converges so slowly that 300 terms are not sufficient to calculate π correctly to two decimal places! However, it can be transformed to

$$\pi = \sum_{k=1}^{\infty}\frac{3^k-1}{4^k}\zeta(k+1), \tag{29}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION (Vardi 1991, pp. 157–158; Flajolet and Vardi 1996), so that the error after k terms is $\approx (3/4)^k$. Newton used

$$\pi = \tfrac{3}{4}\sqrt{3} + 24\int_0^{1/4}\sqrt{x-x^2}\,dx \tag{30}$$

$$= \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \ldots\right) \tag{31}$$

(Borwein *et al.* 1989). Using Euler's CONVERGENCE IMPROVEMENT transformation gives

$$\frac{\pi}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(n!)^2 2^{n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{n!}{(2n+1)!!}$$

$$= 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \ldots \tag{32}$$

$$= 1 + \frac{1}{3}\left(1+\frac{2}{5}\left(1+\frac{3}{7}\left(1+\frac{4}{9}(1+\ldots)\right)\right)\right) \tag{33}$$

(Beeler *et al.* 1972, Item 120). This corresponds to plugging $x = 1/\sqrt{2}$ into the POWER SERIES for the HYPERGEOMETRIC FUNCTION ${}_2F_1(a,b;c;x)$,

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = \sum_{i=0}^{\infty}\frac{(2x)^{2i+1}(i!)^2}{2(2i+1)!} = {}_2F_1(1,1;\tfrac{3}{2};x^2)x. \tag{34}$$

Despite the convergence improvement, series (33) converges at only one bit/term. At the cost of a SQUARE ROOT, Gosper has noted that $x = 1/2$ gives 2 bits/term,

$$\tfrac{1}{9}\sqrt{3}\pi = \frac{1}{2}\sum_{i=1}^{\infty}\frac{(i!)^2}{(2i+1)!}, \tag{35}$$

and $x = \sin(\pi/10)$ gives almost 3.39 bits/term,

$$\frac{\pi}{5\sqrt{\phi+2}} = \frac{1}{2}\sum_{i=0}^{\infty}\frac{(i!)^2}{\phi^{2i+1}(2i+1)!}, \tag{36}$$

where ϕ is the GOLDEN RATIO. Gosper also obtained

$$\pi = 3 + \frac{1}{60}\left(8 + \frac{2\cdot 3}{7\cdot 8\cdot 3}\left(13 + \frac{3\cdot 5}{10\cdot 11\cdot 3}\right.\right. \\ \left.\left.\times\left(18 + \frac{4\cdot 7}{13\cdot 14\cdot 3}(23+\ldots)\right)\right)\right). \quad (37)$$

An infinite sum due to Ramanujan is

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}} \quad (38)$$

(Borwein *et al.* 1989). Further sums are given in Ramanujan (1913–14),

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(1123+21460n)(2n-1)!!(4n-1)!!}{882^{2n+1}32^n(n!)^3} \quad (39)$$

and

$$\frac{1}{\pi} = \sqrt{8}\sum_{n=0}^{\infty} \frac{(1103+26390n)(2n-1)!!(4n-1)!!}{99^{4n+2}32^n(n!)^3} \\ = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}} \quad (40)$$

(Beeler *et al.* 1972, Item 139; Borwein *et al.* 1989). Equation (40) is derived from a modular identity of order 58, although a first derivation was not presented prior to Borwein and Borwein (1987). The above series both give

$$\pi \approx \frac{2206\sqrt{2}}{9801} = 3.14159273001\ldots \quad (41)$$

as the first approximation and provide, respectively, about 6 and 8 decimal places per term. Such series exist because of the rationality of various modular invariants. The general form of the series is

$$\sum_{n=0}^{\infty}[a(t)+nb(t)]\frac{(6n)!}{(3n)!(n!)^3}\frac{1}{[j(t)]^n} = \frac{\sqrt{-j(t)}}{\pi}, \quad (42)$$

where t is a QUADRATIC FORM DISCRIMINANT, $j(t)$ is the j-FUNCTION,

$$b(t) = \sqrt{t[1728-j(t)]} \quad (43)$$

$$a(t) = \frac{b(t)}{6}\left\{1 - \frac{E_4(t)}{E_6(t)}\left[E_2(t) - \frac{6}{\pi\sqrt{t}}\right]\right\}, \quad (44)$$

and the E_i are RAMANUJAN-EISENSTEIN SERIES. A CLASS NUMBER p field involves pth degree ALGEBRAIC INTEGERS of the constants $A = a(t)$, $B = b(t)$, and $C = c(t)$. The fastest converging series that uses only INTEGER terms corresponds to the largest CLASS NUMBER 1 discriminant of $d = -163$ and was formulated by the Chudnovsky brothers (1987). The 163 appearing here is the same one appearing in the fact that $e^{\pi\sqrt{163}}$ (the RAMANUJAN CONSTANT) is very nearly an INTEGER. The series is given by

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!(13591409+545140134n)}{(n!)^3(3n)!(640320^3)^{n+1/2}} \\ = \frac{163\cdot 8\cdot 27\cdot 7\cdot 11\cdot 19\cdot 127}{640320^{3/2}} \\ \times\sum_{n=0}^{\infty}\left(\frac{13591409}{163\cdot 2\cdot 9\cdot 7\cdot 11\cdot 19\cdot 127}+n\right) \\ \times\frac{(6n)!}{(3n)!(n!)^3}\frac{(-1)^n}{640320^{3n}} \quad (45)$$

(Borwein and Borwein 1993). This series gives 14 digits accurately per term. The same equation in another form was given by the Chudnovsky brothers (1987) and is used by *Mathematica*® (Wolfram Research, Champaign, IL) to calculate π (Vardi 1991),

$$\pi = \frac{426880\sqrt{10005}}{A[{}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;B) - C\,{}_3F_2(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;B)]}, \quad (46)$$

where

$$A \equiv 13591409 \quad (47)$$

$$B \equiv -\tfrac{1}{151931373056000} \quad (48)$$

$$C \equiv \tfrac{30285563}{1651969144908540723200}. \quad (49)$$

The best formula for CLASS NUMBER 2 (largest discriminant -427) is

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!(A+Bn)}{(n!)^3(3n)!C^{n+1/2}}, \quad (50)$$

where

$$A \equiv 212175710912\sqrt{61} + 1657145277365 \quad (51)$$

$$B \equiv 13773980892672\sqrt{61} + 107578229802750 \quad (52)$$

$$C \equiv [5280(236674+30303\sqrt{61}]^3 \quad (53)$$

(Borwein and Borwein 1993). This series adds about 25 digits for each additional term. The fastest converging series for CLASS NUMBER 3 corresponds to $d = -907$ and gives 37–38 digits per term. The fastest converging CLASS NUMBER 4 series corresponds to $d = -1555$ and is

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(n!)^3}\frac{A+nB}{C^{3n}}, \quad (54)$$

where

$$
\begin{aligned}
A = {} & 63365028312971999585426220 \\
& + 28337702140800842046825600\sqrt{5} \\
& + 384\sqrt{5}(10891728551171178200467 4\cdots \\
& \cdots 3621239520916038565601 7 + 48790290865788102 2\cdots \\
& \cdots 5077338534541688721351255040\sqrt{5}\,)^{1/2}
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
B = {} & 7849910453496627210289749000 \\
& + 3510586678260932028965606400\sqrt{5} \\
& + 2515968\sqrt{3110}(6260208323789001 6\cdots \\
& \cdots 3699332265444402088216 1 + 27996502730604429 6\cdots \\
& \cdots 5772068907188251902 35\sqrt{5}\,)^{1/2}
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
C = {} & -214772995063512240 - 96049403338648032\sqrt{5} \\
& - 1296\sqrt{5}(10985234579463550323713318473 \\
& + 4912746253692362754607395912\sqrt{5}\,)^{1/2}.
\end{aligned}
\tag{57}
$$

This gives 50 digits per term. Borwein and Borwein (1993) have developed a general ALGORITHM for generating such series for arbitrary CLASS NUMBER. Bellard gives the exotic formula

$$
\pi = \frac{1}{740025}\left[\sum_{n=1}^{\infty} \frac{3P(n)}{\binom{7n}{2n}2^{n-1}} - 20379280\right], \tag{58}
$$

where

$$
\begin{aligned}
P(n) \equiv {} & -885673181n^5 + 3125347237n^4 - 2942969225n^3 \\
& + 1031962795n^2 - 196882274n + 10996648.
\end{aligned}
\tag{59}
$$

A complete listing of Ramanujan's series for $1/\pi$ found in his second and third notebooks is given by Berndt (1994, pp. 352–354),

$$
\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6n+1)(\tfrac{1}{2})_n{}^3}{4^n(n!)^3} \tag{60}
$$

$$
\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(42n+5)(\tfrac{1}{2})_n{}^3}{(64)^n(n!)^3} \tag{61}
$$

$$
\frac{32}{\pi} = \sum_{n=0}^{\infty} \frac{(42\sqrt{5}\,n + 5\sqrt{5} + 30n - 1)(\tfrac{1}{2})_n{}^3}{(64)^n(n!)^3} \times \left(\frac{\sqrt{5}-1}{2}\right)^{8n} \tag{62}
$$

$$
\frac{27}{4\pi} = \sum_{n=0}^{\infty} \frac{(15n+2)(\tfrac{1}{2})_n(\tfrac{1}{3})_n(\tfrac{2}{3})_n}{(n!)^3}\left(\frac{2}{27}\right)^n \tag{63}
$$

$$
\frac{15\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \frac{(33n+4)(\tfrac{1}{2})_n(\tfrac{1}{3})_n(\tfrac{2}{3})_n}{(n!)^3}\left(\frac{4}{125}\right)^n \tag{64}
$$

$$
\frac{5\sqrt{5}}{2\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(11n+1)(\tfrac{1}{2})_n(\tfrac{1}{6})_n(\tfrac{5}{6})_n}{(n!)^3}\left(\frac{4}{125}\right)^n \tag{65}
$$

$$
\frac{85\sqrt{85}}{18\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(133n+8)(\tfrac{1}{2})_n(\tfrac{1}{6})_n(\tfrac{5}{6})_n}{(n!)^3}\left(\frac{4}{85}\right)^n \tag{66}
$$

$$
\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(20n+3)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3 2^{2n+1}} \tag{67}
$$

$$
\frac{4}{\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n(28n+3)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3 3^n 4^{n+1}} \tag{68}
$$

$$
\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(260n+23)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3(18)^{2n+1}} \tag{69}
$$

$$
\frac{4}{\pi\sqrt{5}} = \sum_{n=0}^{\infty} \frac{(-1)^n(644n+41)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3 5^n(72)^{2n+1}} \tag{70}
$$

$$
\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n(21460n+1123)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3(882)^{2n+1}} \tag{71}
$$

$$
\frac{2\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \frac{(8n+1)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3 9^n} \tag{72}
$$

$$
\frac{1}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(10n+1)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3 9^{2n+1}} \tag{73}
$$

$$
\frac{1}{3\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(40n+3)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3(49)^{2n+1}} \tag{74}
$$

$$
\frac{2}{\pi\sqrt{11}} = \sum_{n=0}^{\infty} \frac{(280n+19)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3(99)^{2n+1}} \tag{75}
$$

$$
\frac{1}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(26390n+1103)(\tfrac{1}{2})_n(\tfrac{1}{4})_n(\tfrac{3}{4})_n}{(n!)^3(99)^{4n+2}}. \tag{76}
$$

These equations were first proved by Borwein and Borwein (1987, pp. 177–187). Borwein and Borwein (1987b, 1988, 1993) proved other equations of this type, and Chudnovsky and Chudnovsky (1987) found similar equations for other transcendental constants.

A SPIGOT ALGORITHM for π is given by Rabinowitz and Wagon (1995). Amazingly, a closed form expression giving a *digit extraction* algorithm which produces digits of π (or π^2) in base-16 was recently discovered by Bailey *et al.* (Bailey *et al.* 1995, Adamchik and Wagon 1997),

$$
\pi = \sum_{n=0}^{\infty}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right)\left(\frac{1}{16}\right)^n, \tag{77}
$$

which can also be written using the shorthand notation

$$
\pi = \sum_{i=1}^{\infty} \frac{p_i}{16^{\lfloor i/8 \rfloor} i} \qquad \{p_i\} = \{\,\overline{4,0,0,-2,-1,-1,0,0}\,\}, \tag{78}
$$

where $\{p_i\}$ is given by the periodic sequence obtained by appending copies of $\{4,0,0,-2,-1,-1,0,0\}$ (in other words, $p_i \equiv p_{[(i-1) \pmod 8)]+1}$ for $i > 8$) and $\lfloor x \rfloor$ is the FLOOR FUNCTION. This expression was discovered using the PSLQ ALGORITHM and is equivalent to

$$
\pi = \int_0^1 \frac{16y - 16}{y^4 - 2y^3 + 4y - 4}\,dy. \tag{79}
$$

A similar formula was subsequently discovered by Ferguson, leading to a 2-D lattice of such formulas which can be generated by these two formulas. A related integral is

$$\pi = \tfrac{22}{7} - \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx \tag{80}$$

(Le Lionnais 1983, p. 22). F. Bellard found the more rapidly converging digit-extraction algorithm (in HEXADECIMAL)

$$\pi = \frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}-\frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}\right). \tag{81}$$

More amazingly still, S. Plouffe has devised an algorithm to compute the nth DIGIT of π in any base in $\mathcal{O}(n^3(\log n)^3)$ steps.

Another identity is

$$\pi^2 = 36\,\mathrm{Li}_2(\tfrac{1}{2}) - 36\,\mathrm{Li}_2(\tfrac{1}{4}) - 12\,\mathrm{Li}_2(\tfrac{1}{8}) + 6\,\mathrm{Li}_2(\tfrac{1}{64}), \tag{82}$$

where L_n is the POLYLOGARITHM. (82) is equivalent to

$$\frac{\pi^2}{36} = \sum_{i=1}^{\infty}\frac{a_i}{2^i i^2} \qquad \{a_i\} = [\overline{1,-3,-2,-3,1,0}] \tag{83}$$

and

$$\pi^2 = 12L_2(\tfrac{1}{2}) + 6(\ln 2)^2 \tag{84}$$

(Bailey *et al.* 1995). Furthermore

$$\pi^2 = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left[\frac{144}{(6k+1)^2}-\frac{216}{(6k+2)^2}-\frac{72}{(6k+3)^2}-\frac{54}{(6k+4)^2}+\frac{9}{(6k+5)^2}\right] \tag{85}$$

and

$$\pi^2 = \sum_{k=0}^{\infty}\frac{1}{16^k}\left[\frac{16}{(8k+1)^2}-\frac{16}{(8k+2)^2}-\frac{8}{(8k+3)^2}-\frac{16}{(8k+4)^2}-\frac{4}{(8k+5)^2}-\frac{4}{(8k+6)^2}+\frac{2}{(8k+7)^2}\right] \tag{86}$$

(Bailey *et al.* 1995, Bailey and Plouffe).

A slew of additional identities due to Ramanujan, Catalan, and Newton are given by Castellanos (1988, pp. 86–88), including several involving sums of FIBONACCI NUMBERS.

Gasper quotes the result

$$\pi = \frac{16}{3}\left[\lim_{x\to\infty} x\,{}_1F_2(\tfrac{1}{2};2,3;-x^2)\right]^{-1}, \tag{87}$$

where ${}_1F_2$ is a GENERALIZED HYPERGEOMETRIC FUNCTION, and transforms it to

$$\pi = \lim_{x\to\infty} 4x\,{}_1F_2(\tfrac{1}{2};\tfrac{3}{2},\tfrac{3}{2};-x^2). \tag{88}$$

Fascinating results due to Gosper include

$$\lim_{n\to\infty}\prod_{i=n}^{2n}\frac{\pi}{2\tan^{-1} i} = 4^{1/\pi} = 1.554682275\ldots \tag{89}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{9}{n\pi+\sqrt{n^2\pi^2-9}}\right) = -\frac{\pi^2}{12e^3} = -0.040948222\ldots. \tag{90}$$

Gosper also gives the curious identity

$$\frac{1}{e}\prod_{n=1}^{\infty}\left(\frac{1}{3n}+1\right)^{3n+1/2} = \frac{3\cdot 3^{1/24}\sqrt{(\frac{1}{3})!}}{2^{5/6}\exp\left[\frac{\gamma}{3}-\frac{\pi\sqrt{3}}{18}+\frac{\sqrt{3}\psi_1(\frac{1}{3})}{12\pi}-\frac{2\zeta'(2)}{\pi^2}-1\right]\pi^{5/6}} = 1.012378552722912\ldots. \tag{91}$$

Another curious fact is the ALMOST INTEGER

$$e^\pi - \pi = 19.999099979\ldots, \tag{92}$$

which can also be written as

$$(\pi+20)^i = -0.9999999992 - 0.0000388927i \approx -1 \tag{93}$$

$$\cos(\ln(\pi+20)) \approx -0.9999999992. \tag{94}$$

Applying COSINE a few more times gives

$$\cos(\pi\cos(\pi\cos(\ln(\pi+20)))) \approx -1 + 3.9321609261\times 10^{-35}. \tag{95}$$

π may also be computed using iterative ALGORITHMS. A quadratically converging ALGORITHM due to Borwein is

$$x_0 = \sqrt{2} \tag{96}$$

$$\pi_0 = 2+\sqrt{2} \tag{97}$$

$$y_1 = 2^{1/4} \tag{98}$$

and

$$x_{n+1} = \frac{1}{2}\left(\sqrt{x_n}+\frac{1}{\sqrt{x_n}}\right) \tag{99}$$

$$y_{n+1} = \frac{y_n\sqrt{x_n}+\frac{1}{\sqrt{x_n}}}{y_n+1} \tag{100}$$

$$\pi_n = \pi_{n-1}\frac{x_n+1}{y_n+1}. \tag{101}$$

π_n decreases monotonically to π with

$$\pi_n - \pi < 10^{-2^{n+1}} \tag{102}$$

for $n \geq 2$. The BRENT-SALAMIN FORMULA is another quadratically converging algorithm which can be used to calculate π. A quadratically convergent algorithm for $\pi/\ln 2$ based on an observation by Salamin is given by defining

$$f(k) = k2^{-k/4}\left[\sum_{n=1}^{\infty} 2^{-k\binom{n}{2}}\right]^2, \tag{103}$$

then writing

$$g_0 \equiv \frac{f(n)}{f(2n)}. \tag{104}$$

Now iterate

$$g_k = \sqrt{\frac{1}{2}\left(g_{k-1} + \frac{1}{g_{k+1}}\right)} \tag{105}$$

to obtain

$$\pi = 2(\ln 2) f(n) \prod_{k=1}^{\infty} g_k. \tag{106}$$

A cubically converging ALGORITHM which converges to the nearest multiple of π to f_0 is the simple iteration

$$f_n = f_{n-1} + \sin(f_{n-1}) \tag{107}$$

(Beeler *et al.* 1972). For example, applying to 23 gives the sequence

$$\{23, 22.1537796, 21.99186453, 21.99114858, \ldots\}, \tag{108}$$

which converges to $7\pi \approx 21.99114858$.

A quartically converging ALGORITHM is obtained by letting

$$y_0 = \sqrt{2} - 1 \tag{109}$$

$$\alpha = 6 - 4\sqrt{2}, \tag{110}$$

then defining

$$y_{n+1} = \frac{1 - (1 - y_n{}^4)^{1/4}}{1 + (1 - y_n{}^4)^{1/4}} \tag{111}$$

$$\alpha_{n+1} = (1 + y_{n+1})^4 \alpha_n - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}{}^2). \tag{112}$$

Then

$$\pi = \lim_{n\to\infty} \frac{1}{\alpha_n} \tag{113}$$

and α_n converges to $1/\pi$ quartically with

$$\alpha_n - \frac{1}{\pi} < 16 \cdot 4^n e^{-2\pi \cdot 4^n} \tag{114}$$

(Borwein and Borwein 1987, Bailey 1988, Borwein *et al.* 1989). This ALGORITHM rests on a MODULAR EQUATION identity of order 4.

A quintically converging ALGORITHM is obtained by letting

$$s_0 = 5(\sqrt{5} - 2) \tag{115}$$

$$\alpha_0 = \tfrac{1}{2}. \tag{116}$$

Then let

$$s_{n+1} = \frac{25}{(z + \frac{x}{z} + 1)^2 s_n}, \tag{117}$$

where

$$x = \frac{5}{s_n} - 1 \tag{118}$$

$$y = (x-1)^2 + 7 \tag{119}$$

$$z = \left[\tfrac{1}{2}x\left(y + \sqrt{y^2 - 4x^3}\right)\right]^{1/5}. \tag{120}$$

Finally, let

$$\alpha_{n+1} = s_n{}^2\alpha_n - 5^n\left[\tfrac{1}{2}(s_n{}^2 - 5) + \sqrt{s_n(s_n{}^2 - 2s_n + 5)}\right], \tag{121}$$

then

$$0 < \alpha_n - \frac{1}{\pi} < 16 \cdot 5^n e^{-\pi 5^n} \tag{122}$$

(Borwein *et al.* 1989). This ALGORITHM rests on a MODULAR EQUATION identity of order 5.

Another ALGORITHM is due to Woon (1995). Define $a(0) \equiv 1$ and

$$a(n) = \sqrt{1 + \left[\sum_{k=0}^{n-1} a(k)\right]^2}. \tag{123}$$

It can be proved by induction that

$$a(n) = \csc\left(\frac{\pi}{2^{n+1}}\right). \tag{124}$$

For $n = 0$, the identity holds. If it holds for $n \leq t$, then

$$a(t+1) = \sqrt{1 + \left[\sum_{k=0}^{t} \csc\left(\frac{\pi}{2^{k+1}}\right)\right]^2}, \tag{125}$$

but

$$\csc\left(\frac{\pi}{2^{k+1}}\right) = \cot\left(\frac{\pi}{2^{k+2}}\right) - \cot\left(\frac{\pi}{2^{k+1}}\right), \tag{126}$$

so

$$\sum_{k=0}^{t} \csc\left(\frac{\pi}{2^{k+1}}\right) = \cot\left(\frac{\pi}{2^{t+2}}\right). \tag{127}$$

Therefore,

$$a(t+1) = \csc\left(\frac{\pi}{2^{t+2}}\right), \tag{128}$$

so the identity holds for $n = t+1$ and, by induction, for all NONNEGATIVE n, and

$$\begin{aligned}\lim_{n\to\infty}\frac{2^{n+1}}{a(n)} &= \lim_{n\to\infty} 2^{n+1}\sin\left(\frac{\pi}{2^{n+1}}\right)\\ &= \lim_{n\to\infty} 2^{n+1}\frac{\pi}{2^{n+1}}\frac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{\frac{\pi}{2^{n+1}}}\\ &= \pi\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \pi. \end{aligned} \tag{129}$$

Other iterative ALGORITHMS are the ARCHIMEDES ALGORITHM, which was derived by Pfaff in 1800, and the BRENT-SALAMIN FORMULA. Borwein *et al.* (1989) discuss pth order iterative algorithms.

KOCHANSKY'S APPROXIMATION is the ROOT of

$$9x^4 - 240x^2 + 1492. \tag{130}$$

given by

$$\pi \approx \sqrt{\tfrac{40}{3} - \sqrt{12}} \approx 3.141533. \tag{131}$$

An approximation involving the GOLDEN MEAN is

$$\pi \approx \tfrac{6}{5}\phi^2 = \frac{6}{5}\left(\frac{\sqrt{5}+1}{2}\right)^2 = \tfrac{3}{5}(3+\sqrt{5}) = 3.14164\ldots. \tag{132}$$

Some approximations due to Ramanujan

$$\pi \approx \frac{19\sqrt{7}}{16} \tag{133}$$

$$\approx \tfrac{7}{3}(1+\tfrac{1}{5}\sqrt{3}) \tag{134}$$

$$\approx \tfrac{9}{5} + \sqrt{\tfrac{9}{5}} \tag{135}$$

$$\approx \left(9^2 + \frac{19^2}{22}\right)^{1/4} = \left(102 - \frac{2222}{22^2}\right)^{1/4} \tag{136}$$

$$\approx (97 + \tfrac{1}{2} - \tfrac{1}{11})^{1/4} = (97 + \tfrac{9}{22})^{1/4} \tag{137}$$

$$\approx \frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) \tag{138}$$

$$\approx \frac{355}{113}\left(1 - \frac{0.0003}{3533}\right) \tag{139}$$

$$\approx \frac{12}{\sqrt{130}}\ln\left[\frac{(3+\sqrt{13})(\sqrt{8}+\sqrt{10})}{2}\right] \tag{140}$$

$$\approx \frac{24}{\sqrt{142}}\ln\left[\frac{\sqrt{10+11\sqrt{2}}+\sqrt{10+7\sqrt{2}}}{2}\right] \tag{141}$$

$$\approx \frac{12}{\sqrt{190}}\ln[(3+\sqrt{10})(\sqrt{8}+\sqrt{10})] \tag{142}$$

$$\approx \frac{12}{\sqrt{310}}\ln\left[\tfrac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\left(5+2\sqrt{10} + \sqrt{61+20\sqrt{10}}\right)\right] \tag{143}$$

$$\approx \frac{4}{\sqrt{522}}\ln\left[\left(\frac{5+\sqrt{29}}{\sqrt{2}}\right)^3(5\sqrt{29}+11\sqrt{6}) \times \left(\sqrt{\frac{9+3\sqrt{6}}{4}}+\sqrt{\frac{5+3\sqrt{6}}{4}}\right)^6\right], \tag{144}$$

which are accurate to 3, 4, 4, 8, 8, 9, 14, 15, 15, 18, 23, 31 digits, respectively (Ramanujan 1913–1914; Hardy 1952, p. 70; Berndt 1994, pp. 48–49 and 88–89).

Castellanos (1988) gives a slew of curious formulas:

$$\pi \approx (2e^3 + e^8)^{1/7} \tag{145}$$

$$\approx \left(\frac{553}{311+1}\right)^2 \tag{146}$$

$$\approx (\tfrac{3}{14})^4(\tfrac{193}{5})^2 \tag{147}$$

$$\approx (\tfrac{296}{167})^2 \tag{148}$$

$$\approx \left(\frac{66^3+86^2}{55^3}\right)^2 \tag{149}$$

$$\approx 1.09999901 \cdot 1.19999911 \cdot 1.39999931 \cdot 1.69999961 \tag{150}$$

$$\approx \frac{47^3+20^3}{30^3} - 1 \tag{151}$$

$$\approx 2 + \sqrt{1 + (\tfrac{413}{750})^2} \tag{152}$$

$$\approx (\tfrac{77729}{254})^{1/5} \tag{153}$$

$$\approx \left(31 + \frac{62^2+14}{28^4}\right)^{1/3} \tag{154}$$

$$\approx \frac{1700^3+82^3-10^3-9^3-6^3-3^3}{69^5} \tag{155}$$

$$\approx \left(95 + \frac{93^4+34^4+17^4+88}{75^4}\right)^{1/4} \tag{156}$$

$$\approx \left(100 - \frac{2125^3+214^3+30^3+37^2}{82^5}\right)^{1/4}, \tag{157}$$

which are accurate to 3, 4, 4, 5, 6, 7, 7, 8, 9, 10, 11, 12, and 13 digits, respectively. An extremely accurate approximation due to Shanks (1982) is

$$\pi \approx \frac{6}{\sqrt{3502}}\ln(2u) + 7.37 \times 10^{-82}, \tag{158}$$

where u is the product of four simple quartic units. A sequence of approximations due to Plouffe includes

$$\pi \approx 43^{7/23} \tag{159}$$

$$\approx \frac{\ln 2198}{\sqrt{6}} \tag{160}$$

$$\approx (\tfrac{13}{4})^{1181/1216} \tag{161}$$

$$\approx \frac{689}{396\ln\left(\frac{689}{396}\right)} \tag{162}$$

$$\approx (\tfrac{2143}{22})^{1/4} \tag{163}$$

$$\approx \sqrt{\frac{9}{67}}\ln 5280 \tag{164}$$

$$\approx (\tfrac{63023}{30510})^{1/3} + \tfrac{1}{4} + \tfrac{1}{2}(\sqrt{5}+1) \tag{165}$$

$$\approx \tfrac{48}{23}\ln\left(\frac{60318}{13387}\right) \tag{166}$$

$$\approx (228 + \tfrac{16}{1329})^{1/41} + 2 \tag{167}$$

$$\approx \tfrac{125}{124} \ln\left(\frac{28102}{1277}\right) \tag{168}$$

$$\approx \tfrac{276694819753963}{226588}^{1/158} + 2 \tag{169}$$

$$\approx \frac{\ln 262537412640768744}{\sqrt{163}}, \tag{170}$$

which are accurate to 4, 5, 7, 7, 8, 9, 10, 11, 11, 11, 23, and 30 digits, respectively.

Ramanujan (1913–14) and Olds (1963) give geometric constructions for 355/113. Gardner (1966, pp. 92–93) gives a geometric construction for $3 + 16/113 = 3.1415929\ldots$. Dixon (1991) gives constructions for $6/5(1+\phi) = 3.141640\ldots$ and $\sqrt{4+(3-\tan(30^\circ))} = 3.141533\ldots$. Constructions for approximations of π are approximations to CIRCLE SQUARING (which is itself impossible).

A short mnemonic for remembering the first eight DECIMAL DIGITS of π is "May I have a large container of coffee?" giving 3.1415926 (Gardner 1959; Gardner 1966, p. 92; Eves 1990, p. 122, Davis 1993, p. 9). A more substantial mnemonic giving 15 digits (3.14159265358979) is "How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics," originally due to Sir James Jeans (Gardner 1966, p. 92; Castellanos 1988, p. 152; Eves 1990, p. 122; Davis 1993, p. 9; Blatner 1997, p. 112). A slight extension of this adds the phrase "All of thy geometry, Herr Planck, is fairly hard," giving 24 digits in all (3.14159265358979323846264).

An even more extensive rhyming mnemonic giving 31 digits is "Now I will a rhyme construct, By chosen words the young instruct. Cunningly devised endeavour, Con it and remember ever. Widths in circle here you see, Sketched out in strange obscurity." (Note that the British spelling of "endeavour" is required here.)

The following stanzas are the first part of a poem written by M. Keith based on Edgar Allen Poe's "The Raven." The entire poem gives 740 digits; the fragment below gives only the first 80 (Blatner 1997, p. 113). Words with ten letters represent the digit 0, and those with 11 or more digits are taken to represent two digits.

Poe, E.: Near a Raven.

Midnights so dreary, tired and weary.
Silently pondering volumes extolling all by-now obsolete lore.
During my rather long nap-the weirdest tap!
An ominous vibrating sound disturbing my chamber's antedoor.
'This,' I whispered quietly, 'I ignore.'

Perfectly, the intellect remembers: the ghostly fires, a glittering ember.
Inflamed by lightning's outbursts, windows cast penumbras upon this floor.
Sorrowful, as one mistreated, unhappy thoughts I heeded:
That inimitable lesson in elegance—Lenore—
is delighting, exciting... nevermore.

An extensive collection of π mnemonics in many languages is maintained by A. P. Hatzipolakis. Other mnemonics in various languages are given by Castellanos (1988) and Blatner (1997, pp. 112–118).

In the following, the word "digit" refers to decimal digit after the decimal point. J. H. Conway has shown that there is a sequence of fewer than 40 FRACTIONS F_1, F_2, ... with the property that if you start with 2^n and repeatedly multiply by the first of the F_i that gives an integral answer, then the next POWER of 2 to occur will be the 2^nth decimal digit of π.

The first occurrence of n 0s appear at digits 32, 307, 601, 13390, 17534, ... The sequence 9999998 occurs at decimal 762 (which is sometimes called the FEYNMAN POINT). This is the largest value of any seven digits in the first million decimals. The first time the BEAST NUMBER 666 appears is decimal 2440. The digits 314159 appear at least six times in the first 10 million decimal places of π (Pickover 1995). In the following, "digit" means digit of $\pi - 3$. The sequence 0123456789 occurs beginning at digits 17,387,594,880, 26,852,899,245, 30,243,957,439, 34,549,153,953, 41,952,536,161, and 43,289,964,000. The sequence 9876543210 occurs beginning at digits 21,981,157,633, 29,832,636,867, 39,232,573,648, 42,140,457,481, and 43,065,796,214. The sequence 27182818284 (the digits of e) occur beginning at digit 45,111,908,393. There are also interesting patterns for $1/\pi$. 0123456789 occurs at 6,214,876,462, 9876543210 occurs at 15,603,388,145 and 51,507,034,812, and 999999999999 occurs at 12,479,021,132 of $1/\pi$.

Scanning the decimal expansion of π until all n-digit numbers have occurred, the last 1-, 2-, ... digit numbers appearing are 0, 68, 483, 6716, 33394, 569540, ... (Sloane's A032510). These end at digits 32, 606, 8555, 99849, 1369564, 14118312, ...

see also ALMOST INTEGER, ARCHIMEDES ALGORITHM, BRENT-SALAMIN FORMULA, BUFFON-LAPLACE NEEDLE PROBLEM, BUFFON'S NEEDLE PROBLEM, CIRCLE, DIRICHLET BETA FUNCTION, DIRICHLET ETA FUNCTION, DIRICHLET LAMBDA FUNCTION, e, EULER-MASCHERONI CONSTANT, GAUSSIAN DISTRIBUTION, MACLAURIN SERIES, MACHIN'S FORMULA, MACHIN-LIKE FORMULAS, RELATIVELY PRIME, RIEMANN ZETA FUNCTION, SPHERE, TRIGONOMETRY

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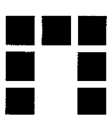
Pi Heptomino



A HEPTOMINO in the shape of the Greek character π.

Piano Mover's Problem

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Given an open subset U in n-D space and two compact subsets C_0 and C_1 of U, where C_1 is derived from C_0 by a continuous motion, is it possible to move C_0 to C_1 while remaining entirely inside U?

see also MOVING LADDER CONSTANT, MOVING SOFA CONSTANT

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Picard's Existence Theorem

If f is a continuous function that satisfies the LIPSCHITZ CONDITION

$$|f(x,t) - f(y,t)| \leq L|x - y|$$

in a surrounding of $(x_0, t_0) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n = \{(x,t) : |x - x_0| < b, |t - t_0| < a\}$, then the differential equation

$$\frac{df}{dx} = f(x,t)$$
$$x(t_0) = x_0$$

has a unique solution $x(t)$ in the interval $|t - t_0| < d$, where $d = \min(a, b/B)$, min denotes the MINIMUM, $B = \sup |f(t,x)|$, and sup denotes the SUPREMUM.

see also ORDINARY DIFFERENTIAL EQUATION

Picard's Little Theorem

Any ENTIRE ANALYTIC FUNCTION whose range omits two points must be a constant.

Picard's Theorem

An ANALYTIC FUNCTION assumes every COMPLEX NUMBER, with possibly one exception, infinitely often in any NEIGHBORHOOD of an ESSENTIAL SINGULARITY.

see also ANALYTIC FUNCTION, ESSENTIAL SINGULARITY, NEIGHBORHOOD

Picard Variety

Let V be a VARIETY, and write $G(V)$ for the set of divisors, $G_l(V)$ for the set of divisors linearly equivalent to 0, and $G_a(V)$ for the group of divisors algebraically equal to 0. Then $G_a(V)/G_l(V)$ is called the Picard variety. The ALBANESE VARIETY is dual to the Picard variety.

see also ALBANESE VARIETY

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Pick's Formula

see PICK'S THEOREM

Pick's Theorem

Let A be the AREA of a simply closed POLYGON whose VERTICES are lattice points. Let B denote the number of LATTICE POINTS on the EDGES and I the number of points in the interior of the POLYGON. Then

$$A = I + \tfrac{1}{2}B - 1.$$

The FORMULA has been generalized to 3-D and higher dimensions using EHRHART POLYNOMIALS.

see also BLICHFELDT'S THEOREM, EHRHART POLYNOMIAL, LATTICE POINT, MINKOWSKI CONVEX BODY THEOREM

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Picone's Theorem

Let $f(x)$ be integrable in $[-1,1]$, let $(1-x^2)f(x)$ be of bounded variation in $[-1,1]$, let M' denote the least upper bound of $|f(x)(1-x^2)|$ in $[-1,1]$, and let V' denote the total variation of $f(x)(1-x^2)$ in $[-1,1]$. Given the function

$$F(x) = F(-1) + \int_1^x f(x)\,dx,$$

then the terms of its LEGENDRE SERIES

$$F(x) \sim \sum_{n=0}^{\infty} a_n P_n(x)$$

$$a_n = \tfrac{1}{2}(2n+1)\int_{-1}^{1} F(x)P_n(x)\,dx,$$

where $P_n(x)$ is a LEGENDRE POLYNOMIAL, satisfy the inequalities

$$|a_n P_n(x)| < \begin{cases} 8\sqrt{\frac{2}{\pi}}\frac{M'+V'}{(1-\delta^2)^{1/4}}n^{-3/2} & \text{for } |x| \leq \delta < 1 \\ 2(M'+V')n^{-1} & \text{for } |x| \leq 1 \end{cases}$$

for $n \geq 1$ (Sansone 1991).

see also JACKSON'S THEOREM, LEGENDRE SERIES

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Pie Cutting

see CIRCLE CUTTING, CYLINDER CUTTING, PANCAKE THEOREM, PIZZA THEOREM

Piecewise Circular Curve

A curve composed exclusively of circular ARCS, e.g., the FLOWER OF LIFE, LENS, REULEAUX TRIANGLE, SEED OF LIFE, and YIN-YANG.

see also ARC, REULEAUX TRIANGLE, YIN-YANG FLOWER OF LIFE, LENS, REULEAUX POLYGON, REULEAUX TRIANGLE, SALINON, SEED OF LIFE, TRIANGLE ARCS, YIN-YANG

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Pigeonhole Principle

see DIRICHLET'S BOX PRINCIPLE

Pillai's Conjecture

For every $k > 1$, there exist only finite many pairs of POWERS (p, p') with p and p' PRIME and $k = p' - p$.

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Pilot Vector

see VECTOR SPHERICAL HARMONIC

Pinch Point

A singular point such that every NEIGHBORHOOD of the point intersects itself. Pinch points are also called Whitney singularities or branch points.

Pinching Theorem

Let $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a. If

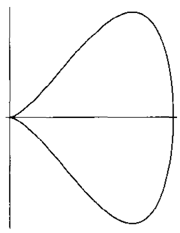
$$\lim_{\Delta x \to a} g(x) = \lim_{\Delta x \to a} h(x) = L,$$

then $\lim_{\Delta x \to a} f(x) = L$.

Pine Cone Number

see FIBONACCI NUMBER

Piriform



A plane curve also called the PEG TOP and given by the CARTESIAN equation

$$a^4y^2 = b^2x^3(2a - x) \tag{1}$$

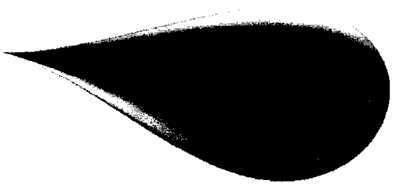
and the parametric curves

$$x = a(1 + \sin t) \tag{2}$$
$$y = b\cos t(1 + \sin t) \tag{3}$$

for $t \in [-\pi/2, \pi/2]$. It was studied by G. de Longchamps in 1886. The generalization to a QUARTIC 3-D surface

$$(x^4 - x^3) + y^2 + z^2 = 0, \tag{4}$$

is shown below (Nordstrand).



see also BUTTERFLY CURVE, DUMBBELL CURVE, EIGHT CURVE, HEART SURFACE, PEAR CURVE

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Pisot-Vijayaraghavan Constants

Let θ be a number greater than 1, λ a POSITIVE number, and

$$(x) \equiv x - \lfloor x \rfloor \tag{1}$$

denote the fractional part of x. Then for a given λ, the sequence of numbers $(\lambda\theta^n)$ for $n = 1, 2, \ldots$ is uniformly distributed in the interval (0, 1) when θ does not belong to a λ-dependent exceptional set S of MEASURE zero (Koksma 1935). Pisot (1938) and Vijayaraghavan (1941) independently studied the exceptional values of θ, and Salem (1943) proposed calling such values Pisot-Vijayaraghavan numbers.

Pisot (1938) proved that if θ is such that there exists a $\lambda \neq 0$ such that the series $\sum_{n=0}^{\infty} \sin^2(\pi\lambda\theta)^n$ converges, then θ is an ALGEBRAIC INTEGER whose conjugates all (except for itself) have modulus < 1, and λ is an algebraic INTEGER of the FIELD $K(\theta)$. Vijayaraghavan (1940) proved that the set of Pisot-Vijayaraghavan numbers has infinitely many limit points. Salem (1944) proved that the set of Pisot-Vijayaraghavan constants is closed. The proof of this theorem is based on the LEMMA that for a Pisot-Vijayaraghavan constant θ, there always exists a number λ such that $1 \leq \lambda < \theta$ and the following inequality is satisfied,

$$\sum_{n=0}^{\infty} \sin^2(\pi\lambda\theta^n) \leq \frac{\pi^2(2\theta+1)^2}{(\theta-1)^2}. \tag{2}$$

The smallest Pisot-Vijayaraghavan constant is given by the POSITIVE ROOT θ_0 of

$$x^3 - x - 1 = 0. \tag{3}$$

This number was identified as the smallest known by Salem (1944), and proved to be the smallest possible by Siegel (1944). Siegel also identified the next smallest Pisot-Vijayaraghavan constant θ_1 as the root of

$$x^4 - x^3 - 1 = 0, \tag{4}$$

showed that θ_1 and θ_2 are isolated in S, and showed that the roots of each POLYNOMIAL

$$x^n(x^2 - x - 1) + x^2 - 1 \qquad n = 1, 2, 3, \ldots \tag{5}$$

$$x^n - \frac{x^{n+1} - 1}{x^2 - 1} \qquad n = 3, 5, 7, \ldots \tag{6}$$

$$x^n - \frac{x^{n-1} - 1}{x - 1} \qquad n = 3, 5, 7, \ldots \tag{7}$$

belong to S, where $\theta_0 = \phi$ (the GOLDEN MEAN) is the accumulation point of the set (in fact, the smallest; Le Lionnais 1983, p. 40). Some small Pisot-Vijayaraghavan constants and their POLYNOMIALS are given in the following table. The latter two entries are from Boyd (1977).

k	Number	Order	Polynomial
0	1.3247179572	3	1 0 −1 −1
1	1.3802775691	4	1 −1 0 0 −1
	1.6216584885	16	1 −2 2 −3 2 −2 1 0 0 1 −1 2 −2 2 −2 1 −1
	1.8374664495	20	1 −2 0 1 −1 0 1 −1 0 1 0 −1 0 1 −1 0 1 −1 0 1 −1

All the points in S less than ϕ are known (Dufresnoy and Pisot 1955). Each point of S is a limit point from both sides of the set T of SALEM CONSTANTS (Salem 1945).

see also SALEM CONSTANTS

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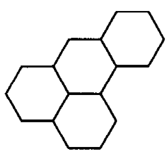
Pistol



A 4-POLYHEX.

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Pitchfork Bifurcation

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a one-parameter family of C^3 map satisfying

$$f(-x, \mu) = -f(x, \mu) \quad (1)$$

$$\left[\frac{\partial f}{\partial x}\right]_{\mu=0,x=0} = 1 \quad (2)$$

$$\left[\frac{\partial f}{\partial x}\right]_{\mu,x} = \left[\frac{\partial f}{\partial x}\right]_{\mu=0,x=\mu} \quad (3)$$

$$\left[\frac{\partial^2 f}{\partial x \partial \mu}\right]_{0,0} > 0 \quad (4)$$

$$\left[\frac{\partial^3 f}{\partial \mu^3}\right]_{\mu=0,x=0} < 0. \quad (5)$$

Then there are intervals having a single stable fixed point and three fixed points (two of which are stable and one of which is unstable). This BIFURCATION is called a pitchfork bifurcation. An example of an equation displaying a pitchfork bifurcation is

$$\dot{x} = \mu x - x^3 \quad (6)$$

(Guckenheimer and Holmes 1997, p. 145).

see also BIFURCATION

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Pivot Theorem

If the VERTICES A, B, and C of TRIANGLE ΔABC lie on sides QR, RP, and PQ of the TRIANGLE ΔPQR, then the three CIRCLES CBP, ACQ, and BAR have a common point. In extended form, it is MIQUEL'S THEOREM.

see also MIQUEL'S THEOREM

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Pivoting

The element in the diagonal of a matrix by which other elements are divided in an algorithm such as GAUSS-JORDAN ELIMINATION is called the pivot element. Partial pivoting is the interchanging of rows and full pivoting is the interchanging of both rows and columns in order to place a particularly "good" element in the diagonal position prior to a particular operation.

see also GAUSS-JORDAN ELIMINATION

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Pizza Theorem

If a circular pizza is divided into 8, 12, 16, ... slices by making cuts at equal angles from an arbitrary point, then the sums of the areas of alternate slices are equal.

Place (Digit)

see DIGIT

Place (Field)

A place ν of a number FIELD k is an ISOMORPHISM class of field maps k onto a dense subfield of a nondiscrete locally compact FIELD k_ν.

In the function field case, let F be a function field of algebraic functions of one variable over a FIELD K. Then by a place in F, we mean a subset p of F which is the IDEAL of nonunits of some VALUATION RING O over K.

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Place (Game)

For n players, $n-1$ games are needed to fairly determine first place, and $n-1+\lg(n-1)$ are needed to fairly determine first and second place.

Planar Bubble Problem

see BUBBLE

Planar Distance

For n points in the PLANE, there are at least

$$N_1 = \sqrt{n - \tfrac{3}{4}} - \tfrac{1}{2}$$

different DISTANCES. The minimum DISTANCE can occur only $\leq 3n-6$ times, and the MAXIMUM DISTANCE can occur $\leq n$ times. Furthermore, no DISTANCE can occur as often as

$$N_2 = \tfrac{1}{4}n(1+\sqrt{8n-7}) < \frac{n^{3/2}}{\sqrt{2}} - \frac{n}{4}$$

times. No set of $n > 6$ points in the PLANE can determine only ISOSCELES TRIANGLES.

see also DISTANCE

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Planar Graph

A GRAPH is planar if it can be drawn in a PLANE without EDGES crossing (i.e., it has CROSSING NUMBER 0). Only planar graphs have DUALS. If G is planar, then G has VERTEX DEGREE ≤ 5. COMPLETE GRAPHS are planar only for $n \leq 4$. The complete BIPARTITE GRAPH $K(3,3)$ in nonplanar. More generally, Kuratowski proved in 1930 that a graph is planar IFF it does not contain within it any graph which can be CONTRACTED to the pentagonal graph $K(5)$ or the hexagonal graph $K(3,3)$. K_5 can be decomposed into a union of two planar graphs, giving it a "DEPTH" of $E(K_5)=2$. Simple CRITERIA for determining the depth of graphs are not known. Beineke and Harary (1964, 1965) have shown that if $n \not\equiv 4 \pmod 6$, then

$$E(K_n) = \left\lfloor \tfrac{1}{6}(n+7) \right\rfloor.$$

The DEPTHS of the graphs K_n for $n=4$, 10, 22, 28, 34, and 40 are 1, 3, 4, 5, 6, and 7 (Meyer 1970).

see also COMPLETE GRAPH, FABRY IMBEDDING, INTEGRAL DRAWING, PLANAR STRAIGHT LINE GRAPH

References
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Planar Point

A point $\mathbf{p}$ on a REGULAR SURFACE $M \in \mathbb{R}^3$ is said to be planar if the GAUSSIAN CURVATURE $K(\mathbf{p})=0$ and $S(\mathbf{p})=0$ (where S is the SHAPE OPERATOR), or equivalently, both of the PRINCIPAL CURVATURES κ_1 and κ_2 are 0.

see also ANTICLASTIC, ELLIPTIC POINT, GAUSSIAN CURVATURE, HYPERBOLIC POINT, PARABOLIC POINT, SYNCLASTIC

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Planar Space

Let (ξ_1, ξ_2) be a locally EUCLIDEAN coordinate system. Then

$$ds^2 = d\xi_1{}^2 + d\xi_2{}^2. \tag{1}$$

Now plug in

$$d\xi_1 = \frac{\partial\xi_1}{\partial x_1}dx_1 + \frac{\partial\xi_1}{\partial x_2}dx_2 \tag{2}$$

$$d\xi_2 = \frac{\partial\xi_2}{\partial x_1}dx_1 + \frac{\partial\xi_2}{\partial x_2}dx_2 \tag{3}$$

to obtain

$$\begin{aligned} ds^2 = {} & \left[\left(\frac{\partial\xi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\xi_2}{\partial x_1}\right)^2\right] dx_1{}^2 \\ & + 2\left[\frac{\partial\xi_1}{\partial x_1}\frac{\partial\xi_1}{\partial x_2} + \frac{\partial\xi_2}{\partial x_1}\frac{\partial\xi_2}{\partial x_2}\right] dx_1\,dx_2 \\ & + \left[\left(\frac{\partial\xi_1}{\partial x_2}\right)^2 + \left(\frac{\partial\xi_2}{\partial x_2}\right)^2\right] dx_2{}^2. \end{aligned} \tag{4}$$

Reading off the COEFFICIENTS from

$$ds^2 = g_{11}\,dx_1{}^2 + 2g_{12}\,dx_1\,dx_2 + g_{22}\,(dx_2)^2 \tag{5}$$

gives

$$g_{11} = \left(\frac{\partial\xi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\xi_2}{\partial x_1}\right)^2 \tag{6}$$

$$g_{12} = \frac{\partial\xi_1}{\partial x_1}\frac{\partial\xi_1}{\partial x_2} + \frac{\partial\xi_2}{\partial x_1}\frac{\partial\xi_2}{\partial x_2} \tag{7}$$

$$g_{22} = \left(\frac{\partial\xi_1}{\partial x_2}\right)^2 + \left(\frac{\partial\xi_2}{\partial x_2}\right)^2. \tag{8}$$

Making a change of coordinates $(x_1, x_2) \to (x'_1, x'_2)$ gives

$$\begin{aligned} g'_{11} &= \left(\frac{\partial \xi_1}{\partial x'_1}\right)^2 + \left(\frac{\partial \xi_2}{\partial x'_1}\right)^2 \\ &= \left(\frac{\partial \xi_1}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial \xi_1}{\partial x_2}\frac{\partial x_2}{\partial x'_1}\right)^2 \\ &\quad + \left(\frac{\partial \xi_2}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial \xi_2}{\partial x_2}\frac{\partial x_2}{\partial x'_1}\right)^2 \\ &= g_{11}\left(\frac{\partial x_1}{\partial x'_1}\right)^2 + 2g_{12}\frac{\partial x_1}{\partial x'_1}\frac{\partial x_2}{\partial x'_1} + g_{22}\left(\frac{\partial x_2}{\partial x'_1}\right)^2 \quad (9) \\ g'_{12} &= \frac{\partial \xi_1}{\partial x_1}\frac{\partial x_1}{\partial x'_1}\frac{\partial \xi_1}{\partial x_2}\frac{\partial x_2}{\partial x'_2} + \frac{\partial \xi_2}{\partial x_1}\frac{\partial x_1}{\partial x'_1}\frac{\partial \xi_2}{\partial x_2}\frac{\partial x_2}{\partial x'_2} \\ &= g_{12}\frac{\partial x_1}{\partial x'_1}\frac{\partial x_2}{\partial x'_2} \quad (10) \\ g'_{22} &= g_{11}\left(\frac{\partial x_1}{\partial x'_1}\right)^2 + 2g_{12}\frac{\partial x_1}{\partial x'_2}\frac{\partial x_2}{\partial x'_2} + g_{22}\left(\frac{\partial x_2}{\partial x'_2}\right)^2. \quad (11) \end{aligned}$$

Planar Straight Line Graph

A PLANAR GRAPH in which only straight line segments are used to connect the VERTICES, where the EDGES may intersect.

see also PLANAR GRAPH

Plancherel's Theorem

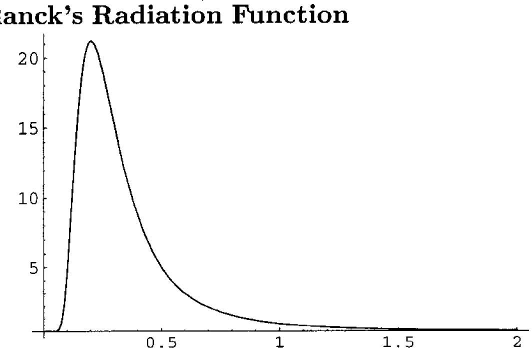
$$\int_{-\infty}^{\infty} f(x)g^*(x)\,dx = \int_{-\infty}^{\infty} F(s)G^*(s)\,ds,$$

where $F(s) \equiv \mathcal{F}[f(x)]$ and $\mathcal{F}$ denotes a FOURIER TRANSFORM. If f and g are real

$$\int_{-\infty}^{\infty} f(x)g(-x)\,dx = \int_{-\infty}^{\infty} F(s)G(s)\,ds.$$

see also FOURIER TRANSFORM, PARSEVAL'S THEOREM

Planck's Radiation Function



The function

$$f(x) = \frac{1}{x^5(e^{1/x} - 1)}.$$

It has a MAXIMUM at $x \approx 0.201405$, where

$$f'(x) = \frac{5x - e^{1/x}(5x-1)}{x^7(e^{1/x}-1)^2} = 0,$$

and inflection points at $x \approx 0.11842$ and $x \approx 0.283757$, where

$$f''(x) = \frac{e^{1/x}(1+e^{1/x}) + 6x(e^{1/x}-1)[e^{1/x}(5x-2) - 5x]}{(e^{1/x}-1)^3 x^9} = 0.$$

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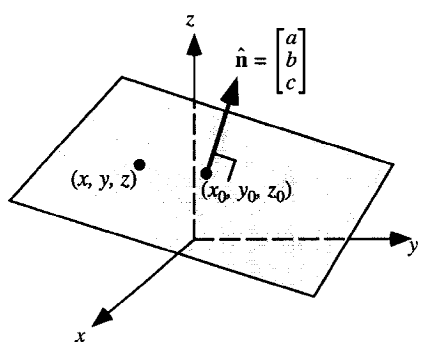
Abramowitz, M. and Stegun, C. A. (Eds.). "Planck's Radiation Function." §27.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 999, 1972.

Plane

A plane is a 2-D SURFACE spanned by two linearly independent vectors. The generalization of the plane to higher DIMENSIONS is called a HYPERPLANE.

In intercept form, a plane passing through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is given by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad (1)$$



The equation of a plane PERPENDICULAR to the NONZERO VECTOR $\hat{\mathbf{n}} = (a, b, c)$ through the point (x_0, y_0, z_0) is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \quad (2)$$

so

$$ax + by + cz + d = 0, \quad (3)$$

where

$$d \equiv -ax_0 - by_0 - cz_0. \quad (4)$$

A plane specified in this form therefore has x-, y-, and z-intercepts at

$$x = -\frac{d}{a} \tag{5}$$

$$y = -\frac{d}{b} \tag{6}$$

$$z = -\frac{d}{c}, \tag{7}$$

and lies at a DISTANCE

$$h = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}} \tag{8}$$

from the ORIGIN.

The plane through P_1 and parallel to (a_1, b_1, c_1) and (a_2, b_2, c_2) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0. \tag{9}$$

The plane through points P_1 and P_2 parallel to direction (a, b, c) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0. \tag{10}$$

The three-point form is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0. \tag{11}$$

The DISTANCE from a point (x_1, y_1, z_1) to a plane

$$Ax + By + Cz + D = 0 \tag{12}$$

is

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}. \tag{13}$$

The DIHEDRAL ANGLE between the planes

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \tag{14}$$

$$A_2 x + B_2 y + C_2 z + D_2 = 0 \tag{15}$$

is

$$\cos\theta = \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{{A_1}^2 + {B_1}^2 + {C_1}^2}\sqrt{{A_2}^2 + {B_2}^2 + {C_2}^2}}. \tag{16}$$

In order to specify the relative distances of $n > 1$ points in the plane, $1 + 2(n-2) = 2n - 3$ coordinates are needed, since the first can always be placed at (0, 0) and the second at $(x, 0)$, where it defines the x-AXIS. The remaining $n - 2$ points need two coordinates each. However, the total number of distances is

$$_nC_2 = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \tfrac{1}{2}n(n-1), \tag{17}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, so the distances between points are subject to m relationships, where

$$m \equiv \tfrac{1}{2}n(n-1) - (2n-3) = \tfrac{1}{2}(n-2)(n-3). \tag{18}$$

For $n = 2$ and $n = 3$, there are no relationships. However, for a QUADRILATERAL (with $n = 4$), there is one (Weinberg 1972).

It is impossible to pick random variables which are uniformly distributed in the plane (Eisenberg and Sullivan 1996). In 4-D, it is possible for four planes to intersect in exactly one point. For every set of n points in the plane, there exists a point O in the plane having the property such that *every* straight line through O has at least 1/3 of the points on each side of it (Honsberger 1985).

Every RIGID motion of the plane is one of the following types (Singer 1995):

1. ROTATION about a fixed point P.
2. TRANSLATION in the direction of a line l.
3. REFLECTION across a line l.
4. Glide-reflections along a line l.

Every RIGID motion of the hyperbolic plane is one of the previous types or a

5. Horocycle rotation.

see also ARGAND PLANE, COMPLEX PLANE, DIHEDRAL ANGLE, ELLIPTIC PLANE, FANO PLANE, HYPERPLANE, MOUFANG PLANE, NIRENBERG'S CONJECTURE, NORMAL SECTION, POINT-PLANE DISTANCE, PROJECTIVE PLANE

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Plane Curve

see CURVE

Plane Cutting

see CIRCLE CUTTING

Plane Division

Consider n intersecting CIRCLES and ELLIPSES. The maximal number of regions in which these divide the PLANE are

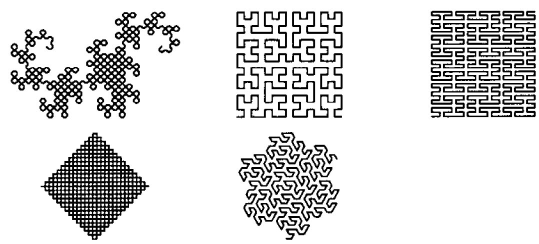
$$N_{\text{circle}} = n^2 - n + 2$$
$$N_{\text{ellipse}} = 2n^2 - 2n + 2.$$

see also ARRANGEMENT, CIRCLE, CUTTING, ELLIPSE, SPACE DIVISION

Plane-Filling Curve

see PLANE-FILLING FUNCTION

Plane-Filling Function



A SPACE-FILLING FUNCTION which maps a 1-D INTERVAL into a 2-D area. Plane-filling functions were thought to be impossible until Hilbert discovered the HILBERT CURVE in 1891.

Plane-filling functions are often (imprecisely) defined to be the "limit" of an infinite sequence of specified curves which "fill" the PLANE without "HOLES," hence the more popular term PLANE-FILLING CURVE. The term "plane-filling function" is preferable to "PLANE-FILLING CURVE" because "curve" informally connotes "GRAPH" (i.e., range) of some continuous function, but the GRAPH of a plane-filling function is a solid patch of 2-space with no evidence of the order in which it was traced (and, for a dense set, retraced). Actually, all that is needed to rigorously define a plane-filling function is an arbitrarily refinable correspondence between contiguous subintervals of the domain and contiguous subareas of the range.

True plane-filling functions are not ONE-TO-ONE. In fact, because they map closed intervals onto closed areas, they cannot help but overfill, revisiting at least twice a dense subset of the filled area. Thus, every point in the filled area has *at least* one inverse image.

see also HILBERT CURVE, PEANO CURVE, PEANO-GOSPER CURVE, SIERPIŃSKI CURVE, SPACE-FILLING FUNCTION, SPACE-FILLING POLYHEDRON

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Plane Geometry

That portion of GEOMETRY dealing with figures in a PLANE, as opposed to SOLID GEOMETRY. Plane geometry deals with the CIRCLE, LINE, POLYGON, etc.

see also CONSTRUCTIBLE POLYGON, GEOMETRIC CONSTRUCTION, GEOMETRY, SOLID GEOMETRY, SPHERICAL GEOMETRY

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Plane Partition

A two-dimensional array of INTEGERS nonincreasing both left to right and top to bottom which add up to a given number, i.e., $n_{ij} \geq n_{i(j+1)}$ and $n_{ij} \geq n_{(i+1)j}$. For example, a planar partition of 2 is given by

$$\begin{matrix} 5 & 4 & 2 & 1 & 1 \\ 3 & 2 & & & \\ 2 & 2. & & & \end{matrix}$$

The GENERATING FUNCTION for the number $\mathrm{PL}(n)$ of planar partitions of n is

$$\sum_{n=0}^{\infty} \mathrm{PL}(n) x^n = \frac{1}{\prod_{k=1}^{\infty} (1 - x^k)^k}$$
$$= 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + \ldots$$

(Sloane's A000219, MacMahon 1912b, Beeler *et al.* 1972, Bender and Knuth 1972). The concept of planar partitions can also be generalized to cubic partitions.

see also PARTITION, SOLID PARTITION

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Plane Symmetry Groups

see WALLPAPER GROUPS

Planted Planar Tree

A planted plane tree (V, E, v, α) is defined as a vertex set V, edges set E, ROOT v, and order relation α on V which satisfies

1. For $x, y \in V$ if $\rho(x) < \rho(y)$, then $x \,\alpha\, y$, where $\rho(x)$ is the length of the path from v to x,
2. If $\{r, s\}, \{x, y\} \in E$, $\rho(r) = \rho(x) = \rho(s) - 1 = \rho(y) - 1$ and $r \,\alpha\, x$, then $s \,\alpha\, y$

(Klarner 1969, Chorneyko and Mohanty 1975). The CATALAN NUMBERS give the number of planar trivalent planted trees.

see also CATALAN NUMBER, TREE

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Plastic Constant

The limiting ratio of the successive terms of the PADOVAN SEQUENCE, $P = 1.32471795\ldots$.

see also PADOVAN SEQUENCE

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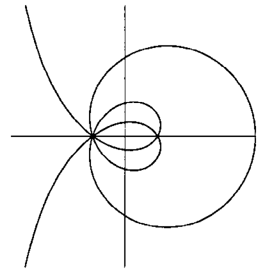
Plat

A BRAID in which strands are intertwined in the center and are free in "handles" on either side of the diagram.

Plateau Curves



A curve studied by the Belgian physicist and mathematician Joseph Plateau. It has Cartesian equation

$$x = \frac{a \sin[(m+n)t]}{\sin[(m-n)t]}$$
$$y = \frac{2a \sin(mt) \sin(nt)}{\sin[(m-n)t]}.$$

If $m = 2n$, the Plateau curve degenerates to a CIRCLE with center $(1, 0)$ and radius 2.

References

MacTutor History of Mathematics Archive. "Plateau Curves." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Plateau.html`.

Plateau's Laws

BUBBLES can meet only at ANGLES of 120° (for two BUBBLES) and 109.5° (for three BUBBLES), where the exact value of 109.5° is the TETRAHEDRAL ANGLE. This was proved by Jean Taylor using MEASURE THEORY to study AREA minimization. The DOUBLE BUBBLE is AREA minimizing, but it is not known the triple BUBBLE is also AREA minimizing. It is also unknown if empty chambers trapped inside can minimize AREA for $n \geq 3$ BUBBLES.

see also BUBBLE, CALCULUS OF VARIATIONS, DOUBLE BUBBLE, PLATEAU'S PROBLEM

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Taylor, J. E. "The Structure of Singularities in Soap-Bubble-Like and Soap-Film-Like Minimal Surfaces." *Ann. Math.* **103**, 489–539, 1976.

Plateau's Problem

The problem in CALCULUS OF VARIATIONS to find the MINIMAL SURFACE of a boundary with specified constraints. In general, there may be one, multiple, or no MINIMAL SURFACES spanning a given closed curve in space.

see also CALCULUS OF VARIATIONS, MINIMAL SURFACE

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Plato's Number

A number appearing in *The Republic* which involves 216 and 12,960,000.

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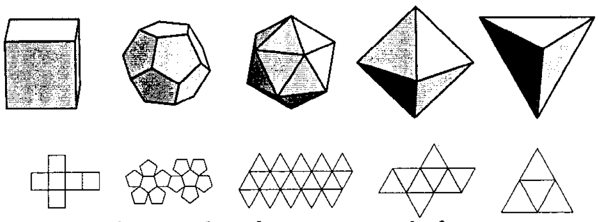
Platonic Solid



A solid with equivalent faces composed of congruent regular convex POLYGONS. There are exactly five such solids: the CUBE, DODECAHEDRON, ICOSAHEDRON, OCTAHEDRON, and TETRAHEDRON, as was proved by Euclid in the last proposition of the *Elements.*

The Platonic solids were known to the ancient Greeks, and were described by Plato in his *Timaeus* ca. 350 BC. In this work, Plato equated the TETRAHEDRON with the "element" fire, the CUBE with earth, the ICOSAHEDRON with water, the OCTAHEDRON with air, and the DODECAHEDRON with the stuff of which the constellations and heavens were made (Cromwell 1997).

The Platonic solids are sometimes also known as the REGULAR POLYHEDRA of COSMIC FIGURES (Cromwell 1997), although the former term is sometimes used to refer collectively to both the Platonic solids *and* KEPLER-POINSOT SOLIDS (Coxeter 1973).

If P is a POLYHEDRON with congruent (convex) regular polygonal faces, then Cromwell (1997, pp. 77–78) shows that the following statements are equivalent.

1. The vertices of P all lie on a SPHERE.
2. All the DIHEDRAL ANGLES are equal.
3. All the VERTEX FIGURES are REGULAR POLYGONS.
4. All the SOLID ANGLES are equivalent.
5. All the vertices are surrounded by the same number of FACES.

Let v (sometimes denoted N_0) be the number of VERTICES, e (or N_1) the number of EDGES, and f (or N_2) the number of FACES. The following table gives the SCHLÄFLI SYMBOL, WYTHOFF SYMBOL, and C&R symbol, the number of vertices v, edges e, and faces f, and the POINT GROUPS for the Platonic solids (Wenninger 1989).

Solid	Schläfli	Wyth.	C&R	v	e	f	Grp
cube	$\{4,3\}$	$3 \mid 2\ 4$	4^3	8	12	6	O_h
dodecahedron	$\{5,3\}$	$3 \mid 2\ 5$	5^3	20	30	12	I_h
icosahedron	$\{3,5\}$	$5 \mid 2\ 3$	3^5	12	30	20	I_h
octahedron	$\{3,4\}$	$4 \mid 2\ 3$	3^4	6	12	8	O_h
tetrahedron	$\{3,3\}$	$3 \mid 2\ 3$	3^3	4	6	4	T_d

Let r be the INRADIUS, ρ the MIDRADIUS, and R the CIRCUMRADIUS. The following two tables give the analytic and numerical values of these distances for Platonic solids with unit side length.

Solid	r	ρ	R
cube	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
dodecahedron	$\frac{1}{20}\sqrt{250+110\sqrt{5}}$	$\frac{1}{4}(3+\sqrt{5})$	$\frac{1}{4}(\sqrt{15}+\sqrt{3})$
icosahedron	$\frac{1}{12}(3\sqrt{3}+\sqrt{15})$	$\frac{1}{4}(1+\sqrt{5})$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$
octahedron	$\frac{1}{6}\sqrt{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$
tetrahedron	$\frac{1}{12}\sqrt{6}$	$\frac{1}{4}\sqrt{2}$	$\frac{1}{4}\sqrt{6}$

Solid	r	ρ	R
cube	0.5	0.70711	0.86603
dodecahedron	1.11352	1.30902	1.40126
icosahedron	0.75576	0.80902	0.95106
octahedron	0.40825	0.5	0.70711
tetrahedron	0.20412	0.35355	0.61237

Finally, let A be the AREA of a single FACE, V be the VOLUME of the solid, the EDGES be of unit length on a side, and α be the DIHEDRAL ANGLE. The following table summarizes these quantities for the Platonic solids.

Solid	A	V	α
cube	1	1	$\frac{1}{2}\pi$
dodecahedron	$\frac{1}{4}\sqrt{25+10\sqrt{5}}$	$\frac{1}{4}(15+7\sqrt{5})$	$\cos^{-1}(-\frac{1}{5}\sqrt{5})$
icosahedron	$\frac{1}{4}\sqrt{3}$	$\frac{5}{12}(3+\sqrt{5})$	$\cos^{-1}(-\frac{1}{3}\sqrt{5})$
octahedron	$\frac{1}{4}\sqrt{3}$	$\frac{1}{3}\sqrt{2}$	$\cos^{-1}(-\frac{1}{3})$
tetrahedron	$\frac{1}{4}\sqrt{3}$	$\frac{1}{12}\sqrt{2}$	$\cos^{-1}(\frac{1}{3})$

The number of EDGES meeting at a VERTEX is $2e/v$. The SCHLÄFLI SYMBOL can be used to specify a Platonic solid. For the solid whose faces are p-gons (denoted $\{p\}$), with q touching at each VERTEX, the symbol is $\{p,q\}$. Given p and q, the number of VERTICES, EDGES, and faces are given by

$$N_0 = \frac{4p}{4-(p-2)(q-2)}$$

$$N_1 = \frac{2pq}{4-(p-2)(q-2)}$$

$$N_2 = \frac{4q}{4-(p-2)(q-2)}.$$

MINIMAL SURFACES for Platonic solid frames are illustrated in Isenberg (1992, pp. 82–83).

see also ARCHIMEDEAN SOLID, CATALAN SOLID, JOHNSON SOLID, KEPLER-POINSOT SOLID, QUASIREGULAR POLYHEDRON, UNIFORM POLYHEDRON

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Platykurtic

A distribution with FISHER KURTOSIS $\gamma_2 < 0$ (and therefore having a flattened shape).

see also FISHER KURTOSIS

Playfair's Axiom

Through any point in space, there is exactly one straight line PARALLEL to a given straight line. This AXIOM is equivalent to the PARALLEL AXIOM.

see also PARALLEL AXIOM

References

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Plethysm

A group theoretic operation which is useful in the study of complex atomic spectra. A plethysm takes a set of functions of a given symmetry type $\{\mu\}$ and forms from them symmetrized products of a given degree r and other symmetry type $\{\nu\}$. A plethysm

$$\{\mu\}\otimes\{\nu\}=\sum\{\lambda\}$$

satisfies the rules

$$A\otimes(BC)=(A\otimes B)(A\otimes C)=A\otimes BA\otimes C,$$

$$A\otimes(B\pm C)=A\otimes B\pm A\otimes C$$

$$(A\otimes B)\otimes C=A\otimes(B\otimes C)$$

$$(A+B)\otimes\{\lambda\}=\sum\Gamma_{\mu\nu\lambda}(A\otimes\{\mu\})(B\otimes\{\nu\}),$$

where $\Gamma_{\mu\nu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\mu\}\{\nu\}$,

$$(A-B)\otimes\{\lambda\}=\sum(-1)^r\Gamma_{\mu\nu\lambda}(A\otimes\{\mu\})(B\otimes\{\tilde\nu\}),$$

where $\{\tilde\nu\}$ is the partition of r conjugate to $\{\nu\}$, and

$$(AB)\otimes\{\lambda\}=\sum g_{\mu\nu\lambda}(A\otimes\{\mu\})(B\otimes\{\nu\}),$$

where $g_{\mu\nu\lambda}$ is the coefficient of $\{\lambda\}$ in the inner product $\{\mu\}\circ\{\nu\}$ (Wybourne 1970).

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Plot

see GRAPH (FUNCTION)

Plouffe's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Define the function

$$\rho(x)\equiv\begin{cases}1 & \text{for } x<0\\ 0 & \text{for } x\geq 0.\end{cases}\tag{1}$$

Let

$$a_n=\sin(2^n)=\begin{cases}\sin 1 & \text{for } n=0\\ 2a_0\sqrt{1-{a_0}^2} & \text{for } n=1\\ 2a_{n-1}(1-2{a_{n-2}}^2) & \text{for } n\geq 2,\end{cases}\tag{2}$$

then

$$\sum_{n=0}^\infty \frac{\rho(a_n)}{2^{n+1}}=\frac{1}{2\pi}.\tag{3}$$

For

$$b_n = \cos(2^n) = \begin{cases} \cos 1 & \text{for } n = 0 \\ 2b_{n-1}^2 - 1 & \text{for } n \geq 1, \end{cases} \quad (4)$$

and

$$\sum_{n=0}^{\infty} \frac{\rho(b_n)}{2^{n+1}} = 0.4756260767\ldots. \quad (5)$$

Letting

$$c_n = \tan(2^n) = \begin{cases} \tan 1 & \text{for } n = 0 \\ \frac{2c_{n-1}}{1-c_{n-1}^2} & \text{for } n \geq 1, \end{cases} \quad (6)$$

then

$$\sum_{n=0}^{\infty} \frac{\rho(c_n)}{2^{n+1}} = \frac{1}{\pi}. \quad (7)$$

Plouffe asked if the above processes could be "inverted." He considered

$$\alpha_n = \sin(2^n \sin^{-1} \tfrac{1}{2}) = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ \frac{1}{2}\sqrt{3} & \text{for } n = 1 \\ 2\alpha_{n-1}(1 - 2\alpha_{n-2}^2) & \text{for } n \geq 2, \end{cases} \quad (8)$$

giving

$$\sum_{n=0}^{\infty} \frac{\rho(\alpha_n)}{2^{n+1}} = \tfrac{1}{12}, \quad (9)$$

and

$$\beta_n = \cos(2^n \cos^{-1} \tfrac{1}{2}) = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ 2\beta_{n-1}^2 - 1 & \text{for } n \geq 1, \end{cases} \quad (10)$$

giving

$$\sum_{n=0}^{\infty} \frac{\rho(\beta_n)}{2^{n+1}} = \tfrac{1}{2}, \quad (11)$$

and

$$\gamma_n = \tan(2^n \tan^{-1} \tfrac{1}{2}) = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ \frac{2\gamma_{n-1}}{1-\gamma_{n-1}^2} & \text{for } n \geq 1, \end{cases} \quad (12)$$

giving

$$\sum_{n=0}^{\infty} \frac{\rho(\alpha_n)}{2^{n+1}} = \frac{1}{\pi} \tan^{-1}(\tfrac{1}{2}). \quad (13)$$

The latter is known as Plouffe's constant (Plouffe 1997). The positions of the 1s in the BINARY expansion of this constant are 3, 6, 8, 9, 10, 13, 21, 23, ... (Sloane's A004715).

Borwein and Girgensohn (1995) extended Plouffe's γ_n to *arbitrary* REAL x, showing that if

$$\xi_n = \tan(2^n \tan^{-1} x) = \begin{cases} x & \text{for } n = 0 \\ \frac{2\xi_{n-1}}{1-\xi_{n-1}^2} & \text{for } n \geq 1 \text{ and } |\xi_{n-1}| \neq 1 \\ -\infty & \text{for } n \geq 1 \text{ and } |\xi_{n-1}| = 1, \end{cases} \quad (14)$$

then

$$\sum_{n=0}^{\infty} \frac{\rho(\xi_n)}{2^{n+1}} = \begin{cases} \frac{\tan^{-1} x}{\pi} & \text{for } x \geq 0 \\ 1 + \frac{\tan^{-1} x}{\pi} & \text{for } x < 0. \end{cases} \quad (15)$$

Borwein and Girgensohn (1995) also give much more general recurrences and formulas.

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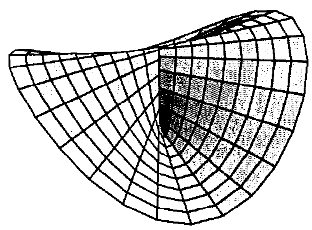
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/plff/plff.html`.
Plouffe, S.. "The Computation of Certain Numbers Using a Ruler and Compass." Dec. 12, 1997. `http://www.research.att.com/~njas/sequences/JIS/compass.html`.

Plücker Characteristics

The CLASS m, ORDER n, number of NODES δ, number of CUSPS κ, number of STATIONARY TANGENTS (INFLECTION POINTS) ι, number of BITANGENTS τ, and GENUS p.

see also ALGEBRAIC CURVE, BITANGENT, CUSP, GENUS (SURFACE), INFLECTION POINT, NODE (ALGEBRAIC CURVE), STATIONARY TANGENT

Plücker's Conoid



A RULED SURFACE sometimes also called the CYLINDROID. von Seggern (1993) gives the general functional form as

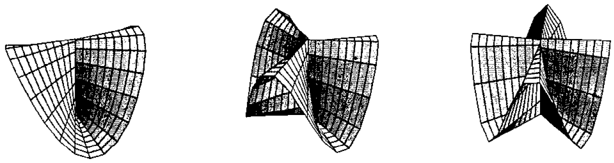
$$ax^2 + by^2 - zx^2 - zy^2 = 0, \quad (1)$$

whereas Fischer (1986) and Gray (1993) give

$$z = \frac{2xy}{(x^2 + y^2)}. \quad (2)$$

A polar parameterization therefore gives

$$x(r, \theta) = r \cos\theta \quad (3)$$
$$y(r, \theta) = r \sin\theta \quad (4)$$
$$z(r, \theta) = 2\cos\theta \sin\theta. \quad (5)$$



A generalization of Plücker's conoid to n folds is given by

$$x(r,\theta) = r\cos\theta \quad (6)$$

$$y(r,\theta) = r\sin\theta \quad (7)$$

$$z(r,\theta) = \sin(n\theta) \quad (8)$$

(Gray 1993). The cylindroid is the inversion of the CROSS-CAP (Pinkall 1986).

see also CROSS-CAP, RIGHT CONOID, RULED SURFACE

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Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 337–339, 1993.

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Plücker's Equations

Relationships between the number of SINGULARITIES of plane algebraic curves. Given a PLANE CURVE,

$$m = n(n-1) - 2\delta - 3\kappa \quad (1)$$

$$n = m(m-1) - 2\tau - 3\iota \quad (2)$$

$$\iota = 3n(n-2) - 6\delta - 8\kappa \quad (3)$$

$$\kappa = 3m(m-2) - 6\tau - 8\iota, \quad (4)$$

where m is the CLASS, n the ORDER, δ the number of NODES, κ the number of CUSPS, ι the number of STATIONARY TANGENTS (INFLECTION POINTS), and τ the number of BITANGENTS. Only three of these equations are LINEARLY INDEPENDENT.

see also ALGEBRAIC CURVE, BIOCHE'S THEOREM, BITANGENT, CUSP, GENUS (SURFACE), INFLECTION POINT, KLEIN'S EQUATION, NODE (ALGEBRAIC CURVE), STATIONARY TANGENT

References

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Plücker Relations

see PLÜCKER'S EQUATIONS

Plumbing

The plumbing of a p-sphere and a q-sphere is defined as the disjoint union of $\mathbb{S}^p \times \mathbb{S}^q$ and $\mathbb{D}^p \times \mathbb{S}^q$ with their common $\mathbb{D}^p \times \mathbb{D}^q$, identified via the identity homeomorphism.

see also HYPERSPHERE

References

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Pluperfect Number

see MULTIPLY PERFECT NUMBER

Plurisubharmonic Function

An upper semicontinuous function whose restrictions to all COMPLEX lines are subharmonic (where defined). These functions were introduced by P. Lelong and Oka in the early 1940s. Examples of such a function are the logarithms of moduli of holomorphic functions.

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Plus

The ADDITION of two quantities, i.e., a plus b. The operation is denoted $a+b$, and the symbol $+$ is called the PLUS SIGN. Floating point ADDITION is sometimes denoted $\oplus$.

see also ADDITION, MINUS, PLUS OR MINUS, TIMES

Plus or Minus

The symbol $\pm$ is used to denote a quantity which should be both added and subtracted, as in $a \pm b$. The symbol can be used to denote a range of uncertainty, or to denote a pair of quantities, such as the roots given by the QUADRATIC FORMULA

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When order is relevant, the symbol $a \mp b$ is also used, so an expression of the form $x \pm y \mp z$ is interpreted as $x+y-z$ or $x-y+z$. In contrast, the expression $x \pm y \pm z$ is interpreted to mean the set of four quantities $x+y+z$, $x-y+z$, $x+y-z$, and $x-y-z$.

see also MINUS, MINUS SIGN, PLUS, PLUS SIGN, SIGN

Plus Perfect Number

see ARMSTRONG NUMBER

Plus Sign

The symbol "+" which is used to denote a POSITIVE number or to indicate ADDITION.

see also ADDITION, MINUS SIGN, SIGN

Plutarch Numbers

In *Moralia,* the Greek biographer and philosopher Plutarch states "Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. (Hipparchus, to be sure, refuted this by showing that on the affirmative side there are 103,049 compound statements, and on the negative side 310,952.)" These numbers are known as the Plutarch numbers. 103,049 can be interpreted as the number s_{10} of BRACKETINGS on ten letters (Stanley 1997), Habsieger *et al.* 1998). Similarly, Plutarch's second number is given by $(s_{10}+s_{11})/2 = 310,954$ (Habsieger *et al.* 1998).

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Pochhammer Symbol

A.k.a. RISING FACTORIAL. For an INTEGER $n > 0$,

$$(a)_n \equiv \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+n-1), \quad (1)$$

where $\Gamma(z)$ is the GAMMA FUNCTION and

$$(a)_0 \equiv 1. \quad (2)$$

The NOTATION conflicts with both that for q-SERIES and that for GAUSSIAN COEFFICIENTS, so context usually serves to distinguish the three. Additional identities are

$$\frac{d}{da}(a)_n = (a)_n[F(a+n-1) - F(a-1)] \quad (3)$$

$$(a)_{n+k} = (a+n)_k(a)_n, \quad (4)$$

where F is the DIGAMMA FUNCTION. The Pochhammer symbol arises in series expansions of HYPERGEOMETRIC FUNCTIONS and GENERALIZED HYPERGEOMETRIC FUNCTIONS.

see also FACTORIAL, GENERALIZED HYPERGEOMETRIC FUNCTION, HARMONIC LOGARITHM, HYPERGEOMETRIC FUNCTION

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Spanier, J. and Oldham, K. B. "The Pochhammer Polynomials $(x)_n$." Ch. 18 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 149–165, 1987.

Pocklington's Criterion

Let p be an ODD PRIME, k be an INTEGER such that $p \nmid k$ and $1 \leq k \leq 2(p+1)$, and

$$N \equiv 2kp + 1.$$

Then the following are equivalent

1. N is PRIME.
2. $\mathrm{GCD}(a^k + 1, N) = 1.$

This is a modified version of the original theorem due to Lehmer.

References

Pocklington, H. C. "The Determination of the Prime or Composite Nature of Large Numbers by Fermat's Theorem." *Proc. Cambridge Phil. Soc.* **18**, 29–30, 1914/16.

Pocklington-Lehmer Test

see POCKLINGTON'S THEOREM

Pocklington's Theorem

Let $n-1 = FR$ where F is the factored part of a number

$$F = {p_1}^{a_1} \cdots {p_r}^{a_r}, \quad (1)$$

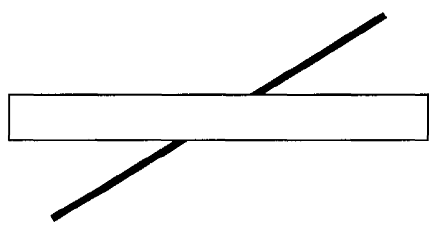
where $(R,F) = 1$, and $R < \sqrt{n}$. If there exists a b_i for $i = 1, \ldots, r$ such that

$${b_i}^{n-1} \equiv 1 \pmod{n} \quad (2)$$

$$\mathrm{GCD}({b_i}^{(n-1)/p_i} - 1, n) = 1, \quad (3)$$

then n is a PRIME.

Poggendorff Illusion



The illusion that the two ends of a straight LINE SEGMENT passing behind an obscuring RECTANGLE are offset when, in fact, they are aligned.

see also ILLUSION, MÜLLER-LYER ILLUSION, PONZO'S ILLUSION, VERTICAL-HORIZONTAL ILLUSION

References

Burmester, E. "Beiträge zu experimentellen Bestimmung geometrisch-optischer Täuschungen." *Z. Psychologie* **12**, 355–394, 1896.

Day, R. H. and Dickenson, R. G. "The Components of the Poggendorff Illusion." *Brit. J. Psychology* **67**, 537–552, 1976.

Fineman, M. "Poggendorff's Illusion." Ch. 19 in *The Nature of Visual Illusion.* New York: Dover, pp. 151–159, 1996.

Pohlke's Theorem

The principal theorem of AXONOMETRY. It states that three segments of arbitrary length $a'x'$, $a'y'$, and $a'z'$ which are drawn in a PLANE from a point a' under arbitrary ANGLES form a parallel projection of three equal segments ax, ay, and az from the ORIGIN of three PERPENDICULAR coordinate axes. However, only one of the segments or one of the ANGLES may vanish.

see also AXONOMETRY

Poincaré-Birkhoff Fixed Point Theorem

For the rational curve of an unperturbed system with ROTATION NUMBER r/s under a map T (for which every point is a FIXED POINT of T^s), only an even number of FIXED POINTS $2ks$ ($k = 1, 2, \ldots$) will remain under perturbation. These FIXED POINTS are alternately stable (ELLIPTIC) and unstable (HYPERBOLIC). Around each elliptic fixed point there is a simultaneous application of the Poincaré-Birkhoff fixed point theorem and the KAM THEOREM, which leads to a self-similar structure on all scales.

The original formulation was: Given a CONFORMAL ONE-TO-ONE transformation from an ANNULUS to itself that advances points on the outer edge positively and on the inner edge negatively, then there are at least two fixed points.

It was conjectured by Poincaré from a consideration of the three-body problem in celestial mechanics and proved by Birkhoff.

Poincaré Conjecture

A SIMPLY CONNECTED 3-MANIFOLD is HOMEOMORPHIC to the 3-SPHERE. The generalized Poincaré conjecture is that a COMPACT n-MANIFOLD is HOMOTOPY equivalent to the n-sphere IFF it is HOMEOMORPHIC to the n-SPHERE. This reduces to the original conjecture for $n = 3$.

The $n = 1$ case of the generalized conjecture is trivial, the $n = 2$ case is classical, $n = 3$ remains open, $n = 4$ was proved by Freedman (1982) (for which he was awarded the 1986 FIELDS MEDAL), $n = 5$ by Zeeman (1961), $n = 6$ by Stallings (1962), and $n \geq 7$ by Smale in 1961 (Smale subsequently extended this proof to include $n \geq 5$.)

see also COMPACT MANIFOLD, HOMEOMORPHIC, HOMOTOPY, MANIFOLD, SIMPLY CONNECTED, SPHERE, THURSTON'S GEOMETRIZATION CONJECTURE

References

Freedman, M. H. "The Topology of Four-Differentiable Manifolds." *J. Diff. Geom.* **17**, 357–453, 1982.

Stallings, J. "The Piecewise-Linear Structure of Euclidean Space." *Proc. Cambridge Philos. Soc.* **58**, 481–488, 1962.

Smale, S. "Generalized Poincaré's Conjecture in Dimensions Greater than Four." *Ann. Math.* **74**, 391–406, 1961.

Zeeman, E. C. "The Generalised Poincaré Conjecture." *Bull. Amer. Math. Soc.* **67**, 270, 1961.

Zeeman, E. C. "The Poincaré Conjecture for $n \geq 5$." In *Topology of 3-Manifolds and Related Topics, Proceedings of the University of Georgia Institute, 1961.* Englewood Cliffs, NJ: Prentice-Hall, pp. 198–204, 1961.

Poincaré Duality

The BETTI NUMBERS of a compact orientable n-MANIFOLD satisfy the relation

$$b_i = b_{n-i}.$$

see also BETTI NUMBER

Poincaré Formula

The POLYHEDRAL FORMULA generalized to a surface of GENUS p.

$$V - E + F = 2 - 2p$$

where V is the number of VERTICES, E is the number of EDGES, F is the number of faces, and

$$\chi \equiv 2 - 2p$$

is called the EULER CHARACTERISTIC.

see also EULER CHARACTERISTIC, GENUS (SURFACE), POLYHEDRAL FORMULA

References

Eppstein, D. "Fourteen Proofs of Euler's Formula: $V - E + F = 2$." `http://www.ics.uci.edu/~eppstein/junkyard/euler`.

Poincaré-Fuchs-Klein Automorphic Function

$$f(z) = \frac{k}{(cz+d)^r} f\left(\frac{az+b}{cz+d}\right)$$

where $\Im(z) > 0$.

see also AUTOMORPHIC FUNCTION

Poincaré Group

see LORENTZ GROUP

Poincaré's Holomorphic Lemma

Solutions to HOLOMORPHIC differential equations are themselves HOLOMORPHIC FUNCTIONS of time, initial conditions, and parameters.

Poincaré-Hopf Index Theorem

The index of a VECTOR FIELD with finitely many zeros on a compact, oriented MANIFOLD is the same as the EULER CHARACTERISTIC of the MANIFOLD.

see also GAUSS-BONNET FORMULA

Poincaré Hyperbolic Disk

A 2-D space having HYPERBOLIC GEOMETRY defined as the 2-BALL $\{x \in \mathbb{R}^2 : |x| < 1\}$, with HYPERBOLIC METRIC

$$\frac{dx^2 + dy^2}{(1 - r^2)^2}.$$

The Poincaré disk is a model for HYPERBOLIC GEOMETRY, and there is an isomorphism between the Poincaré disk model and the KLEIN-BELTRAMI MODEL.

see also ELLIPTIC PLANE, HYPERBOLIC GEOMETRY, HYPERBOLIC METRIC, KLEIN-BELTRAMI MODEL

Poincaré's Lemma

Let $\wedge$ denote the WEDGE PRODUCT and D the EXTERIOR DERIVATIVE. Then

$$D^2 t = \frac{\partial}{\partial x} \wedge \left(\frac{\partial}{\partial x} \wedge t\right) = \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}\right) \wedge t = 0.$$

see also DIFFERENTIAL FORM, EXTERIOR DERIVATIVE, POINCARÉ'S HOLOMORPHIC LEMMA, WEDGE PRODUCT

Poincaré Manifold

A nonsimply connected 3-manifold also called a DODECAHEDRAL SPACE.

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 245, 290, and 308, 1976.

Poincaré Metric

The METRIC

$$ds^2 = \frac{dx^2 + dy^2}{(1 - |z|^2)^2}$$

of the POINCARÉ HYPERBOLIC DISK.

see also POINCARÉ HYPERBOLIC DISK

Poincaré Separation Theorem

Let $\{\mathbf{y}^k\}$ be a set of orthonormal vectors with $k = 1, 2, \ldots, K$, such that the INNER PRODUCT $(\mathbf{y}^k, \mathbf{y}^k) = 1$. Then set

$$\mathbf{x} = \sum_{k=1}^{K} u_k \mathbf{y}^k \qquad (1)$$

so that for any SQUARE MATRIX A for which the product $\mathsf{A}\mathbf{x}$ is defined, the corresponding QUADRATIC FORM is

$$(\mathbf{x}, \mathsf{A}\mathbf{x}) = \sum_{k,l=1}^{K} u_k u_l (\mathbf{y}^k, \mathsf{A}\mathbf{y}^l). \qquad (2)$$

Then if

$$\mathbf{B}_k = (\mathbf{y}^k, \mathsf{A}\mathbf{y}^l) \qquad (3)$$

for $k, l = 1, 2, \ldots, K$, it follows that

$$\lambda_i(\mathbf{B}_K) \leq \lambda_1(\mathsf{A}) \qquad (4)$$

$$\lambda_{K-j}(\mathbf{B}_K) \geq \lambda_{N-j}(\mathsf{A}) \qquad (5)$$

for $i = 1, 2, \ldots, K$ and $j = 0, 1, \ldots, K - 1$.

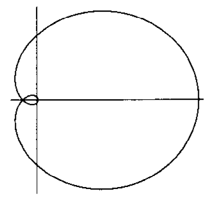
References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1120, 1979.

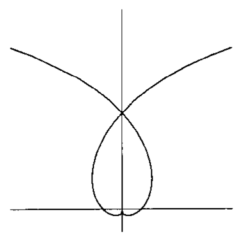
Poinsot Solid

see KEPLER-POINSOT SOLID

Poinsot's Spirals



$r \sinh(n\theta) = a.$

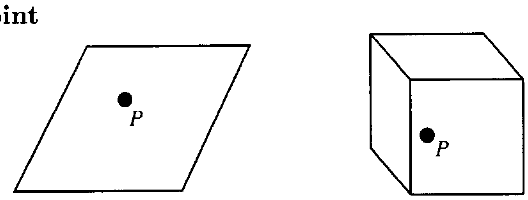


$r \operatorname{csch}(n\theta) = a.$

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 192 and 194, 1972.

Point



A 0-DIMENSIONAL mathematical object which can be specified in n-D space using n coordinates. Although the notion of a point is intuitively rather clear, the mathematical machinery used to deal with points and point-like objects can be surprisingly slippery. This difficulty was encountered by none other than Euclid himself who, in his *Elements*, gave the vague definition of a point as "that which has no part."

The basic geometric structures of higher DIMENSIONAL geometry—the LINE, PLANE, SPACE, and HYPERSPACE—are all built up of infinite numbers of points arranged in particular ways.

see also ACCUMULATION POINT, ANTIGONAL POINTS, ANTIHOMOLOGOUS POINTS, APOLLONIUS POINT, BOUNDARY POINT, BRANCH POINT, BRIANCHON POINT, BROCARD MIDPOINT, BROCARD POINTS,

CANTOR-DEDEKIND AXIOM, CENTER, CIRCLE LATTICE POINTS, CONCUR, CONCURRENT, CONGRUENT INCIRCLES POINT, CONGRUENT ISOSCELIZERS POINT, CONJUGATE POINTS, CRITICAL POINT, CRUCIAL POINT, CUBE POINT PICKING, CUSP POINT, DE LONGCHAMPS POINT, DOUBLE POINT, ECKARDT POINT, ELKIES POINT, ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), ELLIPTIC FIXED POINT (MAP), ELLIPTIC POINT, EQUAL DETOUR POINT, EQUAL PARALLELIANS POINT, EQUICHORDAL POINT, EQUILIBRIUM POINT, EQUIPRODUCT POINT, EQUIRECIPROCAL POINT, EVANS POINT, EXETER POINT, EXMEDIAN POINT, FAGNANO'S POINT, FAR-OUT POINT, FEJES TÓTH'S PROBLEM, FERMAT POINT, FEUERBACH POINT, FEYNMAN POINT, FIXED POINT, FLETCHER POINT, GERGONNE POINT, GREBE POINT, GRIFFITHS POINTS, HARMONIC CONJUGATE POINTS, HERMIT POINT, HOFSTADTER POINT, HOMOLOGOUS POINTS, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), HYPERBOLIC FIXED POINT (MAP), HYPERBOLIC POINT, IDEAL POINT, IMAGINARY POINT, INVARIANT POINT, INVERSE POINTS, ISODYNAMIC POINTS, ISOLATED POINT, ISOPERIMETRIC POINT, ISOTOMIC CONJUGATE POINT, LATTICE POINT, LEMOINE POINT, LIMIT POINT, MALFATTI POINTS, MEDIAN POINT, MID-ARC POINTS, MIDPOINT, MIQUEL POINT, NAGEL POINT, NAPOLEON POINTS, NOBBS POINTS, OLDKNOW POINTS, ONLY CRITICAL POINT IN TOWN TEST, ORDINARY POINT, PARABOLIC POINT, PARRY POINT, PEDAL POINT, PERIODIC POINT, PLANAR POINT, POINT AT INFINITY, POINT-LINE DISTANCE—2-D, POINT-LINE DISTANCE—3-D, POINT-QUADRATIC DISTANCE, POINT-PLANE DISTANCE, POINT-SET TOPOLOGY, POINTWISE DIMENSION, POLICEMAN ON POINT DUTY CURVE, POWER POINT, RADIAL POINT, RADIANT POINT, RATIONAL POINT, RIGBY POINTS, SADDLE POINT (GAME), SADDLE POINT (FUNCTION), SALIENT POINT, SCHIFFLER POINT, SELF-HOMOLOGOUS POINT, SIMILARITY POINT, SINGULAR POINT (ALGEBRAIC CURVE), SINGULAR POINT (FUNCTION), SODDY POINTS, SPECIAL POINT, STATIONARY POINT, STEINER POINTS, SYLVESTER'S FOUR-POINT PROBLEM, SYMMEDIAN POINT, SYMMETRIC POINTS, TARRY POINT, TORRICELLI POINT, TRISECTED PERIMETER POINT, UMBILIC POINT, UNIT POINT, VANISHING POINT, VISIBLE POINT, WEIERSTRAß POINT, WILD POINT, YFF POINTS

References

Casey, J. "The Point." Ch. 1 in *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples, 2nd ed., rev. enl.* Dublin: Hodges, Figgis, & Co., pp. 1–29, 1893.

Point Estimator

An ESTIMATOR of the actual values of population.

Point Groups

The symmetry groups possible in a crystal lattice without the translation symmetry element. Although an isolated object may have an arbitrary SCHÖNFLIES SYMBOL, the requirement that symmetry be present in a lattice requires that only 1, 2, 3, and 6-fold symmetry axes are possible (the CRYSTALLOGRAPHY RESTRICTION), which restricts the number of possible point groups to 32: C_i, C_s, C_1, C_2, C_3, C_4, C_6, C_{2h}, C_{3h}, C_{4h}, C_{6h}, C_{2v}, C_{3v}, C_{4v}, C_{6v}, D_2, D_3, D_4, D_6 (the DIHEDRAL GROUPS), D_{2h}, D_{3h}, D_{4h}, D_{6h}, D_{2d}, D_{3d}, O, O_h (the OCTAHEDRAL GROUP), S_4, S_6, T, T_h, and T_d (the TETRAHEDRAL GROUP).

see also CRYSTALLOGRAPHY RESTRICTION, DIHEDRAL GROUP, GROUP, GROUP THEORY, HERMANN-MAUGUIN SYMBOL, LATTICE GROUPS, OCTAHEDRAL GROUP, SCHÖNFLIES SYMBOL, SPACE GROUPS, TETRAHEDRAL GROUP

References

Arfken, G. "Crystallographic Point and Space Groups." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 248–249, 1985.

Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 379, 1990.

Lomont, J. S. "Crystallographic Point Groups." §4.4 in *Applications of Finite Groups.* New York: Dover, pp. 132–146, 1993.

Point at Infinity

P is the point on the line AB such that $\overline{PA}/\overline{PB} = 1$. It can also be thought of as the point of intersection of two PARALLEL lines.

see also LINE AT INFINITY

References

Behnke, H.; Bachmann, F.; Fladt, K.; and Suss, W. (Eds.). Ch. 7 in *Fundamentals of Mathematics, Vol. 3: Points at Infinity.* Cambridge, MA: MIT Press, 1974.

Point-Line Distance—2-D

Given a line $ax + by + c = 0$ and a point (x_0, y_0), in slope-intercept form, the equation of the line is

$$y = -\frac{a}{b}x - \frac{c}{b}, \tag{1}$$

so the line has SLOPE $-a/b$. Points on the line have the vector coordinates

$$\begin{bmatrix} x \\ -\frac{a}{b}x - \frac{c}{d} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{c}{d} \end{bmatrix} - \frac{1}{b}\begin{bmatrix} -b \\ a \end{bmatrix}. \tag{2}$$

Therefore, the VECTOR

$$\begin{bmatrix} -b \\ a \end{bmatrix} \tag{3}$$

is PARALLEL to the line, and the VECTOR

$$\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{4}$$

is PERPENDICULAR to it. Now, a VECTOR from the point to the line is given by

$$\mathbf{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \tag{5}$$

Projecting $\mathbf{r}$ onto $\mathbf{v}$,

$$\begin{aligned} |\mathrm{proj}_{\mathbf{v}}\mathbf{r}| &= \frac{|\mathbf{v} \cdot \mathbf{r}|}{|\mathbf{v}|} = |\hat{\mathbf{v}} \cdot \mathbf{r}| = \frac{|a(x - x_0) + b(y - y_0)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax + by - ax_0 - by_0|}{\sqrt{a^2 + b^2}} \\ &= \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}. \end{aligned} \tag{6}$$

If the line is represented by the endpoints of a VECTOR (x_1, y_1) and (x_2, y_2), then the PERPENDICULAR VECTOR is

$$\mathbf{v} = \begin{bmatrix} y_2 - y_1 \\ -(x_2 - x_1) \end{bmatrix} \tag{7}$$

$$\hat{\mathbf{v}} = \frac{1}{s} \begin{bmatrix} y_2 - y_1 \\ -(x_2 - x_1) \end{bmatrix}, \tag{8}$$

where

$$s = |\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{9}$$

so the distance is

$$d = |\hat{\mathbf{v}} \cdot \mathbf{r}| = \frac{|(y_2 - y_1)(x_0 - x_1) - (x_2 - x_1)(y_0 - y_1)|}{s}. \tag{10}$$

The distance from a point (x_1, y_1) to the line $y = a + bx$ can be computed using VECTOR algebra. Let $\mathbf{L}$ be a VECTOR in the same direction as the line

$$\mathbf{L} = \begin{bmatrix} x \\ a + bx \end{bmatrix} - \begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} x \\ bx \end{bmatrix} \tag{11}$$

$$\hat{\mathbf{L}} = \frac{1}{\sqrt{b^2 + 1}} \begin{bmatrix} 1 \\ b \end{bmatrix}. \tag{12}$$

A given point on the line is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} 0 \\ -a \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 - a \end{bmatrix}, \tag{13}$$

so the point-line distance is

$$\begin{aligned} \mathbf{r} &= (\mathbf{x} \cdot \hat{\mathbf{L}})\hat{\mathbf{L}} - \mathbf{x} \\ &= \frac{1}{1 + b^2} \left(\begin{bmatrix} x_1 \\ y_1 - a \end{bmatrix} \cdot \begin{bmatrix} 1 \\ v \end{bmatrix} \right) \begin{bmatrix} 1 \\ b \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 - a \end{bmatrix} \\ &= \frac{x_1 + b(y_1 - a)}{1 + b^2} \begin{bmatrix} 1 \\ b \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 - a \end{bmatrix} \\ &= \frac{1}{1 + b^2} \begin{bmatrix} b(y_1 - a) - b^2 x_1 \\ bx_1 + b^2 y_1 - ab^2 - y_1 + a - b^2 y_1 + ab^2 \end{bmatrix} \\ &= \frac{1}{1 + b^2} \begin{bmatrix} b[(y_1 - a) - bx_1] \\ -[(y_1 - a) - bx_1] \end{bmatrix} \\ &= \frac{y_1 - (a + bx_1)}{1 + b^2} \begin{bmatrix} b \\ -1 \end{bmatrix}. \end{aligned} \tag{14}$$

Therefore,

$$d = |\mathbf{r}| = \frac{|y_1 - (a + bx_1)|}{1 + b^2} \sqrt{1 + b^2} = \frac{|y_1 - (a + bx_1)|}{\sqrt{1 + b^2}}. \tag{15}$$

This result can also be obtained much more simply by noting that the PERPENDICULAR distance is just $\cos\theta$ times the vertical distance $|y_1 - (a + bx_1)|$. But the SLOPE b is just $\tan\theta$, so

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \frac{1}{\cos^2\theta}, \tag{16}$$

and

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + b^2}}. \tag{17}$$

The PERPENDICULAR distance is then

$$d = \frac{|y_1 - (a + bx_1)|}{\sqrt{1 + b^2}}, \tag{18}$$

the same result as before.

see also LINE, POINT, POINT-LINE DISTANCE—3-D

Point-Line Distance—3-D

A line in 3-D is given by the parametric VECTOR

$$\mathbf{v} = \begin{bmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{bmatrix}. \tag{1}$$

The distance between a point on the line with parameter t and the point (x_1, y_1, z_1) is therefore

$$r^2 = (x_1 - x_0 - at)^2 + (y_1 - y_0 - bt)^2 + (z_1 - z_0 - ct)^2. \tag{2}$$

To minimize the distance, take

$$\frac{\partial(r^2)}{\partial t} = -2a(x_1 - x_0 - at) - 2b(y_1 - y_0 - bt) - 2c(z_1 - z_0 - ct) = 0 \tag{3}$$

$$a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) - t(a^2 + b^2 + c^2) = 0 \tag{4}$$

$$t = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{a^2 + b^2 + c^2}, \tag{5}$$

so the minimum distance is found by plugging (5) into (2) and taking the SQUARE ROOT.

see also LINE, POINT, POINT-LINE DISTANCE—2-D

Point Picking

see 18-POINT PROBLEM, BALL TRIANGLE PICKING, CUBE POINT PICKING, CUBE TRIANGLE PICKING, DISCREPANCY THEOREM, ISOSCELES TRIANGLE, OBTUSE TRIANGLE, PLANAR DISTANCE, SYLVESTER'S FOUR-POINT PROBLEM

Point-Plane Distance

Given a PLANE

$$ax + by + cz + d = 0 \tag{1}$$

and a point (x_0, y_0, z_0), the NORMAL to the PLANE is given by

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{2}$$

and a VECTOR from the plane to the point is given by

$$\mathbf{w} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}. \tag{3}$$

Projecting $\mathbf{w}$ onto $\mathbf{v}$,

$$\begin{aligned}|\mathrm{proj}_{\mathbf{v}}\mathbf{w}| &= \frac{|\mathbf{v}\cdot\mathbf{w}|}{|\mathbf{v}|} \\ &= \frac{|a(x-x_0)+b(y-y_0)+c(z-z_0)+d|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax+by+cz-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}. \end{aligned}\tag{4}$$

Point-Point Distance—1-D

Given a unit LINE SEGMENT $[0,1]$, pick two points at random on it. Call the first point x_1 and the second point x_2. Find the distribution of distances d between points. The probability of the points being a (POSITIVE) distance d apart (i.e., without regard to ordering) is given by

$$\begin{aligned}P(d) &= \frac{\int_0^1\int_0^1 \delta(d-|x_2-x_1|)\,dx_1\,dx_2}{\int_0^1\int_0^1 dx_1\,dx_2} \\ &= (1-d)[H(1-d)-H(d-1)+H(d)-H(-d)] \\ &= \begin{cases} 2(1-d) & \text{for } 0 \le d \le 1 \\ 0 & \text{otherwise,}\end{cases}\end{aligned}\tag{1}$$

where δ is the DIRAC DELTA FUNCTION and H is the HEAVISIDE STEP FUNCTION. The MOMENTS are then

$$\begin{aligned}\mu'_m &= \int_0^1 d^m P(d)\,dd = 2\int_0^1 d^m(1-d)\,dd \\ &= 2\left[\frac{d^{m+1}}{m+1} - \frac{d^{m+2}}{m+2}\right]_0^1 \\ &= 2\left(\frac{1}{m+1} - \frac{1}{m+2}\right) = 2\left[\frac{(m+2)-(m+1)}{(m+1)(m+2)}\right] \\ &= \frac{2}{(m+1)(m+2)} \\ &= \begin{cases} \frac{1}{(n+1)(2n+1)} & \text{for } m = 2n \\ \frac{1}{(n+1)(2n+3)} & \text{for } m = 2n+1,\end{cases}\end{aligned}\tag{2}$$

giving MOMENTS about 0

$$\mu'_1 = \tfrac{1}{3} \tag{3}$$
$$\mu'_2 = \tfrac{1}{6} \tag{4}$$
$$\mu'_3 = \tfrac{1}{10} \tag{5}$$
$$\mu'_4 = \tfrac{1}{15}. \tag{6}$$

The MOMENTS can also be computed directly without explicit knowledge of the distribution

$$\begin{aligned}\mu'_1 &= \frac{\int_0^1\int_0^1 |x_2-x_1|\,dx_1\,dx_2}{\int_0^1\int_0^1 dx_1\,dx_2} \\ &= \int_0^1\int_0^1 |x_2-x_1|\,dx_1\,dx_2 \\ &= \underset{x_2-x_1>0}{\int_0^1\int_0^1}(x_2-x_1)\,dx_1\,dx_2 \\ &\quad + \underset{x_2-x_1<0}{\int_0^1\int_0^1}(x_1-x_2)\,dx_1\,dx_2 \\ &= \int_0^1\int_{x_1}^1 (x_2-x_1)\,dx_1\,dx_2 \\ &\quad + \int_0^1\int_0^{x_1}(x_2-x_1)\,dx_1\,dx_2 \\ &= \int_0^1 \left[\tfrac{1}{2}{x_2}^2 - x_1x_2\right]_{x_1}^1 dx_1 \\ &\quad + \int_0^1 \left[x_1x_2 - \tfrac{1}{2}{x_2}^2\right]_0^{x_1} dx_1 \\ &= \int_0^1 \left[(\tfrac{1}{2}-x_1) - (\tfrac{1}{2}{x_1}^2 - {x_1}^2)\right] dx_1 \\ &\quad + \int_0^1 \left[({x_1}^2 - \tfrac{1}{2}{x_1}^2) - (0-0)\right] dx_1 \\ &= \int_0^1 (\tfrac{1}{2} - x_1 + {x_1}^2)\,dx_1 = [\tfrac{1}{2}x_1 - \tfrac{1}{2}{x_1}^2 + \tfrac{1}{3}{x_1}^3]_0^1 \\ &= (\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3}) - (0-0+0) = \tfrac{1}{3}\end{aligned}\tag{7}$$

$$\begin{aligned}\mu'_2 &= \int_0^1\int_0^1 (|x_2-x_1|)^2\,dx_2\,dx_1 \\ &= \int_0^1\int_0^1 (x_2-x_1)^2\,dx_1\,dx_2 \\ &= \int_0^1\int_0^1 ({x_2}^2 - 2x_1x_2 + {x_1}^2)\,dx_1\,dx_2 \\ &= \int_0^1 [\tfrac{1}{3}{x_2}^3 - x_1{x_2}^2 + {x_1}^2x_2]_0^1\,dx_1 \\ &= \int_0^1 (\tfrac{1}{3} - x_1 + {x_1}^2)\,dx_1 = [\tfrac{1}{3}{x_1}^3 - \tfrac{1}{2}{x_1}^2 + \tfrac{1}{3}x_1]_0^1 \\ &= \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{1}{6}.\end{aligned}\tag{8}$$

The MOMENTS about the MEAN are therefore

$$\mu_2 = \mu_2' - \mu_1'^2 = \tfrac{1}{6} - (\tfrac{1}{3})^2 = \tfrac{1}{18} \quad (9)$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = \tfrac{1}{135} \quad (10)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \tfrac{1}{135}, \quad (11)$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \mu_1' = \tfrac{1}{3} \quad (12)$$

$$\sigma^2 = \mu_2 = \tfrac{1}{18} \quad (13)$$

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \tfrac{2}{5}\sqrt{2} \quad (14)$$

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = -\tfrac{3}{5}. \quad (15)$$

The probability distribution of the distance between two points randomly picked on a LINE SEGMENT is germane to the problem of determining the access time of computer hard drives. In fact, the average access time for a hard drive is precisely the time required to seek across 1/3 of the tracks (Benedict 1995).

see also POINT-POINT DISTANCE—2-D, POINT-POINT DISTANCE—3-D, POINT-QUADRATIC DISTANCE, TETRAHEDRON INSCRIBING, TRIANGLE INSCRIBING IN A CIRCLE

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 930–931, 1985.

Benedict, B. *Using Norton Utilities for the Macintosh.* Indianapolis, IN: Que, pp. B-8–B-9, 1995.

Point-Point Distance—2-D

Given two points in the PLANE, find the curve which minimizes the distance between them. The LINE ELEMENT is given by

$$ds = \sqrt{dx^2 + dy^2}, \quad (1)$$

so the ARC LENGTH between the points x_1 and x_2 is

$$L = \int ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx, \quad (2)$$

where $y' \equiv dy/dx$ and the quantity we are minimizing is

$$f = \sqrt{1 + y'^2}. \quad (3)$$

Finding the derivatives gives

$$\frac{\partial f}{\partial y} = 0 \quad (4)$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}[(1 + y'^2)^{-1/2} y'], \quad (5)$$

so the EULER-LAGRANGE DIFFERENTIAL EQUATION becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0. \quad (6)$$

Integrating and rearranging,

$$\frac{y'}{\sqrt{1 + y'^2}} = c \quad (7)$$

$$y'^2 = c^2(1 + y'^2) \quad (8)$$

$$y'^2(1 - c^2) = c^2 \quad (9)$$

$$y' = \frac{c}{\sqrt{1 - c^2}} \equiv a. \quad (10)$$

The solution is therefore

$$y = ax + b, \quad (11)$$

which is a straight LINE. Now verify that the ARC LENGTH is indeed the straight-line distance between the points. a and b are determined from

$$y_1 = ax_1 + b \quad (12)$$

$$y_2 = ax_2 + b. \quad (13)$$

Writing (12) and (13) as a MATRIX EQUATION gives

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{x_1 - x_2} \begin{bmatrix} 1 & -1 \\ -x_2 & x_1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (15)$$

so

$$a = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \quad (16)$$

$$b = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2} \quad (17)$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dy = (x_2 - x_1)\sqrt{1 + a^2} = (x_2 - x_1)\sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \quad (18)$$

as expected.

The shortest distance between two points on a SPHERE is the so-called GREAT CIRCLE distance.

see also CALCULUS OF VARIATIONS, GREAT CIRCLE, POINT-POINT DISTANCE—1-D, POINT-POINT DISTANCE—3-D, POINT-QUADRATIC DISTANCE, TETRAHEDRON INSCRIBING, TRIANGLE INSCRIBING IN A CIRCLE

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 930–931, 1985.

Point-Point Distance—3-D

The LINE ELEMENT is

$$ds = \sqrt{dx^2 + dy^2 + dz^2}, \tag{1}$$

so the ARC LENGTH between the points x_1 and x_2 is

$$L = \int ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2 + z'^2}\, dx \tag{2}$$

and the quantity we are minimizing is

$$f = \sqrt{1 + y'^2 + z'^2}. \tag{3}$$

Finding the derivatives gives

$$\frac{\partial f}{\partial y} = 0 \tag{4}$$

$$\frac{\partial f}{\partial z} = 0 \tag{5}$$

and

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2 + z'^2}} \tag{6}$$

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{1 + y'^2 + z'^2}}, \tag{7}$$

so the EULER-LAGRANGE DIFFERENTIAL EQUATIONS become

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0 \tag{8}$$

$$\frac{d}{dx}\left(\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right) = 0. \tag{9}$$

These give

$$\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = c_1 \tag{10}$$

$$\frac{z'}{\sqrt{1 + y'^2 + z'^2}} = c_2. \tag{11}$$

Taking the ratio,

$$z' = \frac{c_2}{c_1} y' \tag{12}$$

$$\frac{y'}{\sqrt{1 + y'^2 + \left(\frac{c_2}{c_1}\right)^2 y'^2}} = c_1 \tag{13}$$

$$y'^2 = {c_1}^2 \left[1 + y'^2 + \left(\frac{c_2}{c_1}\right)^2 y'^2\right] = {c_1}^2 + y'^2({c_1}^2 + {c_2}^2), \tag{14}$$

which gives

$$y'^2 = \frac{{c_1}^2}{1 - {c_1}^2 - {c_2}^2} \equiv {a_1}^2 \tag{15}$$

$$z'^2 = \left(\frac{c_2}{c_1}\right)^2 y'^2 = \frac{{c_2}^2}{1 - {c_1}^2 - {c_2}^2} \equiv {b_1}^2. \tag{16}$$

Therefore, $y' = a_1$ and $z' = b_1$, so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ a_1 x + a_0 \\ b_1 x + b_0 \end{bmatrix}, \tag{17}$$

which is the parametric representation of a straight line with parameter $x \in [x_1, x_2]$. Verifying the ARC LENGTH gives

$$L = \sqrt{1 + {a_1}^2 + {b_1}^2}\,(x_2 - x_1) \tag{18}$$

where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}. \tag{20}$$

see also POINT-POINT DISTANCE—1-D, POINT-POINT DISTANCE—2-D, POINT-QUADRATIC DISTANCE

Point Probability

The portion of the probability distribution which has a P-VALUE *equal to* the observed P-VALUE.

see also TAIL PROBABILITY

Point-Quadratic Distance

Find the minimum distance between a point in the plane (x_0, y_0) and a quadratic PLANE CURVE

$$y = a_0 + a_1 x + a_2 x^2. \tag{1}$$

The square of the distance is

$$\begin{aligned} r^2 &= (x - x_0)^2 + (y - y_0)^2 \\ &= (x - x_0)^2 + (a_0 + a_1 x + a_2 x^2 - y_0)^2. \end{aligned} \tag{2}$$

Minimizing the distance squared is the equivalent to minimizing the distance (since r^2 and $|r|$ have minima at the same point), so take

$$\frac{\partial(r^2)}{\partial x} = 2(x - x_0) + 2(a_0 + a_1 x + a_2 x^2 - y_0)(a_1 + 2a_2 x) = 0 \tag{3}$$

$$\begin{aligned} x - x_0 + a_0 a_1 + {a_1}^2 + a_1 a_2 x^2 - a_1 y_0 + 2a_0 a_2 x \\ + 2a_1 a_2 x^2 + 2{a_2}^2 x^3 - 2a_2 y_0 x = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} 2{a_2}^2 x^3 + 3a_1 a_2 x^2 + ({a_1}^2 + 2a_0 a_2 - 2a_2 y_0 + 1)x \\ + (a_0 a_1 - a_1 y_0 - x_0) = 0. \end{aligned} \tag{5}$$

Minimizing the distance therefore requires solution of a CUBIC EQUATION.

see also POINT-POINT DISTANCE—1-D, POINT-POINT DISTANCE—2-D, POINT-POINT DISTANCE—3-D

Point-Set Topology

The low-level language of TOPOLOGY, which is not really considered a separate "branch" of TOPOLOGY. Point-set topology, also called set-theoretic topology or general topology, is the study of the general abstract nature of continuity or "closeness" on SPACES. Basic point-set topological notions are ones like CONTINUITY, DIMENSION, COMPACTNESS, and CONNECTEDNESS. The INTERMEDIATE VALUE THEOREM (which states that if a path in the real line connects two numbers, then it passes over every point between the two) is a basic topological result. Others are that EUCLIDEAN n-space is HOMEOMORPHIC to EUCLIDEAN m-space IFF $m = n$, and that REAL valued functions achieve maxima and minima on COMPACT SETS.

Foundational point-set topological questions are ones like "when can a topology on a space be derived from a metric?" Point-set topology deals with differing notions of continuity and compares them, as well as dealing with their properties. Point-set topology is also the ground-level of inquiry into the geometrical properties of spaces and continuous functions between them, and in that sense, it is the foundation on which the remainder of topology (ALGEBRAIC, DIFFERENTIAL, and LOW-DIMENSIONAL) stands.

see also ALGEBRAIC TOPOLOGY, DIFFERENTIAL TOPOLOGY, LOW-DIMENSIONAL TOPOLOGY, TOPOLOGY

References

Sutherland, W. A. *An Introduction to Metric & Topological Spaces.* New York: Oxford University Press, 1975.

Points Problem

see SHARING PROBLEM

Pointwise Dimension

$$D_P(\mathbf{x}) \equiv \lim_{\epsilon\to 0} \frac{\ln \mu(B_\epsilon(\mathbf{x}))}{\ln \epsilon},$$

where $B_\epsilon(\mathbf{x})$ is an n-D BALL of RADIUS ϵ centered at $\mathbf{x}$ and μ is the PROBABILITY MEASURE.

see also BALL, PROBABILITY MEASURE

References

Nayfeh, A. H. and Balachandran, B. *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods.* New York: Wiley, pp. 541–545, 1995.

Poisson's Bessel Function Formula

For $\Re[\nu] > -1/2$,

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{2}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^{\pi/2} \cos(z\cos t)\sin^{2\nu} t\,dt,$$

where $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND, and $\Gamma(z)$ is the GAMMA FUNCTION.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1472, 1980.

Poisson Bracket

Let F and G be infinitely differentiable functions of x and p. Then the Poisson bracket is defined by

$$(F,G) = \sum_{\nu=1}^{n} \left(\frac{\partial F}{\partial p_\nu}\frac{\partial G}{\partial x_p} - \frac{\partial G}{\partial p_\nu}\frac{\partial F}{\partial x_\nu} \right).$$

If F and G are functions of x and p only, then the LAGRANGE BRACKET $[F,G]$ collapses the Poisson bracket (F,G).

see also LAGRANGE BRACKET, LIE BRACKET

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1004, 1980.

Poisson-Charlier Function

$$\rho_n(\nu,x) \equiv \frac{(1+\nu-n)_n}{\sqrt{n!x^n}}\, {}_1F_1(-n;1+\nu-n;x),$$

where $(a)_n$ is a POCHHAMMER SYMBOL and ${}_1F_1(a;b;z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION.

see also POISSON-CHARLIER POLYNOMIAL

Poisson-Charlier Polynomial

Polynomials $p_n(x)$ which belong to the distribution $d\alpha(x)$ where $\alpha(x)$ is a STEP FUNCTION with JUMP

$$j(x) = e^{-a}a^x(x!)^{-1} \tag{1}$$

at $x = 0, 1, \ldots$ for $a > 0$.

$$\begin{aligned} p_n(x) &= a^{n/2}(n!)^{-1/2}\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\nu!a^{-\nu}\binom{x}{\nu} && (2)\\ &= a^{n/2}(n!)^{-1/2}(-1)^n[j(x)]^{-1}\Delta^n j(x-n) && (3)\\ &= a^{-n/2}\sqrt{n!}\,L_n^{x-n}(a), && (4) \end{aligned}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, $L_n^k(x)$ is an associated LAGUERRE POLYNOMIAL, and

$$\Delta f(x) = f(x+1) - f(x) \tag{5}$$

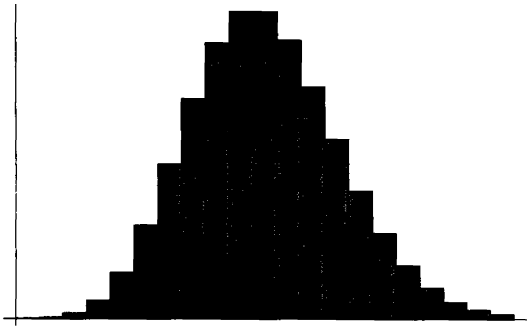
$$\begin{aligned} \Delta^n f(x) &= \Delta[\Delta^{n-1}f(x)]\\ &= f(x+n) - \binom{n}{1}f(x+n-1) + \ldots + (-1)^n f(x). \qquad (6) \end{aligned}$$

see also POISSON-CHARLIER FUNCTION

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 34–35, 1975.

Poisson Distribution



A Poisson distribution is a distribution with the following properties:

1. The number of changes in nonoverlapping intervals are independent for all intervals.
2. The probability of exactly one change in a sufficiently small interval $h \equiv 1/n$ is $P = \nu h \equiv \nu/n$, where ν is the probability of one change and n is the number of TRIALS.
3. The probability of two or more changes in a sufficiently small interval h is essentially 0.

The probability of k changes in a given interval is then given by the limit of the BINOMIAL DISTRIBUTION

$$P(k) = \frac{n!}{k!(n-k)!}\left(\frac{\nu}{n}\right)^k\left(1-\frac{\nu}{n}\right)^{n-k} \tag{1}$$

as the number of trials becomes very large,

$$\lim_{n\to\infty} P(k) =$$

$$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-k-1)}{n^k}\frac{\nu^k}{k!}\left(1-\frac{\nu}{n}\right)^n\left(1-\frac{\nu}{n}\right)^{-k}$$
$$= (1)\left(\frac{\nu^k}{k!}\right)(e^{-\nu})(1) = \frac{\nu^k e^{-\nu}}{k!}. \tag{2}$$

This should be normalized so that the sum of probabilities equals 1. Indeed,

$$\sum_{k=0}^{\infty} P(k) = e^{\nu}\sum_{k=0}^{\infty}\frac{\nu^k}{k!} = e^{\nu}e^{-\nu} = 1, \tag{3}$$

as required. The ratio of probabilities is given by

$$\frac{P(k=i+1)}{P(k=i)} = \frac{\frac{\nu^{i+1}e^{-\nu}}{(i+1)!}}{\frac{i!}{e^{-\nu}\nu^i}} = \frac{\nu}{i+1}. \tag{4}$$

The MOMENT-GENERATING FUNCTION of this distribution is

$$M(t) = \sum_{k=0}^{\infty} e^{tk}\frac{\nu^k e^{-\nu}}{k!} = e^{-\nu}\sum_{k=0}^{\infty}\frac{(\nu e^t)^k}{k!}$$
$$= e^{-\nu}e^{\nu e^t} = e^{\nu(e^t-1)} \tag{5}$$

$$M'(t) = \nu e^t e^{\nu(e^t-1)} \tag{6}$$

$$M''(t) = (\nu e^t)^2 e^{\nu(e^t-1)} + \nu e^t e^{\nu(e^t-1)} \tag{7}$$

$$R(t) \equiv \ln M(t) = \nu(e^t-1) \tag{8}$$

$$R'(t) = \nu e^t \tag{9}$$

$$R''(t) = \nu e^t, \tag{10}$$

so

$$\mu = R'(0) = \nu \tag{11}$$

$$\sigma^2 = R''(0) = \nu. \tag{12}$$

The MOMENTS about zero can also be computed directly

$$\mu_2' = \nu(1+\nu) \tag{13}$$

$$\mu_3' = \nu(1+3\nu+\nu^2) \tag{14}$$

$$\mu_4' = \nu(1+7\nu+6\nu^2+\nu^3), \tag{15}$$

as can the MOMENTS about the MEAN.

$$\mu_1 = \nu \tag{16}$$

$$\mu_2 = \nu \tag{17}$$

$$\mu_3 = \nu \tag{18}$$

$$\mu_4 = \nu(1+3\nu), \tag{19}$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \nu \tag{20}$$

$$\sigma^2 = \nu \tag{21}$$

$$\gamma_1 \equiv \frac{\mu_3}{\sigma^3} = \frac{\nu}{\nu^{3/2}} = \nu^{-1/2} \tag{22}$$

$$\gamma_2 \equiv \frac{\mu_4}{\sigma^4} - 3 = \frac{\nu(1+3\nu)}{\nu} - 3$$
$$= \frac{\nu + 3\nu^2 - 3\nu^2}{\nu^2} = \nu^{-1}. \tag{23}$$

The CHARACTERISTIC FUNCTION is

$$\phi(t) = e^{m(e^{it}-1)} \tag{24}$$

and the CUMULANT-GENERATING FUNCTION is

$$K(h) = \nu(e^h - 1) = \nu(h + \tfrac{1}{2!}h^2 + \tfrac{1}{3!}h^3 + \ldots), \tag{25}$$

so

$$\kappa_r = \nu. \tag{26}$$

The Poisson distribution can also be expressed in terms of

$$\lambda \equiv \frac{\nu}{x}, \tag{27}$$

the rate of changes, so that

$$P(k) = \frac{(\lambda x)^k e^{-\lambda x}}{k!}. \tag{28}$$

The MOMENT-GENERATING FUNCTION of a Poisson distribution in two variables is given by

$$M(t) = e^{(\nu_1+\nu_2)(e^t-1)}. \tag{29}$$

If the independent variables $x_1, x_2, \ldots, x_N$ have Poisson distributions with parameters $\mu_1, \mu_2, \ldots, \mu_N$, then

$$X = \sum_{j=1}^{N} x_j \tag{30}$$

has a Poisson distribution with parameter

$$\mu = \sum_{j=1}^{N} \mu_j. \tag{31}$$

This can be seen since the CUMULANT-GENERATING FUNCTION is

$$K_j(h) = \mu_j(e^h - 1), \tag{32}$$

$$K \equiv \sum_j K_j(h) = (e^h - 1)\sum_j \mu_j = \mu(e^h - 1). \tag{33}$$

References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 532, 1987.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Incomplete Gamma Function, Error Function, Chi-Square Probability Function, Cumulative Poisson Function." §6.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 209–214, 1992.
Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, p. 111–112, 1992.

Poisson's Equation

A second-order PARTIAL DIFFERENTIAL EQUATION arising in physics:

$$\nabla^2\psi = -4\pi\rho.$$

If $\rho = 0$, it reduces LAPLACE'S EQUATION. It is also related to the HELMHOLTZ DIFFERENTIAL EQUATION

$$\nabla^2\psi + k^2\psi = 0.$$

see also HELMHOLTZ DIFFERENTIAL EQUATION, LAPLACE'S EQUATION

References
Arfken, G. "Gauss's Law, Poisson's Equation." §1.14 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 74–78, 1985.

Poisson's Harmonic Function Formula

Let $\phi(z)$ be a HARMONIC FUNCTION. Then

$$\phi(z_0) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta)\phi(z_0 + re^{i\theta})\,d\theta, \tag{1}$$

where $R = |z_0|$ and $K(r,\theta)$ is the POISSON KERNEL. For a CIRCLE,

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(a\cos\phi, a\sin\phi) \frac{a^2 - R^2}{a^2 + R^2 - 2ar\cos(\theta - \phi)}\,d\phi. \tag{2}$$

For a SPHERE,

$$u(x,y,z) = \frac{1}{4\pi a}\iint_S u\frac{a^2 - R^2}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}}\,dS, \tag{3}$$

where

$$\cos\theta \equiv \mathbf{x}\cdot\boldsymbol{\xi}. \tag{4}$$

see also CIRCLE, HARMONIC FUNCTION, POISSON KERNEL, SPHERE

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 373–374, 1953.

Poisson Integral

A.k.a. BESSEL'S SECOND INTEGRAL.

$$J_n(z) = \frac{\left(\frac{1}{2}\right)^n}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi \cos(z\cos\theta)\sin^{2n}\theta\,d\theta,$$

where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND and $\Gamma(x)$ is a GAMMA FUNCTION. It can be derived from SONINE'S INTEGRAL. With $n = 0$, the integral becomes PARSEVAL'S INTEGRAL.

see also BESSEL FUNCTION OF THE FIRST KIND, PARSEVAL'S INTEGRAL, SONINE'S INTEGRAL

Poisson Integral Representation

$$j_n(z) = \frac{z^n}{2^{n+1}n!}\int_0^\pi \cos(z\cos\theta)\sin^{2n+1}\theta\,d\theta,$$

where $j_n(z)$ is a SPHERICAL BESSEL FUNCTION OF THE FIRST KIND.

Poisson Kernel

In 2-D,

$$\begin{aligned} K(r,\theta) &\equiv \Re\left(\frac{R + re^{i\theta}}{R - re^{i\theta}}\right) \\ &= \Re\left[\frac{(R + re^{i\theta})(R - re^{-i\theta})}{(R - re^{i\theta})(R - re^{-i\theta})}\right] \\ &= \Re\left[\frac{R^2 - rR(e^{i\theta} - e^{-i\theta}) - r^2}{R^2 - rR(e^{i\theta} + e^{-i\theta}) + r^2}\right] \\ &= \Re\left[\frac{R^2 + 2irR\sin\theta - r^2}{R^2 - 2Rr\cos\theta + r^2}\right] \\ &= \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}. \end{aligned} \tag{1}$$

In 3-D,

$$u(\mathbf{y}) = \frac{R(R^2 - a^2)}{4\pi} \times \int_0^{2\pi}\int_0^\pi \frac{f(\theta,\phi)\sin\theta\,d\theta\,d\phi}{(R^2 + a^2 - 2aR\cos\gamma)^{3/2}}, \tag{2}$$

where $a = |\mathbf{y}|$ and

$$\cos\gamma = \mathbf{y} \cdot \begin{bmatrix} R\cos\theta\sin\phi \\ R\sin\theta\sin\phi \\ R\cos\phi \end{bmatrix}. \qquad (3)$$

The Poisson kernel for the n-BALL is

$$P(\mathbf{x},\mathbf{z}) = \frac{1}{2-n}(D_\mathbf{n}\mathbf{v})(\mathbf{z}), \qquad (4)$$

where $D_\mathbf{n}$ is the outward normal derivative at point $\mathbf{z}$ on a unit n-SPHERE and

$$\mathbf{v}(\mathbf{z}) = |\mathbf{z}-\mathbf{x}|^{2-n} - |\mathbf{x}|^{2-n}\left|\frac{\mathbf{x}}{|\mathbf{x}|^2}\right|^{2-n}. \qquad (5)$$

see also POISSON'S HARMONIC FUNCTION FORMULA

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Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1090, 1979.

Poisson Manifold

A smooth MANIFOLD with a POISSON BRACKET defined on its FUNCTION SPACE.

Poisson Sum Formula

A special case of the general result

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} e^{2\pi ikx}\int_{-\infty}^{\infty} f(x_1)e^{-2\pi ikx}\,dx_1 \qquad (1)$$

with $x=0$, yielding

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1)e^{-2\pi ikx}\,dx_1. \qquad (2)$$

An alternate form is

$$\frac{1}{2}+\sum_{n=1}^{\infty} e^{-(nx)^2} = \frac{\sqrt{\pi}}{x}\left[\frac{1}{2}+\sum_{n=1}^{\infty} e^{-(n\pi/x)^2}\right]. \qquad (3)$$

Another formula called the Poisson summation formula is

$$\sqrt{\alpha}\,[\tfrac{1}{2}\phi(0)+\phi(\alpha)+\phi(2\alpha)+\ldots] = \sqrt{\beta}[\tfrac{1}{2}\psi(0)+\psi(\beta)+\psi(2\beta)+\ldots], \qquad (4)$$

where

$$\psi(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \psi(t)\cos(xt)\,dt \qquad (5)$$

$$\alpha\beta = 2\pi. \qquad (6)$$

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 466–467, 1953.

Poisson Trials

A number s TRIALS in which the probability of success p_i varies from trial to trial. Let x be the number of successes, then

$$\operatorname{var}(x) = spq - s{\sigma_p}^2, \qquad (1)$$

where ${\sigma_p}^2$ is the VARIANCE of p_i and $q \equiv (1-p)$. Uspensky has shown that

$$P(s,x) = \beta\frac{m^x e^{-m}}{x!}, \qquad (2)$$

where

$$\beta = [1-\theta g(x)]e^{h(x)} \qquad (3)$$

$$g(x) = \frac{(s-x)m^3}{3(s-m)^3} + \frac{x^3}{2s(s-x)} \qquad (4)$$

$$h(x) = \frac{mx}{s} - \frac{m^2}{2s^2}(s-x) - \frac{x(x-1)}{2s} = p\left[\frac{x}{2}\left(1+\frac{1}{m}\right) - \frac{(x-m)^2}{2m}\right] \qquad (5)$$

and $\theta \in (0,1)$. The probability that the number of successes is at least x is given by

$$Q_m(x) = \sum_{r=x}^{\infty}\frac{m^r e^{-m}}{r!}. \qquad (6)$$

Uspensky gives the true probability that there are at least x successes in s trials as

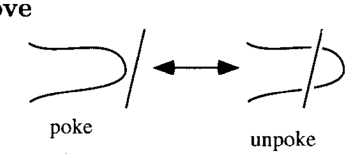
$$P_{ms}(x) = Q_m(x) + \Delta, \qquad (7)$$

where

$$|\Delta| < \begin{cases} (e^\chi - 1)Q_m(x+1) & \text{for } Q_m(x+1) \geq \frac{1}{2} \\ (e^\chi - 1)[1-Q_m(x+1)] & \text{for } Q_m(x+1) \leq \frac{1}{2} \end{cases} \qquad (8)$$

$$\chi = \frac{m+\frac{1}{4}+\frac{m^3}{s}}{2(s-m)}. \qquad (9)$$

Poke Move



The REIDEMEISTER MOVE of type II.

see also REIDEMEISTER MOVES

Poker

Poker is a CARD game played with a normal deck of 52 CARDS. Sometimes, additional cards called "jokers" are also used. In straight or draw poker, each player is normally dealt a hand of five cards. Depending on the variant, players then discard and redraw CARDS, trying to improve their hands. Bets are placed at each discard step. The number of possible distinct five-card hands is

$$N = \binom{52}{5} = 2{,}598{,}960,$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

There are special names for specific types of hands. A royal flush is an ace, king, queen, jack, and 10, all of one suit. A straight flush is five consecutive cards all of the same suit (but not a royal flush), where an ace may count as either high or low. A full house is three-of-a-kind and a pair. A flush is five cards of the same suit (but not a royal flush or straight flush). A straight is five consecutive cards (but not a royal flush or straight flush), where an ace may again count as either high or low.

The probabilities of being dealt five-card poker hands of a given type (before discarding and with no jokers) on the initial deal are given below (Packel 1981). As usual, for a hand with probability P, the ODDS against being dealt it are $(1/r) - 1 : 1$.

Hand	Exact Probability
royal flush	$\frac{4}{N} = \frac{1}{649{,}740}$
straight flush	$\frac{4(10)-4}{N} = \frac{3}{216{,}580}$
four of a kind	$\frac{13(48)}{N} = \frac{1}{4{,}165}$
full house	$\frac{13\binom{4}{3}12\binom{4}{2}}{N} = \frac{6}{4{,}165}$
flush	$\frac{4\binom{13}{5}-36-4}{N} = \frac{1{,}277}{649{,}740}$
straight	$\frac{10(4^5)-36-4}{N} = \frac{5}{1{,}274}$
three of a kind	$\frac{13\binom{4}{3}\frac{(48)(44)}{2!}}{N} = \frac{88}{4{,}165}$
two pair	$\frac{\frac{13\binom{4}{2}12\binom{4}{2}}{2!}44}{N} = \frac{198}{4{,}165}$
one pair	$\frac{13\binom{4}{2}\frac{(48)(44)(40)}{3!}}{N} = \frac{352}{833}$

Hand	Probability	Odds
royal flush	1.54×10^{-6}	649,739.0:1
straight flush	1.39×10^{-5}	72,192.3:1
four of a kind	2.40×10^{-4}	4,164.0:1
full house	1.44×10^{-3}	693.2:1
flush	1.97×10^{-3}	507.8:1
straight	3.92×10^{-3}	253.8:1
three of a kind	0.0211	46.3:1
two pair	0.0475	20.0:1
one pair	0.423	1.366:1

Gadbois (1996) gives probabilities for hands if two jokers are included, and points out that it is *impossible* to rank hands in any single way which is consistent with the relative frequency of the hands.

see also BRIDGE CARD GAME, CARDS

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Conway, J. H. and Guy, R. K. "Choice Numbers with Repetitions." In *The Book of Numbers.* New York: Springer-Verlag, pp. 70–71, 1996.

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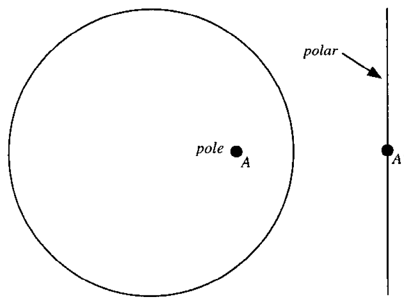
Polar



If two points A and A' are INVERSE with respect to a CIRCLE (the INVERSION CIRCLE), then the straight line through A' which is PERPENDICULAR to the line of the points AA' is called the polar of A with respect to the CIRCLE, and A is called the POLE of the polar.

see also APOLLONIUS' PROBLEM, INVERSE POINTS, INVERSION CIRCLE, POLARITY, POLE, TRILINEAR POLAR

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Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 157, 1965.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 100–106, 1929.

Polar Angle

The ANGLE a point makes from the ORIGIN as measured from the x-AXIS.

see also POLAR COORDINATES

Polar Circle

Given a TRIANGLE, the polar circle has center at the ORTHOCENTER H. Call H_i the FEET of the ALTITUDE. Then the RADIUS is

$$r^2 = \overline{HA_1}\cdot\overline{HH_1} = \overline{HA_2}\cdot\overline{HH_2} = \overline{HA_2}\cdot\overline{HH_2} \quad (1)$$

$$= -4R^2 \cos\alpha_1 \cos\alpha_2 \cos\alpha_3 \quad (2)$$

$$= \tfrac{1}{2}({a_1}^2 + {a_2}^2 + {a_3}^2) - 4R^2, \quad (3)$$

where R is the CIRCUMRADIUS, α_i the VERTEX angles, and a_i the corresponding side lengths.

A TRIANGLE is self-conjugate with respect to its polar circle. Also, the RADICAL AXIS of any two polar circles is the ALTITUDE from the third VERTEX. Any two polar circles of an ORTHOCENTRIC SYSTEM are orthogonal. The polar circles of the triangles of a COMPLETE QUADRILATERAL constitute a COAXAL SYSTEM conjugate to that of the circles on the diagonals.

see also COAXAL SYSTEM, ORTHOCENTRIC SYSTEM, POLAR, POLE, RADICAL AXIS

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Polar Coordinates

The polar coordinates r and θ are defined by

$$x = r\cos\theta \tag{1}$$

$$y = r\sin\theta. \tag{2}$$

In terms of x and y,

$$r = \sqrt{x^2+y^2} \tag{3}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{4}$$

The ARC LENGTH of a polar curve given by $r = r(\theta)$ is

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{5}$$

The LINE ELEMENT is given by

$$ds^2 = r^2\, d\theta^2, \tag{6}$$

and the AREA element by

$$dA = r\, dr\, d\theta. \tag{7}$$

The AREA enclosed by a polar curve $r = r(\theta)$ is

$$A = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta. \tag{8}$$

The SLOPE of a polar function $r = r(\theta)$ at the point (r, θ) is given by

$$m = \frac{r + \tan\theta \frac{dr}{d\theta}}{-r\tan\theta + \frac{dr}{d\theta}}. \tag{9}$$

The ANGLE between the tangent and radial line at the point (r, θ) is

$$\psi = \tan^{-1}\left(\frac{r}{\frac{dr}{d\theta}}\right). \tag{10}$$

A polar curve is symmetric about the x-axis if replacing θ by $-\theta$ in its equation produces an equivalent equation, symmetric about the y-axis if replacing θ by $\pi - \theta$ in its equation produces an equivalent equation, and symmetric about the origin if replacing r by $-r$ in its equation produces an equivalent equation.

In Cartesian coordinates, the POSITION VECTOR and its derivatives are

$$\mathbf{r} = \sqrt{x^2+y^2}\,\hat{\mathbf{r}} \tag{11}$$

$$\dot{\mathbf{r}} = \dot{\hat{\mathbf{r}}}\sqrt{x^2+y^2} + \hat{\mathbf{r}}\,(x^2+y^2)^{-1/2}(x\dot{x}+y\dot{y}) \tag{12}$$

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2+y^2}} \tag{13}$$

$$\begin{aligned}\dot{\hat{\mathbf{r}}} &= \frac{\dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}}}{\sqrt{x^2+y^2}} \\ &\quad - \tfrac{1}{2}(x^2+y^2)^{-3/2}(2)(x\dot{x}+y\dot{y})(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}) \\ &= \frac{(x\dot{y}-y\dot{x})(x\hat{\mathbf{y}}-y\hat{\mathbf{x}})}{(x^2+y^2)^{3/2}}.\end{aligned} \tag{14}$$

In polar coordinates, the UNIT VECTORS and their derivatives are

$$\mathbf{r} \equiv \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \tag{15}$$

$$\hat{\mathbf{r}} \equiv \frac{\frac{d\mathbf{r}}{dr}}{\left|\frac{d\mathbf{r}}{dr}\right|} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{16}$$

$$\hat{\boldsymbol{\theta}} \equiv \frac{\frac{d\boldsymbol{\theta}}{d\theta}}{\left|\frac{d\boldsymbol{\theta}}{d\theta}\right|} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{17}$$

$$\dot{\hat{\mathbf{r}}} = \begin{bmatrix} -\sin\theta\dot{\theta} \\ \cos\theta\dot{\theta} \end{bmatrix} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \tag{18}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} -\cos\theta\dot{\theta} \\ -\sin\theta\dot{\theta} \end{bmatrix} = -\dot{\theta}\hat{\mathbf{r}} \tag{19}$$

$$\dot{\mathbf{r}} = \begin{bmatrix} -r\sin\theta\dot{\theta} + \cos\theta\dot{r} \\ r\cos\theta\dot{\theta} + \sin\theta\dot{r} \end{bmatrix} = r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}} \tag{20}$$

$$\begin{aligned}\ddot{\mathbf{r}} &= \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} + \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}} \\ &= \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}}) + \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned} \tag{21}$$

see also CARDIOID, CIRCLE, CISSOID, CONCHOID, CURVILINEAR COORDINATES, CYLINDRICAL COORDINATES, EQUIANGULAR SPIRAL, LEMNISCATE, LIMAÇON, ROSE

Polar Line

see POLAR

Polarity

A projective CORRELATION of period two. In a polarity, a is called the POLAR of A, and A the POLE a.

see also CHASLES'S THEOREM, CORRELATION, POLAR, POLE (GEOMETRY)

Pole

A COMPLEX function f has a pole of order m at z_0 if, in the LAURENT SERIES, $a_n = 0$ for $n < -m$ and $a_m \neq 0$. Equivalently, f has a pole of order n at z_0 if n is the smallest POSITIVE INTEGER for which $(z - z_0)^n f(z)$ is differentiable at z_0. If $f(\pm\infty) \neq \pm\infty$, there is no pole at $\pm\infty$. Otherwise, the order of the pole is the greatest POSITIVE COEFFICIENT in the LAURENT SERIES.

This is equivalent to finding the smallest n such that

$$\frac{(z - z_0)^n}{f(z)}$$

is differentiable at 0.

see also LAURENT SERIES, RESIDUE (COMPLEX ANALYSIS)

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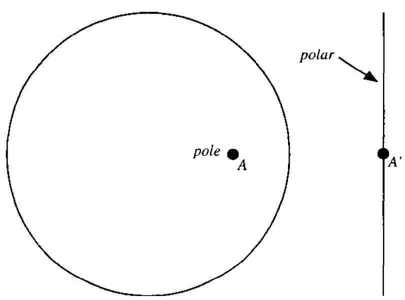
Pole (Geometry)



If two points A and A' are INVERSE with respect to a CIRCLE (the INVERSION CIRCLE), then the straight line through A' which is PERPENDICULAR to the line of the points AA' is called the POLAR of the A with respect to the CIRCLE, and A is called the pole of the POLAR.

see also INVERSE POINTS, INVERSION CIRCLE, POLAR, POLARITY, TRILINEAR POLAR

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Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 157, 1965.

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Pole (Origin)

see ORIGIN

Policeman on Point Duty Curve

see CRUCIFORM

Polignac's Conjecture

see DE POLIGNAC'S CONJECTURE

Polish Space

The HOMEOMORPHIC image of a so-called "complete separable" METRIC SPACE. The continuous image of a Polish space is called a SOUSLIN SET.

see also DESCRIPTIVE SET THEORY, STANDARD SPACE

Pollaczek Polynomial

Let $a > |b|$, and write

$$h(\theta) = \frac{a\cos\theta + b}{2\sin\theta}. \tag{1}$$

Then define $P_n(x; a, b)$ by the GENERATING FUNCTION

$$\begin{aligned} f(x,w) = f(\cos\theta, w) &= \sum_{n=0}^{\infty} P_n(x;a,b) w^n \\ &= (1 - we^{i\theta})^{-1/2+ih(\theta)}(1 - we^{i\theta})^{-1/2-ih(\theta)}. \end{aligned} \tag{2}$$

The GENERATING FUNCTION may also be written

$$f(x,w) = (1 - 2xw + w^2)^{-1/2} \exp\left[(ax+b)\sum_{m=1}^{\infty} \frac{w^m}{m} U_{m-1}(x)\right], \tag{3}$$

where $U_m(x)$ is a CHEBYSHEV POLYNOMIAL OF THE SECOND KIND. They satisfy the RECURRENCE RELATION

$$nP_n(x;a,b) = [(2n - 1 + 2a)x + 2b]P_{n-1}(x;a,b) - (n-1)P_{n-2}(x;a,b) \tag{4}$$

for $n = 2, 3, \ldots$ with

$$P_0 = 1 \tag{5}$$

$$P_1 = (2a+1)x + 2b. \tag{6}$$

In terms of the HYPERGEOMETRIC FUNCTION ${}_2F_1(a,b;c;x)$,

$$P_n(\cos\theta; a; b) = e^{in\theta}{}_2F_1(-n, \tfrac{1}{2} + ih(\theta); 1; 1 - e^{-2i\theta}). \tag{7}$$

They obey the orthogonality relation

$$\int_{-1}^{1} P_n(x;a,b)P_m(x;a,b)w(x;a,b)\,dx = [n + \tfrac{1}{2}(a+1)]^{-1}\delta_{nm}, \tag{8}$$

where δ_{nm} is the KRONECKER DELTA, for $n, m = 0, 1, \ldots$, with the WEIGHT FUNCTION

$$w(\cos\theta; a, b) = e^{(2\theta-\pi)h(\theta)}\{\cosh[\pi h(\theta)]\}^{-1}. \tag{9}$$

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 393–400, 1975.

Pollard Monte Carlo Factorization Method

see POLLARD ρ FACTORIZATION METHOD

Pollard $p-1$ Factorization Method

A PRIME FACTORIZATION ALGORITHM which can be implemented in a single-step or double-step form. In the single-step version, PRIMES p are found if $p-1$ is a product of small PRIMES by finding an m such that

$$m \equiv c^q \pmod{n},$$

where $p-1|q$, with q a large number and $(c,n)=1$. Then since $p-1|q$, $m \equiv 1 \pmod{p}$, so $p|m-1$. There is therefore a good chance that $n \nmid m-1$, in which case $\mathrm{GCD}(m-1,n)$ (where GCD is the GREATEST COMMON DIVISOR) will be a nontrivial divisor of n.

In the double-step version, a PRIMES p can be factored if $p-1$ is a product of small PRIMES and a single larger PRIME.

see also PRIME FACTORIZATION ALGORITHMS, WILLIAMS $p+1$ FACTORIZATION METHOD

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Bressoud, D. M. *Factorization and Prime Testing.* New York: Springer-Verlag, pp. 67–69, 1989.

Pollard, J. M. "Theorems on Factorization and Primality Testing." *Proc. Cambridge Phil. Soc.* **76**, 521–528, 1974.

Pollard ρ Factorization Method

A PRIME FACTORIZATION ALGORITHM also known as POLLARD MONTE CARLO FACTORIZATION METHOD. Let $x_0 = 2$, then compute

$$x_{i+1} = x_i{}^2 - x_i + 1 \pmod{n}.$$

If $\mathrm{GCD}(x_{2i} - x_i, n) > 1$, then n is COMPOSITE and its factors are found. In modified form, it becomes BRENT'S FACTORIZATION METHOD. In practice, almost any unfactorable POLYNOMIAL can be used for the iteration ($x^2 - 2$, however, cannot). Under worst conditions, the ALGORITHM can be very slow.

see also BRENT'S FACTORIZATION METHOD, PRIME FACTORIZATION ALGORITHMS

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Poloidal Field

A VECTOR FIELD resembling a magnetic multipole which has a component along the z-AXIS of a SPHERE and continues along lines of LONGITUDE.

see also DIVERGENCELESS FIELD, TOROIDAL FIELD

References

Stacey, F. D. *Physics of the Earth, 2nd ed.* New York: Wiley, p. 239, 1977.

Pólya-Burnside Lemma

see PÓLYA ENUMERATION THEOREM

Pólya Conjecture

Let n be a POSITIVE INTEGER and $r(n)$ the number of (not necessarily distinct) PRIME FACTORS of n (with $r(1) = 0$). Let $O(m)$ be the number of POSITIVE INTEGERS $\leq m$ with an ODD number of PRIME factors, and $E(m)$ the number of POSITIVE INTEGERS $\leq m$ with an EVEN number of PRIME factors. Pólya conjectured that

$$L(m) \equiv E(m) - O(m) = \sum_{n=1}^{m} \lambda(n)$$

is ≤ 0, where $\lambda(n)$ is the LIOUVILLE FUNCTION.

The conjecture was made in 1919, and disproven by Haselgrove (1958) using a method due to Ingham (1942). Lehman (1960) found the first explicit counterexample, $L(906{,}180{,}359) = 1$, and the smallest counterexample $m = 906{,}150{,}257$ was found by Tanaka (1980). The first n for which $L(n) = 0$ are $n = 2$, 4, 6, 10, 16, 26, 40, 96, 586, 906150256, ... (Tanaka 1980, Sloane's A028488). It is unknown if $L(x)$ changes sign infinitely often (Tanaka 1980).

see also ANDRICA'S CONJECTURE, LIOUVILLE FUNCTION, PRIME FACTORS

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Ingham, A. E. "On Two Conjectures in the Theory of Numbers." *Amer. J. Math.* **64**, 313–319, 1942.

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Pólya Distribution

see NEGATIVE BINOMIAL DISTRIBUTION

Pólya Enumeration Theorem

A very general theorem which allows the number of discrete combinatorial objects of a given type to be enumerated (counted) as a function of their "order." The most common application is in the counting of the number of GRAPHS of n nodes, TREES and ROOTED TREES with n branches, GROUPS of order n, etc. The theorem is an extension of BURNSIDE'S LEMMA and is sometimes also called the PÓLYA-BURNSIDE LEMMA.

see also BURNSIDE'S LEMMA, GRAPH (GRAPH THEORY), GROUP, ROOTED TREE, TREE

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Pólya Polynomial

The POLYNOMIAL giving the number of colorings, with m colors, of a structure defined by a PERMUTATION GROUP.

see also PERMUTATION GROUP, PÓLYA ENUMERATION THEOREM

Pólya's Random Walk Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $p(d)$ be the probability that a RANDOM WALK on a d-D lattice returns to the origin. Pólya (1921) proved that

$$p(1) = p(2) = 1, \tag{1}$$

but

$$p(d) < 1 \tag{2}$$

for $d > 2$. Watson (1939), McCrea and Whipple (1940), Domb (1954), and Glasser and Zucker (1977) showed that

$$p(3) = 1 - \frac{1}{u(3)} = 0.3405373296\ldots, \tag{3}$$

where

$$u(3) = \frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dx\,dy\,dz}{3-\cos x-\cos y-\cos z} \tag{4}$$

$$= \frac{12}{\pi^2}(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})$$
$$\times\{K[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})]\}^2 \tag{5}$$
$$= 3(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})$$
$$\times\left[1+2\sum_{k=1}^{\infty}\exp(-k^2\pi\sqrt{6})\right]^4 \tag{6}$$
$$= \frac{\sqrt{6}}{32\pi^3}\Gamma(\tfrac{1}{24})\Gamma(\tfrac{5}{24})\Gamma(\tfrac{7}{24})\Gamma(\tfrac{11}{24}) \tag{7}$$
$$= 1.5163860592\ldots, \tag{8}$$

where $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND and $\Gamma(z)$ is the GAMMA FUNCTION. Closed forms for $d > 3$ are not known, but Montroll (1956) showed that

$$p(d) = 1 - [u(d)]^{-1}, \tag{9}$$

where

$$u(d) = \frac{d}{(2\pi)^d}\underbrace{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}}_{d}\left(d-\sum_{k=1}^{d}\cos x_k\right)^{-1}$$
$$\times\, dx_1\,dx_2\cdots dx_d$$
$$= \int_0^\infty\left[I_0\left(\frac{t}{d}\right)\right]^d e^{-t}\,dt, \tag{10}$$

and $I_0(z)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND. Numerical values from Montroll (1956) and Flajolet (Finch) are

d	$p(d)$
4	0.20
5	0.136
6	0.105
7	0.0858
8	0.0729

see also RANDOM WALK

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Glasser, M. L. and Zucker, I. J. "Extended Watson Integrals for the Cubic Lattices." *Proc. Nat. Acad. Sci. U.S.A.* **74**, 1800–1801, 1977.

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Montroll, E. W. "Random Walks in Multidimensional Spaces, Especially on Periodic Lattices." *J. SIAM* **4**, 241–260, 1956.

Watson, G. N. "Three Triple Integrals." *Quart. J. Math., Oxford Ser. 2* **10**, 266–276, 1939.

Pólya-Vinogradov Inequality

Let χ be a nonprincipal character (mod q). Then

$$\sum_{n=M+1}^{M+N}\chi(n) \ll \sqrt{q}\ln q,$$

where $\ll$ indicates MUCH LESS than.

References

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Polyabolo

An analog of the POLYOMINO composed of n ISOSCELES RIGHT TRIANGLES joined along edges of the same length. The number of polyaboloes composed of n triangles are 1, 3, 4, 14, 30, 107, 318, 1106, 3671, ... (Sloane's A006074).

see also DIABOLO, HEXABOLO, PENTABOLO, TETRABOLO, TRIABOLO

References

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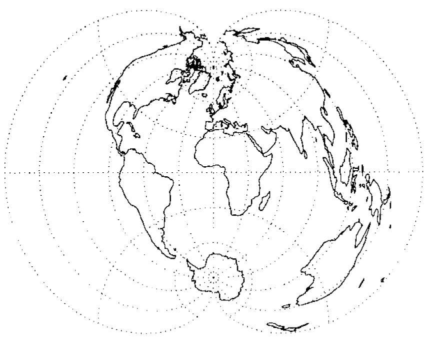
Polyconic Projection



$$x = \cot\phi \sin E \tag{1}$$

$$y = (\phi - \phi_0) + \cot\phi(1 - \cos E), \tag{2}$$

where

$$E = (\lambda - \lambda_0)\sin\phi. \tag{3}$$

The inverse FORMULAS are

$$\lambda = \frac{\sin^{-1}(x \tan\phi)}{\sin\phi} + \lambda_0, \tag{4}$$

and ϕ is determined from

$$\Delta\phi = -\frac{A(\phi\tan\phi + 1) - \phi - \frac{1}{2}(\phi^2 + B)\tan\phi}{\frac{\phi - A}{\tan\phi} - 1}, \tag{5}$$

where $\phi_0 = A$ and

$$A = \phi_0 + y \tag{6}$$

$$B = x^2 + A^2. \tag{7}$$

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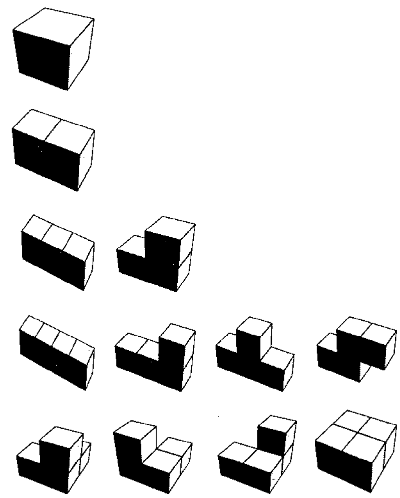
Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 124–137, 1987.

Polycube



3-D generalization of the POLYOMINOES to n-D. The number of polycubes $N(n)$ composed of n CUBES are 1, 1, 2, 8, 29, 166, 1023, ... (Sloane's A000162, Ball and Coxeter 1987).

see also CONWAY PUZZLE, CUBE DISSECTION, DIABOLICAL CUBE, SLOTHOUBER-GRAATSMA PUZZLE, SOMA CUBE

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 112–113, 1987.

Gardner, M. *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 76–77, 1961.

Gardner, M. "Polycubes." Ch. 3 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, 1986.

Sloane, N. J. A. Sequence A000162/M1845 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Polydisk

Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a point in $\mathbb{C}^n$, then the open polydisk is defined by

$$S = \{z : |z_j - c_j| < |z_j^0 - c_j|\}$$

for $j = 1, \ldots, n$.

see also DISK, OPEN DISK

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 100, 1980.

Polygamma Function

The polygamma function is sometimes denoted $F_m(z)$, and sometimes $\psi_m(z)$. In $F_m(z)$ notation,

$$F_m(z) \equiv \frac{d^{m+1}}{dz^{m+1}} \ln z! \tag{1}$$

$$= (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(z+n)^{m+1}} \tag{2}$$

$$= (-1)^{m+1} m! \zeta(m+1, z), \tag{3}$$

where $\zeta(a, z)$ is the HURWITZ ZETA FUNCTION. In the ψ_m NOTATION (the form returned by the `PolyGamma[m,z]` function in *Mathematica*®; Wolfram Research, Champaign, IL),

$$\psi_m(z) = \frac{d^{m+1}}{dz^{m+1}} \ln[\Gamma(z)] = \frac{d^m}{dz^m} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d^m}{dz^m} \Psi(z), \tag{4}$$

where $\Gamma(z)$ is the GAMMA FUNCTION and $\Psi(z)$ is the DIGAMMA FUNCTION. $\psi_m(z)$ is therefore related to $F_m(z)$ by

$$\psi_m(z) = F_m(z-1). \tag{5}$$

The function $\psi_0(z)$ is equivalent to the DIGAMMA FUNCTION $\Psi(z)$. Note that Morse and Feshbach (1953) adopt a notation no longer in standard use in which Morse and Feshbach's $\psi_m(z)$ is equal to the above $\psi_{m-1}(z)$.

The polygamma function obeys the RECURRENCE RELATION

$$\psi_n(z+1) = \psi_n(z) + (-1)^n n! z^{-n-1}, \tag{6}$$

the reflection FORMULA

$$\psi_n(1-z) + (-1)^{n+1}\psi_n(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot(\pi z), \tag{7}$$

and the multiplication FORMULA,

$$\psi_n(mz) = \delta_{n0} \ln m + \frac{1}{m^{n+1}} \sum_{k=1}^{m-1} \psi_n\left(z + \frac{k}{m}\right), \tag{8}$$

where δ_{mn} is the KRONECKER DELTA.

In general, special values for integral indices are given by

$$\psi_n(1) = (-1)^{n+1} n! \zeta(n+1) \tag{9}$$

$$\psi_n(\tfrac{1}{2}) = (-1)^{n+1} n! (2^{n+1} - 1)\zeta(n+1), \tag{10}$$

giving

$$\psi_1(\tfrac{1}{2}) = \tfrac{1}{2}\pi^2 \tag{11}$$

$$\psi_1(1) = \zeta(2) = \tfrac{1}{6}\pi^2 \tag{12}$$

$$\psi_2(1) = -2\zeta(3), \tag{13}$$

$$\psi_3(\tfrac{1}{2}) = \pi^4 \tag{14}$$

and so on.

R. Manzoni has shown that the polygamma function can be expressed in terms of CLAUSEN FUNCTIONS for RATIONAL arguments and integer index. Special cases are given by

$$\psi_1(\tfrac{1}{3}) = \tfrac{2}{3}\pi^2 + \tfrac{3}{2}\sqrt{3}[\mathrm{Cl}_2(\tfrac{2}{3}\pi) - \mathrm{Cl}_2(\tfrac{4}{3}\pi)] \tag{15}$$

$$\psi_1(\tfrac{2}{3}) = \tfrac{2}{3}\pi^2 - \tfrac{3}{2}\sqrt{3}[\mathrm{Cl}_2(\tfrac{2}{3}\pi) - \mathrm{Cl}_2(\tfrac{4}{3}\pi)] \tag{16}$$

$$\psi_1(\tfrac{1}{4}) = \pi^2 + 4[\mathrm{Cl}_2(\tfrac{1}{2}\pi) - \mathrm{Cl}_2(\tfrac{3}{2}\pi)] \tag{17}$$

$$\psi_1(\tfrac{3}{4}) = \pi^2 - 4[\mathrm{Cl}_2(\tfrac{1}{2}\pi) - \mathrm{Cl}_2(\tfrac{3}{2}\pi)]. \tag{18}$$

$$\psi_2(\tfrac{1}{2}) = -8[\mathrm{Cl}_3(0) - \mathrm{Cl}_3(\pi)]. \tag{19}$$

$$\psi_2(\tfrac{1}{3}) = -\frac{4\pi^3}{3\sqrt{3}} - 18\,\mathrm{Cl}_3(0) + 9[\mathrm{Cl}_3(\tfrac{2}{3}\pi) + \mathrm{Cl}_3(\tfrac{4}{3}\pi)] \tag{20}$$

$$\psi_2(\tfrac{2}{3}) = \frac{4\pi^3}{3\sqrt{3}} - 18\,\mathrm{Cl}_3(0) + 9[\mathrm{Cl}_3(\tfrac{2}{3}\pi) + \mathrm{Cl}_3(\tfrac{4}{3}\pi)] \tag{21}$$

$$\psi_2(\tfrac{1}{4}) = -2\pi^3 - 32[\mathrm{Cl}_3(0) - \mathrm{Cl}_3(\pi)] \tag{22}$$

$$\psi_3(\tfrac{3}{4}) = 2\pi^3 - 32[\mathrm{Cl}_3(0) - \mathrm{Cl}_3(\pi)] \tag{23}$$

$$\psi_3(\tfrac{1}{3}) = \tfrac{8}{3}\pi^4 + 81\sqrt{3}[\mathrm{Cl}_4(\tfrac{2}{3}\pi) - \mathrm{Cl}_4(\tfrac{4}{3}\pi)] \tag{24}$$

$$\psi_3(\tfrac{2}{3}) = \tfrac{8}{3}\pi^4 - 81\sqrt{3}[\mathrm{Cl}_4(\tfrac{2}{3}\pi) - \mathrm{Cl}_4(\tfrac{4}{3}\pi)] \tag{25}$$

$$\psi_3(\tfrac{1}{4}) = 8\pi^4 + 384[\mathrm{Cl}_4(\tfrac{1}{2}\pi) - \mathrm{Cl}_4(\tfrac{3}{2}\pi)] \tag{26}$$

$$\psi_3(\tfrac{3}{4}) = 8\pi^4 - 384[\mathrm{Cl}_4(\tfrac{1}{2}\pi) - \mathrm{Cl}_4(\tfrac{3}{2}\pi)]. \tag{27}$$

see also CLAUSEN FUNCTION, DIGAMMA FUNCTION, GAMMA FUNCTION, STIRLING'S SERIES

References

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Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 422–424, 1953.

Polygenic Function

A function which has infinitely many DERIVATIVES at a point. If a function is not polygenic, it is MONOGENIC.

see also MONOGENIC FUNCTION

References

Newman, J. R. *The World of Mathematics, Vol. 3.* New York: Simon & Schuster, p. 2003, 1956.

Polygon

A closed plane figure with n sides. If all sides and angles are equivalent, the polygon is called regular. Regular polygons can be CONVEX or STAR. The word derives from the Greek *poly* (many) and *gonu* (knee).

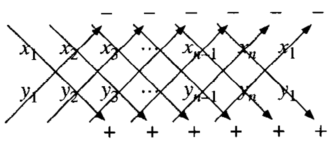
The AREA of a polygon with VERTICES (x_1, y_1), ..., (x_n, y_n) is

$$A = \frac{1}{2}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix} \right), \tag{1}$$

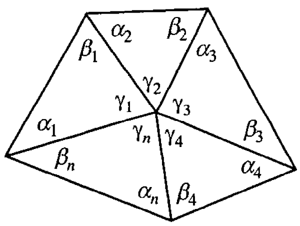
which can be written

$$A = \tfrac{1}{2}(x_1y_2 + x_2y_1 + \ldots + x_{n-1}y_n + x_ny_1 - y_1x_2 - y_2x_3 - \ldots - y_{n+1}x_n - y_nx_1), \tag{2}$$

where the signs can be found from the following diagram.



The AREA of a polygon is defined to be POSITIVE if the points are arranged in a counterclockwise order, and NEGATIVE if they are in clockwise order (Beyer 1987).



The sum I of internal angles in the above diagram of a dissected PENTAGON is

$$I = \sum_{i=1}^{n}(\alpha_i + \beta_i) = \sum_{i=1}^{n}(\alpha_i + \beta_i + \gamma_i) - \sum_{i=1}^{n}\gamma_i. \tag{3}$$

But

$$\sum_{i=1}^{n}\gamma_i = 360° \tag{4}$$

and the sum of ANGLES of the n TRIANGLES is

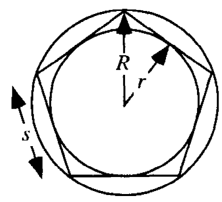
$$\sum_{i=1}^{n}(\alpha_i + \beta_i + \gamma_i) = \sum_{i=1}^{n}(180°) = n(180°). \tag{5}$$

Therefore,

$$I = n(180°) - 360° = (n-2)180°. \tag{6}$$

Let n be the number of sides. The *regular* n-gon is then denoted $\{n\}$.

n	$\{n\}$
2	digon
3	equilateral triangle (trigon)
4	square (quadrilateral, tetragon)
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon (enneagon)
10	decagon
11	undecagon (hendecagon)
12	dodecagon
13	tridecagon (triskaidecagon)
14	tetradecagon (tetrakaidecagon)
15	pentadecagon (pentakaidecagon)
16	hexadecagon (hexakaidecagon)
17	heptadecagon (heptakaidecagon)
18	octadecagon (octakaidecagon)
19	enneadecagon (enneakaidecagon)
20	icosagon
30	triacontagon
40	tetracontagon
50	pentacontagon
60	hexacontagon
70	heptacontagon
80	octacontagon
90	enneacontagon
100	hectogon
10000	myriagon



$n = 5$

Let s be the side length, r be the INRADIUS, and R the CIRCUMRADIUS. Then

$$s = 2r\tan\left(\frac{\pi}{n}\right) = 2R\sin\left(\frac{\pi}{n}\right) \tag{7}$$

$$r = \tfrac{1}{2}s\cot\left(\frac{\pi}{n}\right) \tag{8}$$

$$R = \tfrac{1}{2}s\csc\left(\frac{\pi}{n}\right) \tag{9}$$

$$A = \tfrac{1}{4}ns^2\cot\left(\frac{\pi}{n}\right) \tag{10}$$

$$= nr^2\tan\left(\frac{\pi}{n}\right) \tag{11}$$

$$= \tfrac{1}{2}nR^2\sin\left(\frac{2\pi}{n}\right). \tag{12}$$

If the number of sides is doubled, then

$$s_{2n} = \sqrt{2R^2 - R\sqrt{4R^2 - {s_n}^2}} \tag{13}$$

$$A_{2n} = \frac{4rA_n}{2r + \sqrt{4r^2 + {s_n}^2}}. \tag{14}$$

Furthermore, if p_k and P_k are the PERIMETERS of the regular polygons inscribed in and circumscribed around a given CIRCLE and a_k and A_k their areas, then

$$P_{2n} = \frac{2p_n P_n}{p_n + P_n} \tag{15}$$

$$p_{2n} = \sqrt{p_n P_{2n}}, \tag{16}$$

and

$$a_{2n} = \sqrt{a_n A_n} \tag{17}$$

$$A_{2n} = \frac{2a_{2n} A_n}{a_{2n} + A_n} \tag{18}$$

(Beyer 1987, p. 125).

COMPASS and STRAIGHTEDGE constructions dating back to Euclid were capable of inscribing regular polygons of 3, 4, 5, 6, 8, 10, 12, 16, 20, 24, 32, 40, 48, 64, ..., sides. However, this listing is not a complete enumeration of "constructible" polygons. In fact, a regular n-gon is constructible only if $\phi(n)$ is a POWER of 2, where ϕ is the TOTIENT FUNCTION (this is a NECESSARY but not SUFFICIENT condition). More specifically, a regular n-gon ($n \geq 3$) can be constructed by STRAIGHTEDGE and COMPASS (i.e., can have trigonometric functions of its ANGLES expressed in terms of finite SQUARE ROOT extractions) IFF

$$n = 2^k p_1 p_2 \cdots p_s, \tag{19}$$

where k is in INTEGER ≥ 0 and the p_i are distinct FERMAT PRIMES. FERMAT NUMBERS are of the form

$$F_m = 2^{2^m} + 1, \tag{20}$$

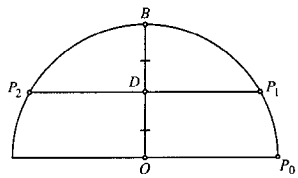
where m is an INTEGER ≥ 0. The only known PRIMES of this form are 3, 5, 17, 257, and 65537.

The fact that this condition was SUFFICIENT was first proved by Gauss in 1796 when he was 19 years old, and it relies on the property of IRREDUCIBLE POLYNOMIALS that ROOTS composed of a finite number of SQUARE ROOT extractions exist only if the order of the equation is of the form 2^h. That this condition was also NECESSARY was not explicitly proven by Gauss, and the first proof of this fact is credited to Wantzel (1836).

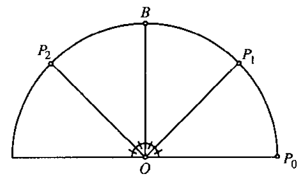
Constructible values of n for $n < 300$ were given by Gauss (Smith 1994), and the first few are 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 136, 160, 170, 192, ... (Sloane's A003401). Gardner (1977) and independently Watkins (Conway and Guy 1996) noticed that the number of sides for constructible polygons with an ODD number of sides is given by the first 32 rows of PASCAL'S TRIANGLE (mod 2) interpreted as BINARY numbers, giving 1, 3, 5, 15, 17, 51, 85, 255, ... (Sloane's A004729, Conway and Guy 1996, p. 140).

1	1	1
1 1	1 1	3
1 2 1	1 0 1	5
1 3 3 1	1 1 1 1	15
1 4 6 4 1	1 0 0 0 1	17
1 5 10 10 5 1	1 1 0 0 1 1	51
1 6 15 20 15 6 1	1 0 1 0 1 0 1	85
1 7 21 35 35 21 7 1	1 1 1 1 1 1 1 1	255
1 8 28 56 70 56 28 8 1	1 0 0 0 0 0 0 0 1	257

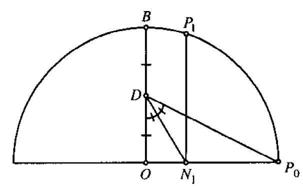
Although constructions for the regular TRIANGLE, SQUARE, PENTAGON, and their derivatives had been given by Euclid, constructions based on the FERMAT PRIMES $\geq$ 17 were unknown to the ancients. The first explicit construction of a HEPTADECAGON (17-gon) was given by Erchinger in about 1800. Richelot and Schwendenwein found constructions for the 257-GON in 1832, and Hermes spent 10 years on the construction of the 65537-GON at Göttingen around 1900 (Coxeter 1969). Constructions for the EQUILATERAL TRIANGLE and SQUARE are trivial (top figures below). Elegant constructions for the PENTAGON and HEPTADECAGON are due to Richmond (1893) (bottom figures below).



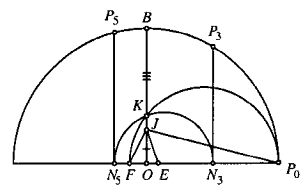
Equilateral Triangle



Square



Pentagon



17-gon

Given a point, a CIRCLE may be constructed of any desired RADIUS, and a DIAMETER drawn through the center. Call the center O, and the right end of the DIAMETER P_0. The DIAMETER PERPENDICULAR to the original DIAMETER may be constructed by finding the PERPENDICULAR BISECTOR. Call the upper endpoint of this PERPENDICULAR DIAMETER B. For the PENTAGON, find the MIDPOINT of OB and call it D. Draw DP_0, and BISECT $\angle ODP_0$, calling the intersection point with OP_0 N_1. Draw N_1P_1 PARALLEL to OB, and the first two points of the PENTAGON are P_0 and P_1. The construction for the HEPTADECAGON is more complicated, but can be accomplished in 17 relatively simple steps. The construction problem has now been automated (Bishop 1978).

see also 257-GON, 65537-GON, ANTHROPOMORPHIC POLYGON, BICENTRIC POLYGON, CARNOT'S POLYGON THEOREM, CHAOS GAME, CONVEX POLYGON, CYCLIC POLYGON, DE MOIVRE NUMBER, DIAGONAL (POLYGON), EQUILATERAL TRIANGLE, EULER'S POLYGON DIVISION PROBLEM, HEPTADECAGON, HEXAGON,

HEXAGRAM, ILLUMINATION PROBLEM, JORDAN POLYGON, LOZENGE, OCTAGON, PARALLELOGRAM, PASCAL'S THEOREM, PENTAGON, PENTAGRAM, PETRIE POLYGON, POLYGON CIRCUMSCRIBING CONSTANT, POLYGON INSCRIBING CONSTANT, POLYGONAL KNOT, POLYGONAL NUMBER, POLYGONAL SPIRAL, POLYGON TRIANGULATION, POLYGRAM, POLYHEDRAL FORMULA, POLYHEDRON, POLYTOPE, QUADRANGLE, QUADRILATERAL, REGULAR POLYGON, REULEAUX POLYGON, RHOMBUS, ROTOR, SIMPLE POLYGON, SIMPLICITY, SQUARE, STAR POLYGON, TRAPEZIUM, TRAPEZOID, TRIANGLE, VISIBILITY, VORONOI POLYGON, WALLACE-BOLYAI-GERWEIN THEOREM

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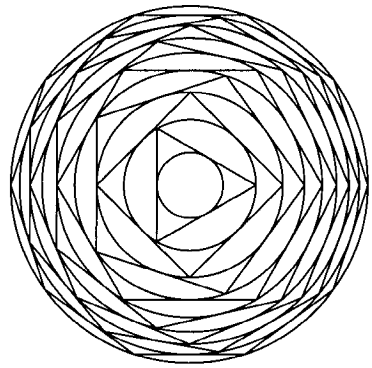
Polygon Circumscribing Constant



If a TRIANGLE is CIRCUMSCRIBED about a CIRCLE, another CIRCLE around the TRIANGLE, a SQUARE outside the CIRCLE, another CIRCLE outside the SQUARE, and so on. From POLYGONS, the CIRCUMRADIUS and INRADIUS for an n-gon are

$$R = \tfrac{1}{2} s \csc\left(\frac{\pi}{n}\right) \tag{1}$$

$$r = \tfrac{1}{2} s \cot\left(\frac{\pi}{n}\right), \tag{2}$$

where s is the side length. Therefore,

$$\frac{R}{r} = \frac{1}{\cos\left(\frac{\pi}{n}\right)} = \sec\left(\frac{\pi}{n}\right), \tag{3}$$

and an infinitely nested set of circumscribed polygons and circles has

$$K \equiv \frac{r_{\text{final circle}}}{r_{\text{initial circle}}} = \sec\left(\frac{\pi}{3}\right)\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{5}\right)\cdots. \tag{4}$$

Kasner and Newman (1989) and Haber (1964) state that $K = 12$, but this is incorrect. Write

$$K = \prod_{n=3}^{\infty} \frac{1}{\cos\left(\frac{\pi}{n}\right)} \tag{5}$$

$$\ln K = -\sum_{n=3}^{\infty} \ln(\cos x). \tag{6}$$

Define

$$y_0(x) \equiv -\ln(\cos x) = \tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 + \tfrac{1}{45}x^6 + \tfrac{17}{2520}x^8 + \ldots. \tag{7}$$

Now define

$$y_1(x) = \tfrac{1}{2}ax^2, \tag{8}$$

with

$$y_1(\tfrac{\pi}{3}) = y_0(\tfrac{\pi}{3}) \tag{9}$$

$$\tfrac{1}{2}a\left(\frac{\pi}{3}\right)^2 = \ln 2, \tag{10}$$

so

$$a = 2\left(\frac{3}{\pi}\right)^2 \ln 2, \tag{11}$$

and

$$y_2(x) = \frac{9\ln 2}{\pi^2}x^2. \quad (12)$$

But $y_2(x) > y_1(x)$ for $x \in (0, \pi/3)$, so

$$\sum_{n=3}^{\infty} y_2\left(\frac{\pi}{n}\right) > -\sum_{n=3}^{\infty} \ln\left[\cos\left(\frac{\pi}{n}\right)\right] \quad (13)$$

$$\begin{aligned}\ln K &< \sum_{n=3}^{\infty} y_2\left(\frac{\pi}{n}\right) \frac{9\ln 2}{\pi^2} \sum_{n=3}^{\infty} \left(\frac{\pi}{n}\right)^2 = 9\ln 2 \sum_{n=3}^{\infty} \frac{1}{n^2} \\ &= 9\ln 2\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{2} \frac{1}{n^2}\right) = 9\ln 2[\zeta(2) - \tfrac{5}{4}] \\ &= 9\ln 2\left(\frac{\pi^2}{6} - \frac{5}{4}\right) = 2.4637 \quad (14)\end{aligned}$$

$$K < e^{2.4637} = 11.75. \quad (15)$$

If the next term is included,

$$y_2(x) = a(\tfrac{1}{2}x^2 + \tfrac{1}{12}x^4). \quad (16)$$

As before,

$$y_2(\tfrac{\pi}{3}) = y_0(\tfrac{\pi}{3}) \quad (17)$$

$$a = \frac{972\ln 2}{\pi^2(54+\pi^2)}, \quad (18)$$

so

$$y_2(x) = \frac{972\ln 2}{\pi^2(54+\pi^2)}(\tfrac{1}{2}x^2 + \tfrac{1}{12}x^4) \quad (19)$$

$$\begin{aligned}\ln K &< \frac{972\ln 2}{\pi^2(54+\pi^2)} \sum_{n=3}^{\infty} \left[\frac{1}{2}\left(\frac{\pi}{n}\right)^2 + \frac{1}{12}\left(\frac{\pi}{n}\right)^4\right] \\ &= \frac{972\ln 2}{\pi^2(54+\pi^2)} \left\{\frac{1}{2}\left[\zeta(2) - \frac{5}{4}\right] + \frac{\pi^2}{12}\left[\zeta(4) - 1 - \frac{1}{2^4}\right]\right\} \\ &= \frac{972\ln 2}{\pi^2(54+\pi^2)} \left[\frac{1}{2}\left(\frac{\pi^2}{6} - \frac{5}{4}\right) + \frac{\pi^2}{12}\left(\frac{\pi^4}{90} - 1 - \frac{1}{2^4}\right)\right] \\ &= \frac{9(8\pi^6 - 45\pi^2 - 5400)\ln 2}{80(\pi^2+54)} = 2.255, \quad (20)\end{aligned}$$

and

$$K < e^{2.255} = 9.535. \quad (21)$$

The process can be automated using computer algebra, and the first few bounds are 11.7485, 9.53528, 8.98034, 8.8016, 8.73832, 8.71483, 8.70585, 8.70235, 8.70097, and 8.70042. In order to obtain this accuracy by direct multiplication of the terms, more than 10,000 terms are needed. The limit is

$$K = 8.700036625\ldots. \quad (22)$$

Bouwkamp (1965) produced the following INFINITE PRODUCT formulas

$$\begin{aligned}K &= \frac{2}{\pi} \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} \left[1 - \frac{1}{m^2(n+\frac{1}{2})^2}\right] \quad (23) \\ &= 6\exp\left\{\sum_{k=1}^{\infty} \frac{[\lambda(2k)-1]2^{2k}[\zeta(2k)-1-2^{-2k}]}{k}\right\}, \quad (24)\end{aligned}$$

where $\zeta(x)$ is the RIEMANN ZETA FUNCTION and $\lambda(x)$ is the DIRICHLET LAMBDA FUNCTION. Bouwkamp (1965) also produced the formula with accelerated convergence

$$\begin{aligned}K &= \tfrac{1}{12}\sqrt{6}\pi^4(1 - \tfrac{1}{2}\pi^2 + \tfrac{1}{24}\pi^4)(1 - \tfrac{1}{8}\pi^2 + \tfrac{1}{384}\pi^4) \\ &\times \csc\left(\frac{\pi^2}{\sqrt{6+2\sqrt{3}}}\right) \csc\left(\frac{\pi^2}{\sqrt{6-2\sqrt{3}}}\right) B, \quad (25)\end{aligned}$$

where

$$B \equiv \prod_{n=3}^{\infty} \left(1 - \frac{\pi^2}{2n^2} + \frac{\pi^4}{24n^4}\right) \sec\left(\frac{\pi}{n}\right) \quad (26)$$

(cited in Pickover 1995).

see also POLYGON INSCRIBING CONSTANT

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Polygon Construction

see GEOMETRIC CONSTRUCTION, GEOMETROGRAPHY, POLYGON, SIMPLICITY

Polygon Division Problem

see EULER'S POLYGON DIVISION PROBLEM

Polygon Fractal

see CHAOS GAME

Polygon Inscribing Constant

If a TRIANGLE is inscribed in a CIRCLE, another CIRCLE inside the TRIANGLE, a SQUARE inside the CIRCLE, another CIRCLE inside the SQUARE, and so on,

$$K' \equiv \frac{r_{\text{final circle}}}{r_{\text{initial circle}}} = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{5}\right)\cdots.$$

Numerically,

$$K' = \frac{1}{K} = \frac{1}{8.7000366252\ldots} = 0.1149420448\ldots,$$

where K is the POLYGON CIRCUMSCRIBING CONSTANT. Kasner and Newman's (1989) assertion that $K = 1/12$ is incorrect.

Let a convex POLYGON be inscribed in a CIRCLE and divided into TRIANGLES from diagonals from one VERTEX. The sum of the RADII of the CIRCLES inscribed in these TRIANGLES is the same independent of the VERTEX chosen (Johnson 1929, p. 193).

see also POLYGON CIRCUMSCRIBING CONSTANT

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Polygon Triangulation

see EULER'S POLYGON DIVISION PROBLEM

Polygonal Knot

A KNOT equivalent to a POLYGON in $\mathbb{R}^3$, also called a TAME KNOT. For a polygonal knot K, there exists a PLANE such that the orthogonal projection π on it satisfies the following conditions:

1. The image $\pi(K)$ has no multiple points other than a FINITE number of double points.
2. The projections of the vertices of K are not double points of $\pi(K)$.

Such a projection $\pi(K)$ is called a regular knot projection.

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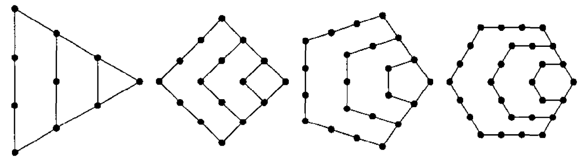
Polygonal Number



A type of FIGURATE NUMBER which is a generalization of TRIANGULAR, SQUARE, etc., numbers to an arbitrary n-gonal number. The above diagrams graphically illustrate the process by which the polygonal numbers are built up. Starting with the nth TRIANGULAR NUMBER T_n, then

$$n + T_{n-1} = T_n. \tag{1}$$

Now note that

$$n + 2T_{n-1} = n^2 = S_n \tag{2}$$

gives the nth SQUARE NUMBER,

$$n + 3T_{n-1} = \tfrac{1}{2}n(3n-1) = P_n, \tag{3}$$

gives the nth PENTAGONAL NUMBER, and so on. The general polygonal number can be written in the form

$$p_r^n = \tfrac{1}{2}r[(r-1)n - 2(r-2)] = \tfrac{1}{2}r[(n-2)r-(n-4)], \tag{4}$$

where p_r^n is the rth n-gonal number. For example, taking $n = 3$ in (4) gives a TRIANGULAR NUMBER, $n = 4$ gives a SQUARE NUMBER, etc.

Fermat proposed that every number is expressible as *at most k k*-gonal numbers (FERMAT'S POLYGONAL NUMBER THEOREM). Fermat claimed to have a proof of this result, although this proof has never been found. Jacobi, Lagrange (1772), and Euler all proved the square case, and Gauss proved the triangular case in 1796. In 1813, Cauchy proved the proposition in its entirety.

An arbitrary number N can be checked to see if it is a n-gonal number as follows. Note the identity

$$\begin{aligned} 8(n-2)p_n^r + (n-4)^2 &= 4r(n-2)[(r-1)n-2(r-2)] \\ &\quad +(n-4)^2 = 4r(r-1)n^2 + r[-8(r-1)-8(r-2)]n \\ &\quad +16r(r-2) + (n^2-8n+16) \\ &= (4r^2-4r+1)n^2 + (-16r^2+24r-8)n \\ &\quad +(16r^2-32r+16) \\ &= (2r-1)^2n^2 - 8(2r^2-3r+1)n + 16(r^2-2r+1) \\ &= (2rn-4r-n+4)^2, \end{aligned} \tag{5}$$

so $8(n-2)N + (n-4)^2 = S^2$ must be a PERFECT SQUARE. Therefore, if it is not, the number cannot be n-gonal. If it is a PERFECT SQUARE, then solving

$$S = 2rn - 4r - n + 4 \tag{6}$$

for the rank r gives

$$r = \frac{S+n-4}{2(n-2)}. \quad (7)$$

An n-gonal number is equal to the sum of the $(n-1)$-gonal number of the same RANK and the TRIANGULAR NUMBER of the previous RANK.

see also CENTERED POLYGONAL NUMBER, DECAGONAL NUMBER, FERMAT'S POLYGONAL NUMBER THEOREM, FIGURATE NUMBER, HEPTAGONAL NUMBER, HEXAGONAL NUMBER, NONAGONAL NUMBER, OCTAGONAL NUMBER, PENTAGONAL NUMBER, PYRAMIDAL NUMBER, SQUARE NUMBER, TRIANGULAR NUMBER

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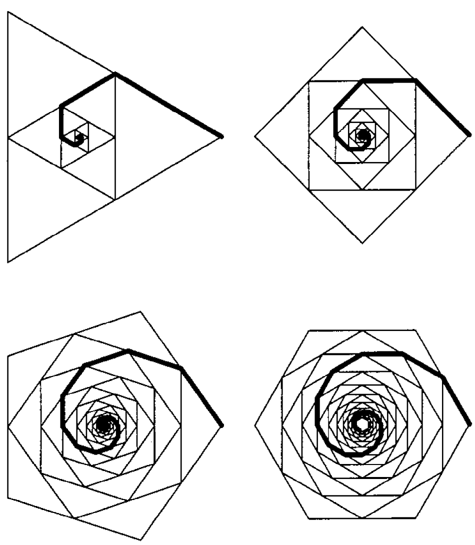
Polygonal Spiral

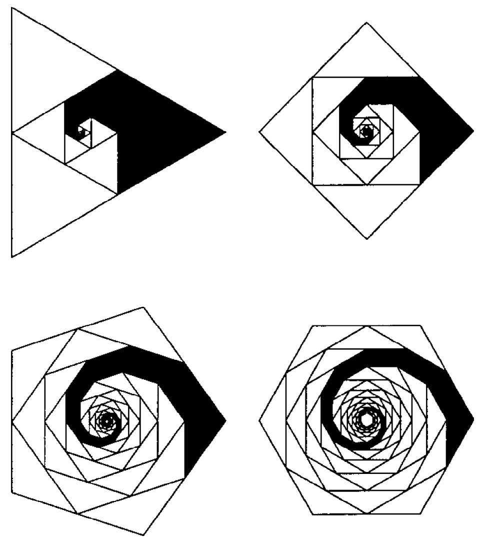


The length of the polygonal spiral is found by noting that the ratio of INRADIUS to CIRCUMRADIUS of a regular POLYGON of n sides is

$$\frac{r}{R} = \frac{\cot\left(\frac{\pi}{n}\right)}{\csc\left(\frac{\pi}{n}\right)} = \cos\left(\frac{\pi}{n}\right). \quad (1)$$

The total length of the spiral for an n-gon with side length s is therefore

$$L = \tfrac{1}{2}s\sum_{k=0}^{\infty}\cos^k\left(\frac{\pi}{n}\right) = \frac{s}{2\left[1-\cos\left(\frac{\pi}{n}\right)\right]}. \quad (2)$$



Consider the solid region obtained by filling in subsequent triangles which the spiral encloses. The AREA of this region, illustrated above for n-gons of side length s, is

$$A = \tfrac{1}{4}s^2\cot\left(\frac{\pi}{n}\right). \quad (3)$$

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Polygram

A self-intersecting STAR FIGURE such as the PENTAGRAM or HEXAGRAM.

n	symbol	polygram
5	{5/2}	pentagram
6	{6/2}	hexagram
7	{7/2}	heptagram
8	{8/3}	octagram
	{8/4}	star of Lakshmi
10	{10/3}	decagram

Polyhedral Formula

A formula relating the number of VERTICES, FACES, and EDGES of a POLYHEDRON (or POLYGON). It was discovered independently by Euler and Descartes, so it is also known as the DESCARTES-EULER POLYHEDRAL FORMULA. The polyhedron need not be CONVEX, but the FORMULA does not hold for STELLATED POLYHEDRA.

$$V + F - E = 2, \quad (1)$$

where $V = N_0$ is the number of VERTICES, $E = N_1$ is the number of EDGES, and $F = N_2$ is the number of FACES. For a proof, see Courant and Robbins (1978, pp. 239–240). The FORMULA can be generalized to n-D POLYTOPES.

$$\Pi_1 : N_0 = 2 \tag{2}$$

$$\Pi_2 : N_0 - N_1 = 0 \tag{3}$$

$$\Pi_3 : N_0 - N_1 + N_2 = 2 \tag{4}$$

$$\Pi_4 : N_0 - N_1 + N_2 - N_3 = 0 \tag{5}$$

$$\Pi_n : N_0 - N_1 + N_2 - \ldots + (-1)^{n-1} N_{n-1} = 1 - (-1)^n. \tag{6}$$

For a proof of this, see Coxeter (1973, pp. 166–171).

see also DEHN INVARIANT, DESCARTES TOTAL ANGULAR DEFECT

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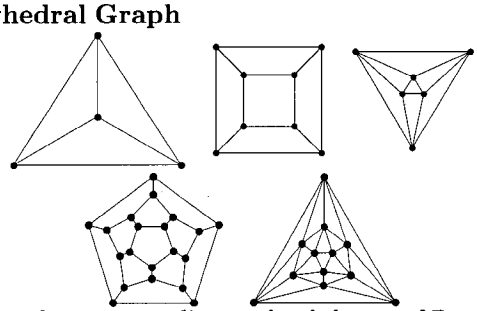
Polyhedral Graph

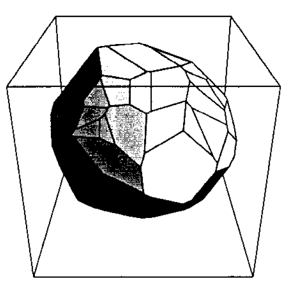


The graphs corresponding to the skeletons of PLATONIC SOLIDS. They are special cases of SCHLEGEL GRAPHS.

see also CUBICAL GRAPH, DODECAHEDRAL GRAPH, ICOSAHEDRAL GRAPH, OCTAHEDRAL GRAPH, SCHLEGEL GRAPH, TETRAHEDRAL GRAPH

Polyhedron

A 3-D solid which consists of a collection of POLYGONS, usually joined at their EDGES. The word derives from the Greek *poly* (many) plus the Indo-European *hedron* (seat). A polyhedron is the 3-D version of the more general POLYTOPE, which can be defined on arbitrary dimensions.



A CONVEX POLYHEDRON can be defined as the set of solutions to a system of linear inequalities

$$\mathbf{mx} \le \mathbf{b},$$

where $\mathbf{m}$ is a real $s \times 3$ MATRIX and $\mathbf{b}$ is a real s-VECTOR. An example is illustrated above. The more simple DODECAHEDRON is given by a system with $s = 12$. In general, given the MATRICES, the VERTICES (and FACES) can be found using VERTEX ENUMERATION.

A polyhedron is said to be regular if its FACES and VERTEX FIGURES are REGULAR (not necessarily CONVEX) polygons (Coxeter 1973, p. 16). Using this definition, there are a total of nine REGULAR POLYHEDRA, five being the CONVEX PLATONIC SOLIDS and four being the CONCAVE (stellated) KEPLER-POINSOT SOLIDS. However, the term "regular polyhedra" is sometimes also used to refer exclusively to the PLATONIC SOLIDS (Cromwell 1997, p. 53). The DUAL POLYHEDRA of the PLATONIC SOLIDS are not new polyhedra, but are themselves PLATONIC SOLIDS.

A CONVEX polyhedron is called SEMIREGULAR if its FACES have a similar arrangement of nonintersecting regular plane CONVEX polygons of two or more different types about each VERTEX (Holden 1991, p. 41). These solids are more commonly called the ARCHIMEDEAN SOLIDS, and there are 13 of them. The DUAL POLYHEDRA of the ARCHIMEDEAN SOLIDS are 13 new (and beautiful) solids, sometimes called the CATALAN SOLIDS.

A QUASIREGULAR POLYHEDRON is the solid region interior to two DUAL REGULAR POLYHEDRA (Coxeter 1973, pp. 17–20). There are only two CONVEX QUASIREGULAR POLYHEDRA: the CUBOCTAHEDRON and ICOSIDODECAHEDRON. There are also infinite families of PRISMS and ANTIPRISMS.

There exist exactly 92 CONVEX POLYHEDRA with REGULAR POLYGONAL faces (and not necessary equivalent vertices). They are known as the JOHNSON SOLIDS. Polyhedra with identical VERTICES related by a symmetry operation are known as UNIFORM POLYHEDRA. There are 75 such polyhedra in which only two faces may meet at an EDGE, and 76 in which any EVEN number of faces may meet. Of these, 37 were discovered by Badoureau in 1881 and 12 by Coxeter and Miller ca. 1930.

Polyhedra can be superposed on each other (with the sides allowed to pass through each other) to yield additional POLYHEDRON COMPOUNDS. Those made from REGULAR POLYHEDRA have symmetries which are especially aesthetically pleasing. The graphs corresponding to polyhedra skeletons are called SCHLEGEL GRAPHS.

Behnke *et al.* (1974) have determined the symmetry groups of all polyhedra symmetric with respect to their VERTICES.

see also ACOPTIC POLYHEDRON, APEIROGON, ARCHIMEDEAN SOLID, CANONICAL POLYHEDRON, CATALAN SOLID, CUBE, DICE, DIGON, DODECAHEDRON, DUAL POLYHEDRON, ECHIDNAHEDRON, FLEXIBLE POLYHEDRON, HEXAHEDRON, HYPERBOLIC POLYHEDRON, ICOSAHEDRON, ISOHEDRON, JOHNSON SOLID, KEPLER-POINSOT SOLID, NOLID, OCTAHEDRON, PETRIE POLYGON, PLATONIC SOLID, POLYHEDRON COLORING, POLYHEDRON COMPOUND, PRISMATOID, QUADRICORN, QUASIREGULAR POLYHEDRON, RIGIDITY THEOREM, SEMIREGULAR POLYHEDRON, SKELETON, TETRAHEDRON, UNIFORM POLYHEDRON, ZONOHEDRON

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Polyhedron Coloring

Define a valid "coloring" to occur when no two faces with a common EDGE share the same color. Given two colors, there is a single way to color an OCTAHEDRON. Given three colors, there is one way to color a CUBE and 144 ways to color an ICOSAHEDRON. Given four-colors, there are two distinct ways to color a TETRAHEDRON and 4 ways to color a DODECAHEDRON. Given five colors, there are four ways to color an ICOSAHEDRON.

see also COLORING, POLYHEDRON

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Polyhedron Compound

Solid	Vertices	Symbol
cube-octahedron	both	
dodec.+icos.	both	
two cubes		
three cubes		
four cubes		
five cubes	dodecahedron	$2\{5,3\}[5\{4,3\}]$
five octahedra	icosidodeca.	$[5\{3,4\}]2\{3,5\}$
five tetrahedra	dodecahedron	$\{5,3\}[5\{3,3\}]2\{3,5\}$
two dodecahedra	both	
great dodecahedron-small stellated dodec.		
great icosahedron-great stellated dodec.	both	
stella octangula	cube	$\{4,3\}[2\{3,3\}]\{3,4\}$
ten tetrahedra	dodecahedron	$2\{5,3\}[10\{3,3\}]2\{3,5\}$

The above table gives some common polyhedron compounds. In Coxeter's NOTATION, d distinct VERTICES of $\{m,n\}$ taken c times are denoted

$$c\{m,n\}[d\{p,q\}],$$

or faces of $\{s,t\}$ e times

$$[d\{p,q\}]e\{s,t\},$$

or both

$$c\{m,n\}[d\{p,q\}]e\{s,t\}.$$

The five TETRAHEDRA can be arranged in a laevo or dextro configuration.

see also CUBE-OCTAHEDRON COMPOUND, DODECAHEDRON-ICOSAHEDRON COMPOUND, OCTAHEDRON 5-COMPOUND, STELLA OCTANGULA, TETRAHEDRON 5-COMPOUND

Polyhedron Dissection

A DISSECTION of one or more polyhedra into other shapes.

see also CUBE DISSECTION, DIABOLICAL CUBE, POLYCUBE, SOMA CUBE, WALLACE-BOLYAI-GERWEIN THEOREM

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Polyhedron Dual

see DUAL POLYHEDRON

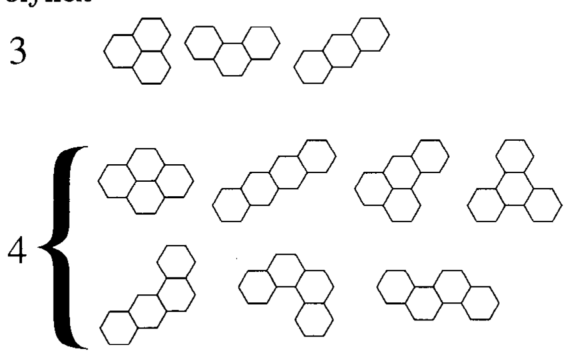
Polyhedron Hinging

see RIGIDITY THEOREM

Polyhedron Packing

see KELVIN'S CONJECTURE, SPACE-FILLING POLYHEDRON

Polyhex



An analog of the POLYOMINOES and POLYIAMONDS in which collections of regular hexagons are arranged with adjacent sides. They are also called HEXES and HEXAS. The number of polyhexes of n hexagons are 1, 1, 2, 7, 22, 82, 333, 1448, 6572, 30490, 143552, 683101, ... (Sloane's A014558). For the 4-hexes (tetrahexes), the possible arrangements are known as the BEE, BAR, PISTOL, PROPELLER, WORM, ARCH, and WAVE.

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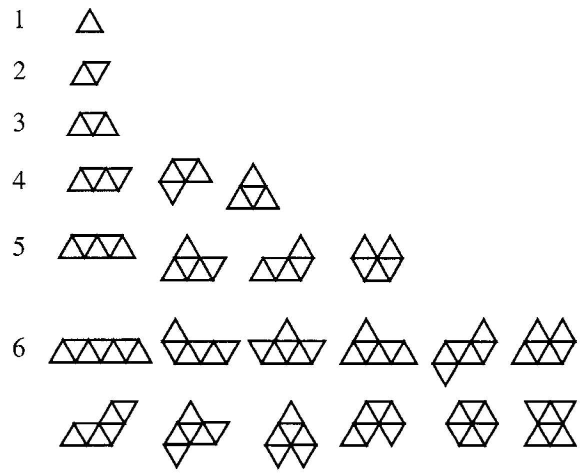
Polyiamond



A generalization of the POLYOMINOES using a collection of equal-sized EQUILATERAL TRIANGLES (instead of SQUARES) arranged with coincident sides. Polyiamonds are sometimes simply known as IAMONDS.

The number of two-sided (i.e., can be picked up and flipped, so MIRROR IMAGE pieces are considered identical) polyiamonds made up of n triangles are 1, 1, 1, 3, 4, 12, 24, 66, 160, 448, ... (Sloane's A000577). The number of one-sided polyiamonds composed of n triangles are 1, 1, 1, 4, 6, 19, 43, 121, ... (Sloane's A006534). No HOLES are possible with fewer than seven triangles.

The top row of 6-polyiamonds in the above figure are known as the BAR, CROOK, CROWN, SPHINX, SNAKE, and YACHT. The bottom row of 6-polyiamonds are known as the CHEVRON, SIGNPOST, LOBSTER, HOOK, HEXAGON, and BUTTERFLY.

see also POLYABOLO, POLYHEX, POLYOMINO

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Polyking

see POLYPLET

Polylogarithm

The function

$$\mathrm{Li}_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \tag{1}$$

Also known as JONQUIÈRE'S FUNCTION. (Note that the NOTATION $\mathrm{Li}(z)$ is also used for the LOGARITHMIC INTEGRAL.) The polylogarithm arises in Feynman Diagram integrals, and the special case $n = 2$ is called the DILOGARITHM. The polylogarithm of NEGATIVE INTEGER order arises in sums of the form

$$\sum_{k=1}^{\infty} k^n r^k = \mathrm{Li}_{-n}(r) = \frac{r}{(1-r)^{n+1}} \sum_{i=1}^{n} \left\langle {n \atop i} \right\rangle r^{n-i}, \tag{2}$$

where $\left\langle {n \atop i} \right\rangle$ is an EULERIAN NUMBER.

The polylogarithm satisfies the fundamental identities

$$-\ln(1-2^{-n}) = \mathrm{Li}_1(2^{-n}) \tag{3}$$

$$\mathrm{Li}_s(-1) = -(1-2^{1-s})\zeta(s), \tag{4}$$

where $\zeta(s)$ is the RIEMANN ZETA FUNCTION. The derivative is therefore given by

$$\frac{d}{ds}\mathrm{Li}_s(-1) = -2^{1-s}\zeta(s)\ln 2 - (1-2^{1-s})\zeta'(s), \tag{5}$$

or in the special case $s = 0$, by

$$\left[\frac{d}{ds}\mathrm{Li}_s(-1)\right]_{s=0} = \ln 2 + \zeta'(0) = \ln 2 - \tfrac{1}{2}\ln(2\pi) = \ln\left(\sqrt{\frac{2}{\pi}}\right). \tag{6}$$

This latter fact provides a remarkable proof of the WALLIS FORMULA.

The polylogarithm identities lead to remarkable expressions. Ramanujan gave the polylogarithm identities

$$\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{6}\mathrm{Li}_2(\tfrac{1}{9}) = \tfrac{1}{18}\pi^2 - \tfrac{1}{6}(\ln 3)^2 \tag{7}$$

$$\mathrm{Li}_2(-\tfrac{1}{2}) + \tfrac{1}{6}\mathrm{Li}_2(\tfrac{1}{9}) = -\tfrac{1}{18}\pi^2 + \ln 2\ln 3 - \tfrac{1}{2}(\ln 2)^2 - \tfrac{1}{3}(\ln 3)^2 \tag{8}$$

$$\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{9}) = \tfrac{1}{18}\pi^2 + 2\ln 2\ln 3 - 2(\ln 2)^2 - \tfrac{2}{3}(\ln 3)^2 \tag{9}$$

$$\mathrm{Li}_2(-\tfrac{1}{3}) - \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{9}) = -\tfrac{1}{18}\pi^2 + \tfrac{1}{6}(\ln 3)^2 \tag{10}$$

$$\mathrm{Li}_2(-\tfrac{1}{8}) + \mathrm{Li}_2(\tfrac{1}{9}) = -\tfrac{1}{2}(\ln \tfrac{9}{8})^2 \tag{11}$$

(Berndt 1994), and Bailey *et al.* show that

$$\pi^2 = 36\,\mathrm{Li}_2(\tfrac{1}{2}) - 36\,\mathrm{Li}_2(\tfrac{1}{4}) - 12\,\mathrm{Li}_2(\tfrac{1}{8}) + 6\,\mathrm{Li}_2(\tfrac{1}{64}) \tag{12}$$

$$12\,\mathrm{Li}_2(\tfrac{1}{2}) = \pi^2 - 6(\ln 2)^2 \tag{13}$$

$$\tfrac{35}{2}\zeta(3) - \pi^2\ln 2 = 36\,\mathrm{Li}_3(\tfrac{1}{2}) - 18\,\mathrm{Li}_3(\tfrac{1}{4}) - 4\,\mathrm{Li}_3(\tfrac{1}{8}) + \mathrm{Li}_3(\tfrac{1}{64}) \tag{14}$$

$$2(\ln 2)^3 - 7\zeta(3) = -24\,\mathrm{Li}_3(\tfrac{1}{2}) + 18\,\mathrm{Li}_3(\tfrac{1}{4}) + 4\,\mathrm{Li}_3(\tfrac{1}{8}) - \mathrm{Li}_3(\tfrac{1}{64}) \tag{15}$$

$$10(\ln 2)^3 - 2\pi^2\ln 2 = -48\,\mathrm{Li}_3(\tfrac{1}{2}) + 54\,\mathrm{Li}_3(\tfrac{1}{4}) + 12\,\mathrm{Li}_3(\tfrac{1}{8}) - 3\,\mathrm{Li}_3(\tfrac{1}{64}), \tag{16}$$

and

$$\frac{\mathrm{Li}_m(\frac{1}{64})}{6^{m-1}} - \frac{\mathrm{Li}_m(\frac{1}{8})}{3^{m-1}} - \frac{2\,\mathrm{Li}_m(\frac{1}{4})}{2^{m-1}} + \frac{4\,\mathrm{Li}_m(\frac{1}{2})}{9} - \frac{5(-\ln 2)^m}{9m!} + \frac{\pi^2(-\ln 2)^{m-2}}{54(m-2)!} - \frac{\pi^4(-\ln 2)^{m-4}}{486(m-4)!} - \frac{403\zeta(5)(-\ln 2)^{m-5}}{1296(m-5)!} = 0. \tag{17}$$

No general ALGORITHM is know for the integration of polylogarithms of functions.

see also DILOGARITHM, EULERIAN NUMBER, LEGENDRE'S CHI-FUNCTION, LOGARITHMIC INTEGRAL, NIELSEN-RAMANUJAN CONSTANTS

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Polymorph

An INTEGER which is expressible in more than one way in the form x^2+Dy^2 or x^2-Dy^2 where x^2 is RELATIVELY PRIME to Dy^2. If the INTEGER is expressible in only one way, it is called a MONOMORPH.

see also ANTIMORPH, IDONEAL NUMBER, MONOMORPH

Polynomial

A POLYNOMIAL is a mathematical expression involving a series of POWERS in one or more variables multiplied by COEFFICIENTS. A POLYNOMIAL in one variable with constant COEFFICIENTS is given by

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0. \tag{1}$$

The highest POWER in a one-variable POLYNOMIAL is called its ORDER. A POLYNOMIAL in two variables with constant COEFFICIENTS is given by

$$a_{nm}x^n y^m + a_{22}x^2y^2 + a_{21}x^2y + a_{12}xy^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00}. \tag{2}$$

Exchanging the COEFFICIENTS of a one-variable POLYNOMIAL end-to-end produces a POLYNOMIAL

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0 \tag{3}$$

whose ROOTS are RECIPROCALS $1/x_i$ of the original ROOTS x_i.

The following table gives special names given to polynomials of low orders.

Order	Polynomial Name
1	linear equation
2	quadratic equation
3	cubic equation
4	quartic equation
5	quintic equation
6	sextic equation

Polynomials of fourth degree may be computed using three multiplications and five additions if a few quantities are calculated first (Press *et al.* 1989):

$$\begin{aligned} &a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\ &= [(Ax+B)^2 + Ax + C][(Ax+B)^2 + D] + E, \end{aligned} \tag{4}$$

where

$$A \equiv (a_4)^{1/4} \tag{5}$$

$$B \equiv \frac{a_3 - A^3}{4A^3} \tag{6}$$

$$D \equiv 3B^2 + 8B^3 + \frac{a_1 A - 2a_2 B}{A^2} \tag{7}$$

$$C \equiv \frac{a_2}{A^2} - 2B - 6B^2 - D \tag{8}$$

$$E \equiv a_0 - B^4 - B^2(C+D) - CD. \tag{9}$$

Similarly, a POLYNOMIAL of fifth degree may be computed with four multiplications and five additions, and a POLYNOMIAL of sixth degree may be computed with four multiplications and seven additions.

Polynomials of orders 1 to 4 are solvable using only algebraic functions and finite square root extraction. A first-order equation is trivially solvable. A second-order equation is soluble using the QUADRATIC EQUATION. A third-order equation is solvable using the CUBIC EQUATION. A fourth-order equation is solvable using the QUARTIC EQUATION. It was proved by Abel using GROUP THEORY that higher order equations cannot be solved by finite root extraction.

However, the general QUINTIC EQUATION may be given in terms of the THETA FUNCTIONS, or HYPERGEOMETRIC FUNCTIONS in one variable. Hermite and Kronecker proved that higher order POLYNOMIALS are not soluble in the same manner. Klein showed that the work of Hermite was implicit in the GROUP properties of the ICOSAHEDRON. Klein's method of solving the quintic in terms of HYPERGEOMETRIC FUNCTIONS in one variable can be extended to the sextic, but for higher order POLYNOMIALS, either HYPERGEOMETRIC FUNCTIONS in several variables or "Siegel functions" must be used. In the 1880s, Poincaré created functions which give the solution to the nth order POLYNOMIAL equation in finite form. These functions turned out to be "natural" generalizations of the ELLIPTIC FUNCTIONS.

Given an nth degree POLYNOMIAL, the ROOTS can be found by finding the EIGENVALUES of the MATRIX

$$\begin{bmatrix} -a_0/a_n & -a_1/a_n & -a_2/a_n & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 1 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{10}$$

This method can be computationally expensive, but is fairly robust at finding close and multiple roots.

Polynomial identities involving sums and differences of like POWERS include

$$x^2 - y^2 = (x-y)(x+y) \tag{11}$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2) \tag{12}$$

$$x^3 + y^3 = (x+y)(x^2-xy+y^2) \tag{13}$$

$$x^4 - y^4 = (x-y)(x+y)(x^2+y^2) \tag{14}$$

$$x^4 + 4y^4 = (x^2+2xy+2y^2)(x^2-2xy+2y^2) \tag{15}$$

$$x^5 - y^5 = (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) \tag{16}$$

$$x^5 + y^5 = (x+y)(x^4-x^3y+x^2y^2-xy^3+y^4) \tag{17}$$

$$x^6 - y^6 = (x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2) \tag{18}$$

$$x^6 + y^6 = (x^2+y^2)(x^4-x^2y^2+y^4). \tag{19}$$

Further identities include

$$x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2) \tag{20}$$

$$\begin{aligned} &(x_1{}^2 - Dy_1{}^2)(x_2{}^2 - Dy_2{}^2) \\ &\quad = (x_1x_2 + Dy_1y_2)^2 - D(x_1y_2 + x_2y_1)^2 \end{aligned} \tag{21}$$

$$\begin{aligned} &(x_1{}^2 + Dy_1{}^2)(x_2{}^2 + Dy_2{}^2) \\ &\quad = (x_1x_2 \pm Dy_1y_2)^2 + D(x_1y_2 \mp x_2y_1)^2. \end{aligned} \tag{22}$$

The identity

$$\begin{aligned} &(X+Y+Z)^7 - (X^7+Y^7+Z^7) = 7(X+Y)(X+Z)(Y+Z) \\ &\times[(X^2+Y^2+Z^2+XY+XZ+YZ)^2 + XYZ(X+Y+Z)] \end{aligned} \tag{23}$$

was used by Lamé in his proof that FERMAT'S LAST THEOREM was true for $n = 7$.

see also APPELL POLYNOMIAL, BERNSTEIN POLYNOMIAL, BESSEL POLYNOMIAL, BEZOUT'S THEOREM, BINOMIAL, BOMBIERI INNER PRODUCT, BOMBIERI NORM, CHEBYSHEV POLYNOMIAL OF THE FIRST KIND, CHEBYSHEV POLYNOMIAL OF THE SECOND KIND, CHRISTOFFEL-DARBOUX FORMULA, CHRISTOFFEL NUMBER, COMPLEX NUMBER, CYCLOTOMIC POLYNOMIAL, DESCARTES' SIGN RULE, DISCRIMINANT (POLYNOMIAL), DURFEE POLYNOMIAL, EHRHART POLYNOMIAL, EULER FOUR-SQUARE IDENTITY, FIBONACCI IDENTITY, FUNDAMENTAL THEOREM OF ALGEBRA, FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS, GAUSS-JACOBI MECHANICAL QUADRATURE, GEGENBAUER POLYNOMIAL, GRAM-SCHMIDT ORTHONORMALIZATION, GREATEST LOWER BOUND, HERMITE POLYNOMIAL, HILBERT POLYNOMIAL, IRREDUCIBLE POLYNOMIAL, ISOBARIC POLYNOMIAL, ISOGRAPH, JENSEN POLYNOMIAL, KERNEL POLYNOMIAL, KRAWTCHOUK POLYNOMIAL, LAGUERRE POLYNOMIAL, LEAST UPPER BOUND, LEGENDRE POLYNOMIAL, LIOUVILLE POLYNOMIAL IDENTITY, LOMMEL POLYNOMIAL, LUKÁCS THEOREM, MONOMIAL, ORTHOGONAL POLYNOMIALS, PERIMETER POLYNOMIAL, POISSON-CHARLIER POLYNOMIAL, POLLACZEK POLYNOMIAL, POLYNOMIAL BAR NORM, QUARTER SQUARES RULE, RAMANUJAN 6-10-8 IDENTITY, ROOT, RUNGE-WALSH THEOREM, SCHLÄFLI POLYNOMIAL, SEPARATION THEOREM, STIELTJES-WIGERT POLYNOMIAL, TRINOMIAL, TRINOMIAL IDENTITY, WEIERSTRAß'S POLYNOMIAL THEOREM, ZERNIKE POLYNOMIAL

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Polynomial Bar Norm

For $p = \sum a_j z^j$, define

$$||P||_1 = \int_0^{2\pi} |P(e^{i\theta})| \frac{d\theta}{2\pi} \qquad |P|_1 = \sum_j |a_j|$$

$$||P||_2 = \sqrt{\int_0^{2\pi} |P(e^{\pi\theta})|^2 \frac{d\theta}{2\pi}} \qquad |P|_2 = \sqrt{\sum_j |a_j|^2}$$

$$||P||_\infty = \max_{|z|=1} |P(z)| \qquad |P|_\infty = \max_j |a_j|,$$

where the $||P||_i$ norms are functions on the UNIT CIRCLE and the $|P|_i$ norms refer to the COEFFICIENTS $a_0, \ldots, a_n$.

see also BOMBIERI NORM, NORM, UNIT CIRCLE

References

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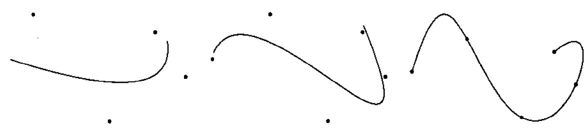
Polynomial Bracket Norm

see BOMBIERI NORM

Polynomial Curve



A curve obtained by fitting POLYNOMIALS to each ordinate of an ordered sequence of points. The above plots show POLYNOMIAL curves where the order of the fitting POLYNOMIAL varies from $p-3$ to $p-1$, where p is the number of points.

Polynomial curves have several undesirable features, including a nonintuitive variation of fitting curve with varying COEFFICIENTS, and numerical instability for high orders. SPLINES such as the BÉZIER CURVE are therefore used more commonly.

see also BÉZIER CURVE, POLYNOMIAL, SPLINE

Polynomial Factor

A FACTOR of a POLYNOMIAL $P(x)$ of degree n is a POLYNOMIAL $Q(x)$ of degree less than n which can be multiplied by another POLYNOMIAL $R(x)$ of degree less than n to yield $P(x)$, i.e., a POLYNOMIAL $Q(x)$ such that

$$P(x) = Q(x)R(x).$$

For example, since

$$x^2 - 1 = (x+1)(x-1),$$

both $x-1$ and $x+1$ are FACTORS of x^2-1. The COEFFICIENTS of factor POLYNOMIALS are often required to be REAL NUMBERS or INTEGERS but could, in general, be COMPLEX NUMBERS.

see also FACTOR, FACTORIZATION, PRIME FACTORIZATION

Polynomial Norm

see BOMBIERI NORM, MATRIX NORM, POLYNOMIAL BAR NORM, VECTOR NORM

Polynomial Remainder Theorem

If the COEFFICIENTS of the POLYNOMIAL

$$d_n x^n + d_{n-1} x^{n-1} + \ldots + d_0 = 0 \quad (1)$$

are specified to be INTEGERS, then integral ROOTS must have a NUMERATOR which is a factor of d_0 and a DENOMINATOR which is a factor of d_n (with either sign possible). This follows since a POLYNOMIAL of ORDER n with k integral ROOTS can be expressed as

$$(a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)(c_{n-k} x^{n-k} + \ldots + c_0) = 0, \quad (2)$$

where the ROOTS are $x_1 = -b_1/a_1$, $x_2 = -b_2/a_2$, ..., and $x_k = -b_k/a_k$. Factoring out the a_is,

$$a_1 a_2 \cdots a_k \left(x - \frac{b_1}{a_1}\right)\left(x - \frac{b_2}{a_2}\right) \cdots \left(x - \frac{b_k}{a_k}\right) \times (c_{n-k} x^{n-k} + \ldots + c_0) = 0. \quad (3)$$

Now, multiplying through,

$$a_1 a_2 \cdots a_k c_{n-k} x^n + \ldots + b_1 b_2 \cdots b_k c_0 = 0, \quad (4)$$

where we have not bothered with the other terms. Since the first and last COEFFICIENTS are d_n and d_0, all the integral roots of (1) are of the form [factors of d_0]/[factors of d_n].

Polynomial Ring

The RING $R[x]$ of POLYNOMIALS in a variable x.

see also POLYNOMIAL, RING

Polynomial Root

If the COEFFICIENTS of the POLYNOMIAL

$$d_n x^n + d_{n-1} x^{n-1} + \ldots + d_0 = 0 \quad (1)$$

are specified to be INTEGERS, then integral roots must have a NUMERATOR which is a factor of d_0 and a DENOMINATOR which is a factor of d_n (with either sign possible). This is known as the POLYNOMIAL REMAINDER THEOREM.

Let the ROOTS of the polynomial

$$P(x) \equiv a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (2)$$

be denoted r_1, r_2, ..., r_n. Then NEWTON'S RELATIONS are

$$\sum r_i = -\frac{a_{n-1}}{a_n} \quad (3)$$

$$\sum r_i r_j = \frac{a_{n-2}}{a_n} \quad (4)$$

$$\sum r_1 r_2 \cdots r_k = (-1)^k \frac{a_{n-k}}{a_n}. \quad (5)$$

These can be derived by writing

$$(x-a)(x-b) = 0 \quad (6)$$

$$\left(\frac{x}{a} - 1\right)\left(\frac{x}{b} - 1\right) = 0 \quad (7)$$

$$\frac{1}{ab} x^2 - \left(\frac{1}{a} - \frac{1}{b}\right) x + 1 = 0. \quad (8)$$

Similarly,

$$\left(\frac{x^2}{a^2} - 1\right)\left(\frac{x^2}{b^2} = 1\right) = 0 \quad (9)$$

$$\frac{x^4}{a^2 b^2} - x^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + 1 = 0. \quad (10)$$

Any POLYNOMIAL can be numerically factored, although different ALGORITHMS have different strengths and weaknesses.

If there are no NEGATIVE ROOTS of a POLYNOMIAL (as can be determined by DESCARTES' SIGN RULE), then the GREATEST LOWER BOUND is 0. Otherwise, write out the COEFFICIENTS, let $n = -1$, and compute the next line. Now, if any COEFFICIENTS are 0, set them to minus the sign of the next higher COEFFICIENT, starting with the second highest order COEFFICIENT. If all the signs alternate, n is the greatest lower bound. If not, then subtract 1 from n, and compute another line. For example, consider the POLYNOMIAL

$$y = 2x^4 + 2x^3 - 7x^2 + x - 7. \quad (11)$$

Performing the above ALGORITHM then gives

0	2	2	−7	1	−7
−1	2	0	−7	8	−15
—	2	−1	−7	8	−15
−2	2	−2	−3	7	−21
−3	2	−4	5	−14	35

so the greatest lower bound is -3.

If there are no POSITIVE ROOTS of a POLYNOMIAL (as can be determined by DESCARTES' SIGN RULE), the LEAST UPPER BOUND is 0. Otherwise, write out the COEFFICIENTS of the POLYNOMIALS, including zeros as necessary. Let $n = 1$. On the line below, write the highest order COEFFICIENT. Starting with the second-highest COEFFICIENT, add n·times the number just written to the original second COEFFICIENT, and write it below the second COEFFICIENT. Continue through order zero. If all the COEFFICIENTS are NONNEGATIVE, the least upper bound is n. If not, add one to x and repeat the process again. For example, take the POLYNOMIAL

$$y = 2x^4 - x^3 - 7x^2 + x - 7. \quad (12)$$

Performing the above ALGORITHM gives

0	2	-1	-7	1	-7
1	2	1	-6	-5	-12
2	2	3	-1	-1	-9
3	2	5	8	25	68

so the LEAST UPPER BOUND is 3.

see also BAIRSTOW'S METHOD, DESCARTES' SIGN RULE, JENKINS-TRAUB METHOD, LAGUERRE'S METHOD, LEHMER-SCHUR METHOD, MAEHLY'S PROCEDURE, MULLER'S METHOD, ROOT, ZASSENHAUS-BERLEKAMP ALGORITHM

Polynomial Series

see MULTINOMIAL SERIES

Polyomino

A generalization of the DOMINO. An n-omino is defined as a collection of n squares of equal size arranged with coincident sides. FREE polyominoes can be picked up and flipped, so mirror image pieces are considered identical, whereas FIXED polyominoes are distinct if they have different chirality or orientation. FIXED polyominoes are also called LATTICE ANIMALS.

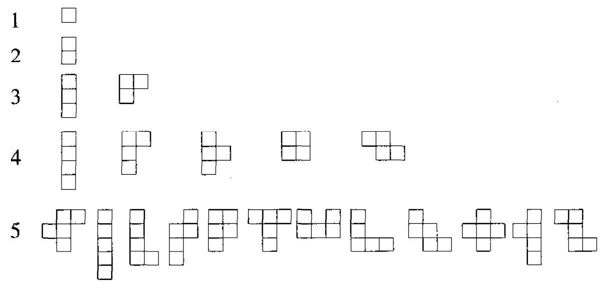
Redelmeier (1981) computed the number of FREE and FIXED polyominoes for $n \leq 24$, and Mertens (1990) gives a simple computer program. The sequence giving the number of FREE polyominoes of each order (Sloane's A000105, Ball and Coxeter 1987) is shown in the second column below, and that for FIXED polyominoes in the third column (Sloane's A014559).

n	Free	Fixed	Pos. Holes
1	1	1	0
2	1	2	0
3	2	6	0
4	5	19	0
5	12	63	0
6	35	216	0
7	108	760	1
8	369	2725	6
9	1285	9910	37
10	4655	39446	384
11	17073	135268	
12	63600	505861	
13	238591	1903890	
14	901971	7204874	
15	3426576	27394666	
16	13079255	104592937	
17	50107909	400795844	
18	192622052	1540820542	
19	742624232	5940738676	
20	2870671950	22964779660	
21	11123060678	88983512783	
22	43191857688	345532572678	
23	168047007728	1344372335524	
24	654999700403	5239988770268	

The best currently known bounds on the number of n-polyominoes are

$$3.72^n < P(n) < 4.65^n$$

(Eden 1961, Klarner 1967, Klarner and Rivest 1973, Ball and Coxeter 1987). For $n = 4$, the quartominoes are called STRAIGHT, L, T, SQUARE, and SKEW. For $n = 5$, the pentominoes are called f, I, L, N, P, T, U, V, W, X, y, and Z (Golomb 1995).



see also DOMINO, HEXOMINO, MONOMINO, PENTOMINO, POLYABOLO, POLYCUBE, POLYHEX, POLYIAMOND, POLYKING, POLYPLET, TETROMINO, TRIOMINO

References

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Klarner, D. A. and Riverst, R. "A Procedure for Improving the Upper Bound for the Number of n-ominoes." *Can. J. Math.* **25**, 585–602, 1973.

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Read, R. C. "Contributions to the Cell Growth Problem." *Canad. J. Math.* **14**, 1–20, 1962.
Redelmeier, D. H. "Counting Polyominoes: Yet Another Attack." *Discrete Math.* **36**, 191–203, 1981.
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Sloane, N. J. A. Sequences A014559 and A000105/M1425 in "An On-Line Version of the Encyclopedia of Integer Sequences."
von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 342–343, 1993.

Polyomino Tiling

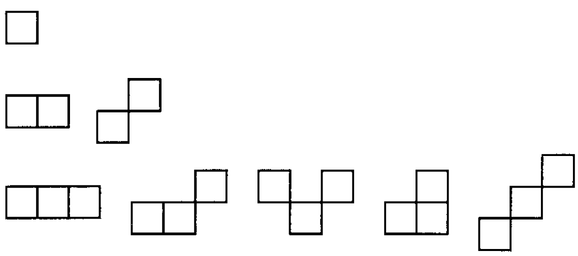
A TILING of the PLANE by specified types of POLYOMINOES. Interestingly, the FIBONACCI NUMBER F_{n+1} gives the number of ways for 2×1 dominoes to cover a $2 \times n$ checkerboard.

see also FIBONACCI NUMBER

References

Gardner, M. "Tiling with Polyominoes, Polyiamonds, and Polyhexes." Ch. 14 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, 1988.

Polyplet



A POLYOMINO-like object made by attaching squares joined either at sides or corners. Because neighboring squares can be in relation to one another as KINGS may move on a CHESSBOARD, polyplets are sometimes also called POLYKINGS. The number of n-polyplets (with holes allowed) are 1, 2, 5, 22, 94, 524, 3031, ... (Sloane's A030222). The number of n-polyplets having bilateral symmetry are 1, 2, 4, 10, 22, 57, 131, ... (Sloane's A030234). The number of n-polyplets not having bilateral symmetry are 0, 0, 1, 12, 72, 467, 2900, ... (Sloane's A030235). The number of fixed n-polyplets are 1, 4, 20, 110, 638, 3832, ... (Sloane's A030232). The number of one-sided n-polyplets are 1, 2, 6, 34, 166, 991, ... (Sloane's A030233).

see also POLYIAMOND, POLYOMINO

References

Sloane, N. J. A. Sequences A030222, A030232, A030233, A030234, and A030235 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Polytope

A convex polytope may be defined as the CONVEX HULL of a finite set of points (which are always bounded), or as the intersection of a finite set of half-spaces. Explicitly, a d-dimensional polytope may be specified as the set of solutions to a system of linear inequalities

$$\mathsf{m}\mathbf{x} \leq \mathbf{b},$$

where m is a real $s \times d$ MATRIX and $\mathbf{b}$ is a real s-VECTOR. The positions of the vertices given by the above equations may be found using a process called VERTEX ENUMERATION.

A regular polytope is a generalization of the PLATONIC SOLIDS to an arbitrary DIMENSION. The NECESSARY condition for the figure with SCHLÄFLI SYMBOL $\{p, q, r\}$ to be a finite polytope is

$$\cos\left(\frac{\pi}{q}\right) < \sin\left(\frac{\pi}{p}\right) \sin\left(\frac{\pi}{r}\right).$$

SUFFICIENCY can be established by consideration of the six figures satisfying this condition. The table below enumerates the six regular polytopes in 4-D (Coxeter 1969, p. 414).

Name	Schläfli Symbol	N_0	N_1	N_2	N_3
regular simplex	$\{3,3,3\}$	5	10	10	5
hypercube	$\{4,3,3\}$	16	32	24	8
16-cell	$\{3,3,4\}$	8	24	32	16
24-cell	$\{3,4,3\}$	24	96	96	24
120-cell	$\{5,3,3\}$	600	1200	720	120
600-cell	$\{3,3,5\}$	120	720	1200	600

Here, N_0 is the number of VERTICES, N_1 the number of EDGES, N_2 the number of FACES, and N_3 the number of cells. These quantities satisfy the identity

$$N_0 - N_1 + N_2 - N_3 = 0,$$

which is a version of the POLYHEDRAL FORMULA.

For n-D with $n \geq 5$, there are only three regular polytopes, the MEASURE POLYTOPE, CROSS POLYTOPE, and regular SIMPLEX (which are analogs of the CUBE, OCTAHEDRON, and TETRAHEDRON).

see also 16-CELL, 24-CELL, 120-CELL, 600-CELL, CROSS POLYTOPE, EDGE (POLYTOPE), FACE, FACET, HYPERCUBE, INCIDENCE MATRIX, MEASURE POLYTOPE, RIDGE, SIMPLEX, TESSERACT, VERTEX (POLYHEDRON)

References

Coxeter, H. S. M. "Regular and Semi-Regular Polytopes I." *Math. Z.* **46**, 380–407, 1940.
Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, 1969.
Eppstein, D. "Polyhedra and Polytopes." `http://www.ics.uci.edu/~eppstein/junkyard/polytope.html`.

Poncelet's Closure Theorem

If an n-sided PONCELET TRANSVERSE constructed for two given CONIC SECTIONS is closed for one point of origin, it is closed for any position of the point of origin. Specifically, given one ELLIPSE inside another, if there exists one CIRCUMINSCRIBED (simultaneously inscribed in the outer and circumscribed on the inner) n-gon, then any point on the boundary of the outer ELLIPSE is the VERTEX of some CIRCUMINSCRIBED n-gon.

References

Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 193, 1965.

Poncelet's Continuity Principle

see PERMANENCE OF MATHEMATICAL RELATIONS PRINCIPLE

Poncelet-Steiner Theorem

All Euclidean GEOMETRIC CONSTRUCTIONS can be carried out with a STRAIGHTEDGE alone if, in addition, one is given the RADIUS of a single CIRCLE and its center. The theorem was suggested by Poncelet in 1822 and proved by Steiner in 1833. A construction using STRAIGHTEDGE alone is called a STEINER CONSTRUCTION.

see also GEOMETRIC CONSTRUCTION, STEINER CONSTRUCTION

References

Dörrie, H. "Steiner's Straight-Edge Problem." §34 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 165–170, 1965.
Steiner, J. *Geometric Constructions with a Ruler, Given a Fixed Circle with Its Center.* New York: Scripta Mathematica, 1950.

Poncelet's Theorem

see PONCELET'S CLOSURE THEOREM

Poncelet Transform

see PONCELET TRANSVERSE

Poncelet Transverse

Let a CIRCLE C_1 lie inside another CIRCLE C_2. From any point on C_2, draw a tangent to C_1 and extend it to C_2. From the point, draw another tangent, etc. For n tangents, the result is called an n-sided PONCELET TRANSFORM.

References

Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 192, 1965.

Pong Hau K'i

A Chinese TIC-TAC-TOE-like game.

see also TIC-TAC-TOE

References

Evans, R. "Pong Hau K'i." *Games and Puzzles* **53**, 19, 1976.
Straffin, P. D. Jr. "Position Graphs for Pong Hau K'i and Mu Torere." *Math. Mag.* **68**, 382–386, 1995.

Pons Asinorum

An elementary theorem in geometry whose name means "ass's bridge." The theorem states that the ANGLES at the base of an ISOSCELES TRIANGLE (defined as a TRIANGLE with two legs of equal length) are equal.

see also ISOSCELES TRIANGLE, PYTHAGOREAN THEOREM

References

Dunham, W. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, p. 38, 1990.

Pontryagin Class

The ith Pontryagin class of a VECTOR BUNDLE is $(-1)^i$ times the ith CHERN CLASS of the complexification of the VECTOR BUNDLE. It is also in the $4i$th cohomology group of the base SPACE involved.

see also CHERN CLASS, STIEFEL-WHITNEY CLASS

Pontryagin Duality

Let G be a locally compact ABELIAN GROUP. Let G^* be the group of all homeomorphisms $G \to R/Z$, in the compact open topology. Then G^* is also a locally compact ABELIAN GROUP, where the asterisk defines a contravariant equivalence of the category of locally compact Abelian groups with itself. The natural mapping $G \to (G^*)^*$, sending g to G, where $G(f) = f(g)$, is an isomorphism and a HOMEOMORPHISM. Under this equivalence, compact groups are sent to discrete groups and vice versa.

see also ABELIAN GROUP, HOMEOMORPHISM

Pontryagin Maximum Principle

A result is CONTROL THEORY. Define

$$H(\psi, x, u) \equiv (\psi, f(x, u)) \equiv \sum_{a=0}^{n} \psi_a f^a(x, u).$$

Then in order for a control $u(t)$ and a trajectory $x(t)$ to be optimal, it is NECESSARY that there exist NONZERO absolutely continuous vector function $\psi(t) = (\psi_0(t), \psi_1(t), \ldots, \psi_n(t))$ corresponding to the functions $u(t)$ and $x(t)$ such that

1. The function $H(\psi(t), x(t), u)$ attains its maximum at the point $u = u(t)$ almost everywhere in the interval $t_0 \leq t \leq t_1$,

$$H(\psi(t), x(t), u(t)) = \max_{u \in U} H(\psi(t), x(t), u).$$

2. At the terminal time t_1, the relations $\psi_0(t_1) \leq 0$ and $H(\psi(t_1), x(t_1), u(t_1)) = 0$ are satisfied.

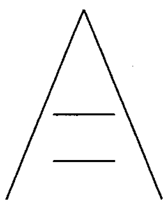
References
Iyanaga, S. and Kawada, Y. (Eds.). "Pontrjagin's Maximum Principle." §88C in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 295–296, 1980.

Pontryagin Number

The Pontryagin number is defined in terms of the PONTRYAGIN CLASS of a MANIFOLD as follows. For any collection of PONTRYAGIN CLASSES such that their cup product has the same DIMENSION as the MANIFOLD, this cup product can be evaluated on the MANIFOLD's FUNDAMENTAL CLASS. The resulting number is called the Pontryagin number for that combination of Pontryagin classes. The most important aspect of Pontryagin numbers is that they are COBORDISM invariant. Together, Pontryagin and STIEFEL-WHITNEY NUMBERS determine an oriented manifold's oriented COBORDISM class.

see also CHERN NUMBER, STIEFEL-WHITNEY NUMBER

Ponzo's Illusion



The upper HORIZONTAL line segment in the above figure appears to be longer than the lower line segment despite the fact that both are the same length.

see also ILLUSION, MÜLLER-LYER ILLUSION, POGGENDORFF ILLUSION, VERTICAL-HORIZONTAL ILLUSION

References
Fineman, M. *The Nature of Visual Illusion.* New York: Dover, p. 153, 1996.

Pop

An action which removes a single element from the top of a QUEUE or STACK, turning the LIST $(a_1, a_2, \ldots, a_n)$ into $(a_2, \ldots, a_n)$ and yielding the element a_1.

see also PUSH, STACK

Population Comparison

Let x_1 and x_2 be the number of successes in variates taken from two populations. Define

$$\hat{p}_1 \equiv \frac{x_1}{n_1} \quad (1)$$

$$\hat{p}_2 \equiv \frac{x_2}{n_2}. \quad (2)$$

The ESTIMATOR of the difference is then $\hat{p}_1 - \hat{p}_2$. Doing a z-TRANSFORM,

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sigma_{\hat{p}_1 - \hat{p}_2}}, \quad (3)$$

where

$$\sigma_{\hat{p}_1 - \hat{p}_2} \equiv \sqrt{{\sigma_{\hat{p}_1}}^2 - {\sigma_{\hat{p}_2}}^2}. \quad (4)$$

The STANDARD ERROR is

$$\mathrm{SE}_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \quad (5)$$

$$\mathrm{SE}_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}} \quad (6)$$

$${s_{\mathrm{pool}}}^2 = \frac{(n_1 - 1){s_1}^2 + (n_2 - 1){s_2}^2}{n_1 + n_2 - 2}. \quad (7)$$

see also z-TRANSFORM

Population Growth

The differential equation describing exponential growth is

$$\frac{dN}{dt} = \frac{N}{\tau}. \quad (1)$$

This can be integrated directly

$$\int_{N_0}^{N} \frac{dN}{N} = \int_0^t \frac{dt}{\tau} \quad (2)$$

$$\ln\left(\frac{N}{N_0}\right) = \frac{t}{\tau}. \quad (3)$$

Exponentiating,

$$N(t) = N_0 e^{t/\tau}. \quad (4)$$

Defining $N(t=1) = N_0 e^{\alpha}$ gives $\tau = 1/\alpha$ in (4), so

$$N(t) = N_0 e^{\alpha t}. \quad (5)$$

The quantity α in this equation is sometimes known as the MALTHUSIAN PARAMETER.

Consider a more complicated growth law

$$\frac{dN}{dt} = \left(\frac{\alpha t - 1}{t}\right) N, \quad (6)$$

where $\alpha > 1$ is a constant. This can also be integrated directly

$$\frac{dN}{N} = \left(\alpha - \frac{1}{t}\right) dt \quad (7)$$

$$\ln N = \alpha t - \ln t + C \quad (8)$$

$$N(t) = \frac{Ce^{\alpha t}}{t}. \quad (9)$$

Note that this expression blows up at $t = 0$. We are given the INITIAL CONDITION that $N(t=1) = N_0 e^{\alpha}$, so $C = N_0$.

$$N(t) = N_0 \frac{e^{\alpha t}}{t}. \quad (10)$$

The t in the DENOMINATOR of (10) greatly suppresses the growth in the long run compared to the simple growth law.

The LOGISTIC GROWTH CURVE, defined by

$$\frac{dN}{dt} = \frac{r(K-N)}{N} \tag{11}$$

is another growth law which frequently arises in biology. It has a rather complicated solution for $N(t)$.

see also GOMPERTZ CURVE, LIFE EXPECTANCY, LOGISTIC GROWTH CURVE, LOTKA-VOLTERRA EQUATIONS, MAKEHAM CURVE, MALTHUSIAN PARAMETER, SURVIVORSHIP CURVE

Porism

An archaic type of mathematical proposition whose purpose is not entirely known.

see also AXIOM, LEMMA, POSTULATE, PRINCIPLE, STEINER'S PORISM, THEOREM

Porter's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

The constant appearing in FORMULAS for the efficiency of the EUCLIDEAN ALGORITHM,

$$\begin{aligned} C &= \frac{6\ln 2}{\pi^2}\left[3\ln 2 + 4\gamma - \frac{24}{\pi^2}\zeta'(2) - 2\right] - \frac{1}{2} \\ &= 1.4670780794\ldots, \end{aligned}$$

where γ is the EULER-MASCHERONI CONSTANT and $\zeta(z)$ is the RIEMANN ZETA FUNCTION.

see also EUCLIDEAN ALGORITHM

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/porter/porter.html`.

Porter, J. W. "On a Theorem of Heilbronn." *Mathematika* **22**, 20–28, 1975.

Pósa's Theorem

Let G be a SIMPLE GRAPH with n VERTICES.

1. If, for every k in $1 \leq k < (n-1)/2$, the number of VERTICES of VALENCY not exceeding k is less than k, and
2. If, for n ODD, the number of VERTICES with VALENCY not exceeding $(n-1)/2$ is less than or equal to $(n-1)/2$,

then G contains a HAMILTONIAN CIRCUIT.

see also HAMILTONIAN CIRCUIT

Poset

see PARTIALLY ORDERED SET

Poset Dimension

The DIMENSION of a POSET $P = (X, \leq)$ is the size of the smallest REALIZER of P. Equivalently, it is the smallest INTEGER d such that P is ISOMORPHIC to a DOMINANCE order in $\mathbb{R}^d$.

see also DIMENSION, DOMINANCE, ISOMORPHIC POSETS, REALIZER

References

Dushnik, B. and Miller, E. W. "Partially Ordered Sets." *Amer. J. Math.* **63**, 600–610, 1941.

Trotter, W. T. *Combinatorics and Partially Ordered Sets: Dimension Theory.* Baltimore, MD: Johns Hopkins University Press, 1992.

Position Four-Vector

The CONTRAVARIANT FOUR-VECTOR arising in special and general relativity,

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \equiv \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix},$$

where c is the speed of light and t is time. Multiplication of two four-vectors gives the spacetime interval

$$\begin{aligned} I = g_{\mu\nu}x^\mu x^\nu &= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \\ &= (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \end{aligned}$$

see also FOUR-VECTOR, LORENTZ TRANSFORMATION, QUATERNION

Position Vector

see RADIUS VECTOR

Positive

A quantity $x > 0$, which may be written with an explicit PLUS SIGN for emphasis, $+x$.

see also NEGATIVE, NONNEGATIVE, PLUS SIGN, ZERO

Positive Definite Function

A POSITIVE definite FUNCTION f on a GROUP G is a FUNCTION for which the MATRIX $\{f(x_i x_j{}^{-1})\}$ is always POSITIVE SEMIDEFINITE HERMITIAN.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Positive Definite Matrix

A MATRIX A is positive definite if

$$(\mathsf{A}\mathbf{v}) \cdot \mathbf{v} > 0 \tag{1}$$

for all VECTORS $\mathbf{v} \neq 0$. All EIGENVALUES of a positive definite matrix are POSITIVE (or, equivalently, the DETERMINANTS associated with *all* upper-left SUBMATRICES are POSITIVE).

The DETERMINANT of a positive definite matrix is POSITIVE, but the converse is not necessarily true (i.e., a matrix with a POSITIVE DETERMINANT is not necessarily positive definite).

A REAL SYMMETRIC MATRIX A is positive definite IFF there exists a REAL nonsingular MATRIX M such that

$$\mathsf{A} = \mathsf{M}\mathsf{M}^{\mathrm{T}}. \tag{2}$$

A 2×2 SYMMETRIC MATRIX

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{3}$$

is positive definite if

$$a{v_1}^2 + 2bv_1v_2 + c{v_2}^2 > 0 \tag{4}$$

for all $\mathbf{v} = (v_1, v_2) \neq 0$.

A HERMITIAN MATRIX A is positive definite if

1. $a_{ii} > 0$ for all i,
2. $a_{ii}a_{ij} > |a_{ij}|^2$ for $i \neq j$,
3. The element of largest modulus must lie on the leading diagonal,
4. $|\mathsf{A}| > 0$.

see also DETERMINANT, EIGENVALUE, HERMITIAN MATRIX, MATRIX, POSITIVE SEMIDEFINITE MATRIX

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1106, 1979.

Positive Definite Quadratic Form

A QUADRATIC FORM $Q(\mathbf{x})$ is said to be positive definite if $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. A REAL QUADRATIC FORM in n variables is positive definite IFF its canonical form is

$$Q(\mathbf{z}) = {z_1}^2 + {z_2}^2 + \ldots + {z_n}^2. \tag{1}$$

A BINARY QUADRATIC FORM

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \tag{2}$$

of two REAL variables is positive definite if it is > 0 for any $(x, y) \neq (0, 0)$, therefore if $a_{11} > 0$ and the DISCRIMINANT $a \equiv a_{11}a_{22} - {a_{12}}^2 > 0$. A BINARY QUADRATIC FORM is positive definite if there exist NONZERO x and y such that

$$(ax^2 + 2bxy + cy^2)^2 \leq \tfrac{4}{3}|ac - b^2| \tag{3}$$

(Le Lionnais 1983).

A QUADRATIC FORM $(\mathbf{x}, \mathsf{A}\mathbf{x})$ is positive definite IFF every EIGENVALUE of A is POSITIVE. A QUADRATIC FORM $Q = (\mathbf{x}, \mathsf{A}\mathbf{x})$ with A a HERMITIAN MATRIX is positive definite if all the principal minors in the top-left corner of A are POSITIVE, in other words

$$a_{11} > 0 \tag{4}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \tag{5}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0. \tag{6}$$

see also INDEFINITE QUADRATIC FORM, POSITIVE SEMIDEFINITE QUADRATIC FORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1106, 1979.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 38, 1983.

Positive Definite Tensor

A TENSOR g whose discriminant satisfies

$$g \equiv g_{11}g_{22} - {g_{12}}^2 > 0.$$

Positive Integer

see $\mathbb{Z}^+$

Positive Semidefinite Matrix

A MATRIX A is positive semidefinite if

$$(\mathsf{A}\mathbf{v}) \cdot \mathbf{v} \geq 0$$

for all $\mathbf{v} \neq 0$.

see also POSITIVE DEFINITE MATRIX

Positive Semidefinite Quadratic Form

A QUADRATIC FORM $Q(\mathbf{x})$ is positive semidefinite if it is never < 0, but is 0 for some $\mathbf{x} \neq \mathbf{0}$. The QUADRATIC FORM, written in the form $(\mathbf{x}, \mathsf{A}\mathbf{x})$, is positive semidefinite IFF every EIGENVALUE of A is NONNEGATIVE.

see also INDEFINITE QUADRATIC FORM, POSITIVE DEFINITE QUADRATIC FORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1106, 1979.

Postage Stamp Problem

Consider a SET $A_k = \{a_1, a_2, \ldots, a_k\}$ of INTEGER denomination postage stamps with $1 = a_1 < a_2 < \ldots < a_k$. Suppose they are to be used on an envelope with room for no more than h stamps. The postage stamp problem then consists of determining the smallest INTEGER $N(h, A_k)$ which cannot be represented by a linear combination $\sum_{i=1}^k x_i a_i$ with $x_i \geq 0$ and $\sum_{i=1}^k x_i < h$.

Exact solutions exist for arbitrary A_k for $k = 2$ and 3. The $k = 2$ solution is

$$n(h, A_2) = (h + 3 - a_2)a_2 - 2$$

for $h \geq a_2 - 2$. The general problem consists of finding

$$n(h, k) = \max_{A_k} n(h, A_k).$$

It is known that

$$n(h, 2) = \left\lfloor \tfrac{1}{4}(h^2 + 6h + 1) \right\rfloor,$$

(Stöhr 1955, Guy 1994), where $\lfloor x \rfloor$ is the FLOOR FUNCTION, the first few values of which are 2, 4, 7, 10, 14, 18, 23, 28, 34, 40, ... (Sloane's A014616).

see also HARMONIOUS GRAPH, STAMP FOLDING

References

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Sloane, N. J. A. Sequence A014616 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Stöhr, A. "Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe I, II." *J. reine angew. Math.* **194**, 111–140, 1955.

Posterior Distribution

see BAYESIAN ANALYSIS

Postnikov System

An iterated FIBRATION of EILENBERG-MAC LANE SPACES. Every TOPOLOGICAL SPACE has this HOMOTOPY type.

see also EILENBERG-MAC LANE SPACE, FIBRATION, HOMOTOPY

Postulate

A statement, also known as an AXIOM, which is taken to be true without PROOF. Postulates are the basic structure from which LEMMAS and THEOREMS are derived. The whole of EUCLIDEAN GEOMETRY, for example, is based on five postulates known as EUCLID'S POSTULATES.

see also ARCHIMEDES' POSTULATE, AXIOM, BERTRAND'S POSTULATE, CONJECTURE, EQUIDISTANCE POSTULATE, EUCLID'S FIFTH POSTULATE, EUCLID'S POSTULATES, LEMMA, PARALLEL POSTULATE, PORISM, PROOF, THEOREM, TRIANGLE POSTULATE

Potato Paradox

You buy 100 pounds of potatoes and are told that they are 99% water. After leaving them outside, you discover that they are now 98% water. The weight of the dehydrated potatoes is then a surprising 50 pounds!

References

Paulos, J. A. *A Mathematician Reads the Newspaper.* New York: BasicBooks, p. 81, 1995.

Potential Function

The term used in physics and engineering for a HARMONIC FUNCTION. Potential functions are extremely useful, for example, in electromagnetism, where they reduce the study of a 3-component VECTOR FIELD to a 1-component SCALAR FUNCTION.

see also HARMONIC FUNCTION, LAPLACE'S EQUATION, SCALAR POTENTIAL, VECTOR POTENTIAL

Potential Theory

The study of HARMONIC FUNCTIONS (also called POTENTIAL FUNCTIONS).

see also HARMONIC FUNCTION, SCALAR POTENTIAL, VECTOR POTENTIAL

References

Kellogg, O. D. *Foundations of Potential Theory.* New York: Dover, 1953.

MacMillan, W. D. *The Theory of the Potential.* New York: Dover, 1958.

Pothenot Problem

see SNELLIUS-POTHENOT PROBLEM

Poulet Number

A FERMAT PSEUDOPRIME to base 2, denoted psp(2), i.e., a COMPOSITE ODD INTEGER such that

$$2^{n-1} \equiv 1 \pmod{n}.$$

The first few Poulet numbers are 341, 561, 645, 1105, 1387, ... (Sloane's A001567). Pomerance *et al.* (1980) computed all 21,853 Poulet numbers less than 25×10^9.

Pomerance has shown that the number of Poulet numbers less than x for sufficiently large x satisfy

$$\exp[(\ln x)^{5/14}] < P_2(x) < x \exp\left(-\frac{\ln x \ln \ln \ln x}{2 \ln \ln x}\right)$$

(Guy 1994).

A Poulet number all of whose DIVISORS d satisfy $d | 2^d - 2$ is called a SUPER-POULET NUMBER. There are an infinite number of Poulet numbers which are not SUPER-POULET NUMBERS. Shanks (1993) calls *any* integer satisfying $2^{n-1} \equiv 1 \pmod{n}$ (i.e., not limited to ODD composite numbers) a FERMATIAN.

see also FERMAT PSEUDOPRIME, PSEUDOPRIME, SUPER-POULET NUMBER

References

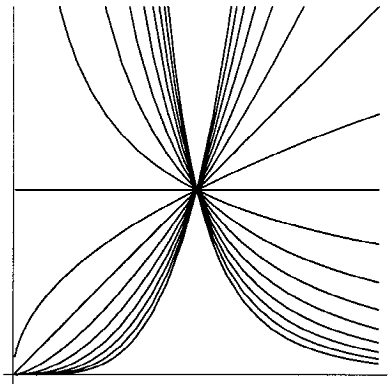
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 28–29, 1994.

Pomerance, C.; Selfridge, J. L.; and Wagstaff, S. S. Jr. "The Pseudoprimes to $25 \cdot 10^9$." *Math. Comput.* **35**, 1003–1026, 1980. Available electronically from `ftp://sable.ox.ac.uk/pub/math/primes/ps2.Z`.

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 115–117, 1993.

Sloane, N. J. A. Sequence A001567/M5441 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Power



The exponent to which a given quantity is raised is known as its POWER. The expression x^a is therefore known as "x to the ath POWER." The rules for combining quantities containing powers are called the EXPONENT LAWS.

Special names given to various powers are listed in the following table.

Power	Name
1/2	square root
1/3	cube root
2	squared
3	cubed

The SUM of pth POWERS of the first n POSITIVE INTEGERS is given by FAULHABER'S FORMULA,

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{k=1}^{p+1}(-1)^{\delta_{kp}}\binom{p+1}{k}B_{p+1-k}n^k,$$

where δ_{kp} is the KRONECKER DELTA, $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, and B_k is a BERNOULLI NUMBER.

Let s_n be the largest INTEGER that is not the SUM of distinct nth powers of POSITIVE INTEGERS (Guy 1994). The first few values for $n = 2, 3, \ldots$ are 128, 12758, 5134240, 67898771, ... (Sloane's A001661).

CATALAN'S CONJECTURE states that 8 and 9 (2^3 and 3^2) are the only consecutive POWERS (excluding 0 and 1), i.e., the only solution to CATALAN'S DIOPHANTINE PROBLEM. This CONJECTURE has not yet been proved or refuted, although R. Tijdeman has proved that there can be only a finite number of exceptions should the CONJECTURE not hold. It is also known that 8 and 9 are the only consecutive CUBIC and SQUARE NUMBERS (in either order). Hyyrö and Mąkowski proved that there do not exist three consecutive POWERS (Ribenboim 1996).

Very few numbers of the form $n^p \pm 1$ are PRIME (where composite powers $p = kb$ need not be considered, since $n^(kb) \pm 1 = (n^k)^b \pm 1$). The only PRIME NUMBERS of the form $n^p - 1$ for $n \leq 100$ and PRIME $2 \leq p \leq 10$ correspond to $n = 2$, i.e., $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, ... The only PRIME NUMBERS of the form $n^p + 1$ for $n \leq 100$ and PRIME $2 \leq p \leq 10$ correspond to $p = 2$ with $n = 1, 2, 4, 6, 10, 14, 16, 20, 24, 26, \ldots$ (Sloane's A005574).

There are no nontrivial solutions to the equation

$$1^n + 2^n + \ldots + m^n = (m+1)^n$$

for $m \leq 10^{2,000,000}$ (Guy 1994, p. 153).

see also APOCALYPTIC NUMBER, BIQUADRATIC NUMBER, CATALAN'S CONJECTURE, CATALAN'S DIOPHANTINE PROBLEM, CUBE ROOT, CUBED, CUBIC NUMBER, EXPONENT, EXPONENT LAWS, FAULHABER'S FORMULA, FIGURATE NUMBER, MOESSNER'S THEOREM, NARCISSISTIC NUMBER, POWER RULE, SQUARE NUMBER, SQUARE ROOT, SQUARED, SUM, WARING'S PROBLEM

References

Barbeau, E. J. *Power Play: A Country Walk through the Magical World of Numbers.* Washington, DC: Math. Assoc. Amer., 1997.

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Guy, R. K. "Diophantine Equations." Ch. D in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 137, 139–198, and 153–154, 1994.

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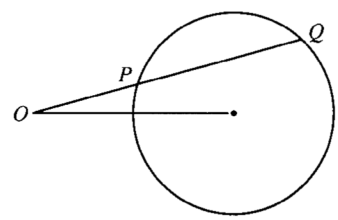
Spanier, J. and Oldham, K. B. "The Integer Powers $(bx+c)^n$ and x^n" and "The Noninteger Powers x^ν." Ch. 11 and 13 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 83–90 and 99–106, 1987.

Power Center

see RADICAL CENTER

Power (Circle)



The POWER of the two points P and Q with respect to a CIRCLE is defined by

$$p \equiv OP \times PQ.$$

Let R be the RADIUS of a CIRCLE and d be the distance between a point P and the circle's center. Then the POWER of the point P relative to the circle is

$$p = d^2 - R^2.$$

If P is outside the CIRCLE, its POWER is POSITIVE and equal to the square of the length of the segment from P to the tangent to the CIRCLE through P. If P is inside the CIRCLE, then the POWER is NEGATIVE and equal to the product of the DIAMETERS through P.

The LOCUS of points having POWER k with regard to a fixed CIRCLE of RADIUS r is a CONCENTRIC CIRCLE of RADIUS $\sqrt{r^2+k}$. The CHORDAL THEOREM states that the LOCUS of points having equal POWER with respect to two given nonconcentric CIRCLES is a line called the RADICAL LINE (or CHORDAL; Dörrie 1965).

see also CHORDAL THEOREM, COAXAL CIRCLES, INVERSE POINTS, INVERSION CIRCLE, INVERSION RADIUS, INVERSIVE DISTANCE, RADICAL LINE

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 27–31, 1967.
Dixon, R. *Mathographics.* New York: Dover, p. 68, 1991.
Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 153, 1965.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 28–34, 1929.
Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., pp. xxii–xxiv, 1995.

Power Curve

The curve with TRILINEAR COORDINATES $a^t : b^t : c^t$ for a given POWER t.

see also POWER

References
Kimberling, C. "Major Centers of Triangles." *Amer. Math. Monthly* **104**, 431–438, 1997.

Power Line

see RADICAL AXIS

Power Point

TRIANGLE centers with TRIANGLE CENTER FUNCTIONS of the form $\alpha = a^n$ are called nth POWER points. The 0th power point is the INCENTER, with TRIANGLE CENTER FUNCTION $\alpha = 1$.

see also INCENTER, TRIANGLE CENTER FUNCTION

References
Groenman, J. T. and Eddy, R. H. "Problem 858 and Solution." *Crux Math.* **10**, 306–307, 1984.
Kimberling, C. "Problem 865." *Crux Math.* **10**, 325–327, 1984.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Power Rule

The DERIVATIVE of the POWER x^n is given by

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

see also CHAIN RULE, DERIVATIVE, EXPONENT LAWS, PRODUCT RULE

References
Anton, H. *Calculus with Analytic Geometry, 2nd ed.* New York: Wiley, p. 131, 1984.

Power Series

A power series in a variable z is an infinite SUM of the form

$$\sum_n^\infty a_i z^i, \tag{1}$$

where $n \geq 0$ and a_i are INTEGERS, REAL NUMBERS, COMPLEX NUMBERS, or any other quantities of a given type.

A CONJECTURE of Pólya is that if a FUNCTION has a POWER series with INTEGER COEFFICIENTS and RADIUS OF CONVERGENCE 1, then either the FUNCTION is RATIONAL or the UNIT CIRCLE is a natural boundary.

A generalized POWER sum $a(h)$ for $h = 0, 1, \ldots$ is given by

$$a(h) = \sum_{i=1}^m A_i(h){\alpha_i}^h, \tag{2}$$

with distinct NONZERO ROOTS α_i, COEFFICIENTS $A_i(h)$ which are POLYNOMIALS of degree $n_i - 1$ for POSITIVE INTEGERS n_i, and $i \in [1, m]$. The generalized POWER sum has order

$$n \equiv \sum_{i=m}^m n_i. \tag{3}$$

For any power series, one of the following is true:

1. The series converges only for $x = 0$.
2. The series converges absolutely for all x.
3. The series converges absolutely for all x in some finite open interval $(-R, R)$ and diverges if $x < -R$ or $x > R$. At the points $x = R$ and $x = -R$, the series may converge absolutely, converge conditionally, or diverge.

To determine the interval of convergence, apply the RATIO TEST for ABSOLUTE CONVERGENCE and solve for x. A POWER series may be differentiated or integrated within the interval of convergence. Convergent power series may be multiplied and divided (if there is no division by zero).

$$\sum_{k=1}^\infty k^{-p}. \tag{4}$$

CONVERGES if $p > 1$ and DIVERGES if $0 < p \leq 1$.

see also BINOMIAL SERIES, CONVERGENCE TESTS, LAURENT SERIES, MACLAURIN SERIES, MULTINOMIAL SERIES, p-SERIES, POLYNOMIAL, POWER SET, QUOTIENT-DIFFERENCE ALGORITHM, RECURRENCE SEQUENCE, SERIES, SERIES REVERSION, TAYLOR SERIES

References

Arfken, G. "Power Series." §5.7 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 313–321, 1985.

Myerson, G. and van der Poorten, A. J. "Some Problems Concerning Recurrence Sequences." *Amer. Math. Monthly* **102**, 698–705, 1995.

Pólya, G. *Mathematics and Plausible Reasoning, Vol. 2: Patterns of Plausible Inference.* Princeton, NJ: Princeton University Press, p. 46, 1954.

Power Set

Given a SET S, the POWER SET of S is the SET of all SUBSETS of S. The order of a POWER set of a SET of order n is 2^n. Power sets are larger than the SETS associated with them.

see also SET, SUBSET

Power Spectrum

For a given signal, the power spectrum gives a plot of the portion of a signal's power (energy per unit time) falling within given frequency bins. The most common way of generating a power spectrum is by using a FOURIER TRANSFORM, but other techniques such as the MAXIMUM ENTROPY METHOD can also be used.

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Power Spectra Estimation Using the FFT" and "Power Spectrum Estimation by the Maximum Entropy (All Poles) Method." §13.4 and 13.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 542–551 and 565–569, 1992.

Power (Statistics)

The probability of getting a positive result for a given test which should produce a positive result.

see also PREDICTIVE VALUE, SENSITIVITY, SPECIFICITY, STATISTICAL TEST

Power Tower

$$a \uparrow\uparrow k \equiv \underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{k},$$

where $\uparrow$ is Knuth's (1976) ARROW NOTATION.

$$a \uparrow^k n = a \uparrow^{k-1} [a \uparrow^k (n-1)].$$

The infinite power tower $x \uparrow\uparrow \infty = x^{x^{\cdot^{\cdot^{\cdot}}}}$ converges IFF $e^{-e} \leq x \leq e^{1/e}$ $(0.0659 \leq x \leq 1.4446)$.

see also ACKERMANN FUNCTION, FERMAT NUMBER, MILLS' CONSTANT

References

Knuth, D. E. "Mathematics and Computer Science: Coping with Finiteness. Advances in our Ability to Compute are Bringing us Substantially Closer to Ultimate Limitations." *Science* **194** 1235–1242, 1976.

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 11 and 226–229, 1991.

Power (Triangle)

The total POWER of a TRIANGLE is defined by

$$P \equiv \tfrac{1}{2}({a_1}^2 + {a_2}^2 + {a_3}^2), \tag{1}$$

where a_i are the side lengths, and the "partial power" is defined by

$$p_1 \equiv \tfrac{1}{2}({a_2}^2 + {a_3}^2 - {a_1}^2). \tag{2}$$

Then

$$P_1 = a_2 a_3 \cos\alpha_1 \tag{3}$$

$$P = p_1 + p_2 + p_3 \tag{4}$$

$$P^2 + {p_1}^2 + {p_2}^2 + {p_3}^2 = {a_1}^4 + {a_2}^4 + {a_3}^4 \tag{5}$$

$$\Delta = \tfrac{1}{2}\sqrt{p_2 p_3 + p_3 p_1 + p_3 p_1} \tag{6}$$

$$p_1 = \overline{A_1 H_2} \cdot \overline{A_1 A_3} \tag{7}$$

$$\frac{a_1 p_1}{\cos\alpha_1} = a_1 a_2 a_3 = 4\Delta R \tag{8}$$

$$p_1 \tan\alpha_1 = p_2 \tan\alpha_2 = p_3 \tan\alpha_3, \tag{9}$$

where Δ is the AREA of the TRIANGLE and H_i are the FEET of the ALTITUDES. Finally, if a side of the TRIANGLE and the value of any partial power are given, then the LOCUS of the third VERTEX is a CIRCLE or straight line.

see also ALTITUDE, FOOT, TRIANGLE

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 260–261, 1929.

Powerfree

see BIQUADRATEFREE, CUBEFREE, PRIME NUMBER, SQUAREFREE

Powerful Number

An INTEGER m such that if $p|m$, then $p^2|m$, is called a powerful number. The first few are 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, ... (Sloane's A001694). Powerful numbers are always of the form a^2b^3 for $a, b \geq 1$.

Not every NATURAL NUMBER is the sum of two powerful numbers, but Heath-Brown (1988) has shown that every sufficiently large NATURAL NUMBER is the sum of at most three powerful numbers. There are infinitely many pairs of consecutive powerful numbers, but Erdős has conjectured that there do not exist three consecutive powerful numbers. The CONJECTURE that there are no powerful number triples implies that there are infinitely many Wieferich primes (Granville 1986, Vardi 1991).

A separate usage of the term powerful number is for numbers which are the sums of the positive powers of their digits. The first few are 1, 2, 3, 4, 5, 6, 7, 8, 9, 24, 43, 63, 89, ... (Sloane's A007532).

References
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Guy, R. K. "Powerful Numbers." §B16 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 67–73, 1994.
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Ribenboim, P. "Catalan's Conjecture." *Amer. Math. Monthly* **103**, 529–538, 1996.
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Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 59–62, 1991.

Practical Number

A number n is practical if for all $k \le n$, k is the sum of distinct proper divisors of n. Defined in 1948 by A. K. Srinivasen. All even PERFECT NUMBERS are practical. The number

$$m = 2^{n-1}(2^n - 1)$$

is practical for all $n = 2, 3, \ldots$. The first few practical numbers are 1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48, 54, 56, ... (Sloane's A005153). G. Melfi has computed twins, triplets, and 5-tuples of practical numbers. The first few 5-tuples are 12, 18, 30, 198, 306, 462, 1482, 2550, 4422, ...

References
Melfi, G. "On Two Conjectures About Practical Numbers." *J. Number Th.* **56**, 205–210, 1996.
Melfi, G. "Practical Numbers." http://www.dm.unipi.it/gauss-pages/melfi/public_html/pratica.html.
Sloane, N. J. A. Sequence A005153/M0991 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Pratt Certificate

A primality certificate based on FERMAT'S LITTLE THEOREM CONVERSE. Although the general idea had been well-established for some time, Pratt became the first to prove that the certificate tree was of polynomial size and could also be verified in polynomial time. He was also the first to observe that the tree implies that PRIMES are in the complexity class NP.

To generate a Pratt certificate, assume that n is a POSITIVE INTEGER and $\{p_i\}$ is the set of PRIME FACTORS of $n-1$. Suppose there exists an INTEGER x (called a "WITNESS") such that $x^{n-1} \equiv 1 \pmod{n}$ but $x^e \not\equiv 1 \pmod{n}$ whenever e is one of $(n-1)/p_i$. Then FERMAT'S LITTLE THEOREM CONVERSE states that n is PRIME (Wagon 1991, pp. 278–279).

By applying FERMAT'S LITTLE THEOREM CONVERSE to n and recursively to each purported factor of $n-1$, a certificate for a given PRIME NUMBER can be generated. Stated another way, the Pratt certificate gives a proof that a number a is a PRIMITIVE ROOT of the multiplicative GROUP (mod p) which, along with the fact that a has order $p-1$, proves that p is a PRIME.

```
7919 —— 7
  2
  37 —— 2
     2
     3 —— 2
          2
  107 —— 2
     2
     53 —— 2
        2
        13 —— 2
           2
           3 —— 2
                2
```

The figure above gives a certificate for the primality of $n = 7919$. The numbers to the right of the dashes are WITNESSES to the numbers to left. The set $\{p_i\}$ for $n-1 = 7918$ is given by $\{2, 37, 107\}$. Since $7^{7918} \equiv 1 \pmod{7919}$ but $7^{7918/2}$, $7^{7918/37}$, $7^{7918/107} \not\equiv 1 \pmod{7919}$, 7 is a WITNESS for 7919. The PRIME divisors of $7918 = 7919 - 1$ are 2, 37, and 107. 2 is a so-called "self-WITNESS" (i.e., it is recognized as a PRIME without further ado), and the remainder of the witnesses are shown as a nested tree. Together, they certify that 7919 is indeed PRIME. Because it requires the FACTORIZATION of $n-1$, the METHOD of Pratt certificates is best applied to small numbers (or those numbers n known to have easily factorable $n-1$).

A Pratt certificate is quicker to generate for small numbers than are other types of primality certificates. The *Mathematica*® (Wolfram Research, Champaign, IL) task `ProvablePrime[n]` therefore generates an ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE only for numbers above a certain limit (10^{10} by default), and a Pratt certificate for smaller numbers.

see also ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE, FERMAT'S LITTLE THEOREM CONVERSE, PRIMALITY CERTIFICATE, WITNESS

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Pratt-Kasapi Theorem

see HOEHN'S THEOREM

Precedes

The relationship x precedes y is written $x \prec y$. The relation x precedes or is equal to y is written $x \preceq y$.

see also SUCCEEDS

Precession

see CURVE OF CONSTANT PRECESSION

Precisely Unless

If A is true precisely unless B, then B implies not-A and not-B implies A. J. H. Conway has suggested the term "UNLESSS" for this state of affairs, by analogy with IFF.

see also IFF, UNLESS

Predicate

A function whose value is either TRUE or FALSE.

see also AND, FALSE, OR, PREDICATE CALCULUS, TRUE, XOR

Predicate Calculus

The branch of formal LOGIC dealing with representing the logical connections between statements as well as the statements themselves.

see also GÖDEL'S INCOMPLETENESS THEOREM, LOGIC, PREDICATE

Predictability

Predictability at a time τ in the future is defined by

$$\frac{R(x(t), x(t+\tau))}{H(x(t))},$$

and linear predictability by

$$\frac{L(x(t), x(t+\tau))}{H(x(t))},$$

where R and L are the REDUNDANCY and LINEAR REDUNDANCY, and H is the ENTROPY.

Prediction Paradox

see UNEXPECTED HANGING PARADOX

Predictive Value

The POSITIVE predictive value is the probability that a test gives a true result for a true statistic. The negative predictive value is the probability that a test gives a false result for a false statistic.

see also POWER (STATISTICS), SENSITIVITY, SPECIFICITY, STATISTICAL TEST

Predictor-Corrector Methods

A general method of integrating ORDINARY DIFFERENTIAL EQUATIONS. It proceeds by extrapolating a polynomial fit to the derivative from the previous points to the new point (the predictor step), then using this to interpolate the derivative (the corrector step). Press *et al.* (1992) opine that predictor-corrector methods have been largely supplanted by the BULIRSCH-STOER and RUNGE-KUTTA METHODS, but predictor-corrector schemes are still in common use.

see also ADAMS' METHOD, GILL'S METHOD, MILNE'S METHOD, RUNGE-KUTTA METHOD

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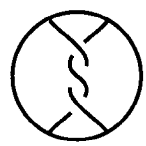
Pretzel Curve

see KNOT CURVE

Pretzel Knot



A KNOT obtained from a TANGLE which can be represented by a FINITE sequence of INTEGERS.

see also TANGLE

References

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Primality Certificate

A short set of data that proves the primality of a number. A certificate can, in general, be checked much more quickly than the time required to generate the certificate. Varieties of primality certificates include the PRATT CERTIFICATE and ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE.

see also ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE, COMPOSITENESS CERTIFICATE, PRATT CERTIFICATE

References

Wagon, S. "Prime Certificates." §8.7 in *Mathematica in Action.* New York: W. H. Freeman, pp. 277–285, 1991.

Primality Test

A test to determine whether or not a given number is PRIME. The RABIN-MILLER STRONG PSEUDOPRIME TEST is a particularly efficient ALGORITHM used by *Mathematica*® version 2.2 (Wolfram Research, Champaign, IL). Like many such algorithms, it is a probabilistic test using PSEUDOPRIMES, and can potentially (although with very small probability) falsely identify a COMPOSITE NUMBER as PRIME (although not vice versa). Unlike PRIME FACTORIZATION, primality testing is believed to be a P-PROBLEM (Wagon 1991). In order to guarantee primality, an almost certainly slower algorithm capable of generating a PRIMALITY CERTIFICATE must be used.

see also ADLEMAN-POMERANCE-RUMELY PRIMALITY TEST, FERMAT'S LITTLE THEOREM CONVERSE, FERMAT'S PRIMALITY TEST, FERMAT'S THEOREM, LUCAS-LEHMER TEST, MILLER'S PRIMALITY TEST, PÉPIN'S TEST, POCKLINGTON'S THEOREM, PROTH'S THEOREM, PSEUDOPRIME, RABIN-MILLER STRONG PSEUDOPRIME TEST, WARD'S PRIMALITY TEST, WILSON'S THEOREM

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Primary

Each factor ${p_i}^{\alpha_i}$ in an INTEGER'S PRIME DECOMPOSITION is called a primary.

Primary Representation

Let π be a unitary REPRESENTATION of a GROUP G on a separable HILBERT SPACE, and let $R(\pi)$ be the smallest weakly closed algebra of bounded linear operators containing all $\pi(g)$ for $g \in G$. Then π is primary if the center of $R(\pi)$ consists of only scalar operations.

References

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Prime

A symbol used to distinguish one quantity x' ("x prime") from another related x. Primes are most commonly used to denote transformed coordinates, conjugate points, and DERIVATIVES.

see also PRIME ALGEBRAIC NUMBER, PRIME NUMBER

Prime Algebraic Number

An irreducible ALGEBRAIC INTEGER which has the property that, if it divides the product of two algebraic INTEGERS, then it DIVIDES at least one of the factors. 1 and -1 are the only INTEGERS which DIVIDE every INTEGER. They are therefore called the PRIME UNITS.

see also ALGEBRAIC INTEGER, PRIME UNIT

Prime Arithmetic Progression

Let the number of PRIMES of the form $mk + n$ less than x be denoted $\pi_{m,n}(x)$. Then

$$\lim_{x\to\infty} \frac{\pi_{a,b}(x)}{\mathrm{Li}(x)} = \frac{1}{\phi(a)},$$

where $\mathrm{Li}(x)$ is the LOGARITHMIC INTEGRAL and $\phi(x)$ is the TOTIENT FUNCTION.

Let P be an increasing arithmetic progression of n PRIMES with minimal difference $d > 0$. If a PRIME $p \leq n$ does not divide d, then the elements of P must assume all residues modulo p, specifically, some element of P must be divisible by p. Whereas P contains only primes, this element must be equal to p.

If $d < n\#$ (where $n\#$ is the PRIMORIAL of n), then some prime $p \leq n$ does not divide d, and that prime p is in P. Thus, in order to determine if P has $d < n\#$, we need only check a finite number of possible P (those with $d < n\#$ and containing prime $p \leq n$) to see if they contain only primes. If not, then $d \geq n\#$. If $d = n\#$, then the elements of P cannot be made to cover all residues of any prime p. The PRIME PATTERNS CONJECTURE then asserts that there are infinitely many arithmetic progressions of primes with difference d.

A computation shows that the smallest possible common difference for a set of n or more PRIMES in arithmetic progression for $n = 1, 2, 3, \ldots$ is 0, 1, 2, 6, 6, 30, 150, 210, 210, 210, 2310, 2310, 30030, 510510, ... (Sloane's A033188, Ribenboim 1989, Dubner and Nelson 1997, Wilson). The values up to $n = 13$ are rigorous, while the remainder are lower bounds which assume the validity of the PRIME PATTERNS CONJECTURE and are simply given by $p_{n-7}\#$, where p_i is the ith PRIME. The smallest first terms of arithmetic progressions of n primes *with minimal differences* are 2, 2, 3, 5, 5, 7, 7, 199, 199, 199, 60858179, 147692845283, 14933623, ... (Sloane's A033189; Wilson).

Smaller first terms are possible for nonminimal n-term progressions. Examples include the 8-term progression $11 + 1210230k$ for $k = 0, 1, \ldots, 7$, the 12-term progression $23143 + 30030k$ for $k = 0, 1, \ldots, 11$ (Golubev 1969, Guy 1994), and the 13-term arithmetic progression $766439 + 510510k$ for $k = 0, 1, \ldots, 12$ (Guy 1994).

The largest known set of primes in ARITHMETIC SEQUENCE is 22,

$$11,410,337,850,553 + 4,609,098,694,200k$$

for $k = 0, 1, \ldots, 21$ (Pritchard *et al.* 1995, UTS School of Mathematical Sciences).

The largest known sequence of *consecutive* PRIMES in ARITHMETIC PROGRESSION (i.e., all the numbers between the first and last term in the progression, except for the members themselves, are composite) is ten, given by

$$100,996,972,469,714,247,637,786,655,587,969,840,329,509,324,689,190,041,803,603,417,758,904,341,703,348,882,159,067,229,719 + 210k$$

for $k = 0, 1, \ldots, 9$, discovered by Harvey Dubner, Tony Forbes, Manfred Toplic, *et al.* on March 2, 1998. This beats the record of nine set on January 15, 1998 by the same investigators,

$$99,679,432,066,701,086,484,490,653,695,853,561,638,982,364,080,991,618,395,774,048,585,529,071,475,461,114,799,677,694,651 + 210k$$

for $k = 0, 1, \ldots, 8$ (two sequences of nine are now known), the progression of eight consecutive primes given by

$$43,804,034,644,029,893,325,717,710,709,965,599,930,101,479,007,432,825,862,362,446,333,961,919,524,977,985,103,251,510,661 + 210k$$

for $k = 0, 1, \ldots, 7$, discovered by Harvey Dubner, Tony Forbes, *et al.* on November 7, 1997 (several are now known), and the progression of seven given by

$$1,089,533,431,247,059,310,875,780,378,922,957,732,908,036,492,993,138,195,385,213,105,561,742,150,447,308,967,213,141,717,486,151 + 210k,$$

for $k = 0, 1, \ldots, 6$, discovered by H. Dubner and H. K. Nelson on Aug. 29, 1995 (Peterson 1995, Dubner and Nelson 1997). The smallest sequence of six consecutive PRIMES in arithmetic progression is

$$121,174,811 + 30k$$

for $k = 0, 1, \ldots, 5$ (Lander and Parkin 1967, Dubner and Nelson 1997). According to Dubner *et al.,* a trillion-fold increase in computer speed is needed before the search for a sequence of 11 consecutive primes is practical, so they expect the ten-primes record to stand for a long time to come.

It is conjectured that there are arbitrarily long sequences of PRIMES in ARITHMETIC PROGRESSION (Guy 1994).

see also ARITHMETIC PROGRESSION, CUNNINGHAM CHAIN, DIRICHLET'S THEOREM, LINNIK'S THEOREM, PRIME CONSTELLATION, PRIME-GENERATING POLYNOMIAL, PRIME NUMBER THEOREM, PRIME PATTERNS CONJECTURE, PRIME QUADRUPLET

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Prime Array

Find the $m \times n$ ARRAY of single digits which contains the maximum possible number of PRIMES, where allowable PRIMES may lie along any horizontal, vertical, or diagonal line. For $m = n = 2$, 11 PRIMES are maximal and are contained in the two distinct arrays

$$A(2,2) = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix},$$

giving the PRIMES (3, 7, 13, 17, 31, 37, 41, 43, 47, 71, 73) and (3, 7, 13, 17, 19, 31, 37, 71, 73, 79, 97), respectively. For the 3×2 array, 18 PRIMES are maximal and are contained in the arrays

$$A(3,2) = \begin{bmatrix} 1 & 1 & 3 \\ 9 & 7 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 7 & 2 \\ 3 & 5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 7 & 2 \\ 4 & 3 & 9 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 7 & 5 \\ 4 & 3 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 7 & 9 \\ 3 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 7 & 9 \\ 4 & 3 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 7 & 9 \\ 4 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 6 \\ 4 & 7 & 9 \end{bmatrix}, \begin{bmatrix} 3 & 7 & 6 \\ 4 & 1 & 9 \end{bmatrix}.$$

The best 3×3, 4×4, and 5×5 prime arrays known were found by C. Rivera and J. Ayala in 1998. They are

$$A(3,3) = \begin{bmatrix} 1 & 1 & 3 \\ 7 & 5 & 4 \\ 9 & 3 & 7 \end{bmatrix},$$

which contains 30 PRIMES,

$$A(4,4) = \begin{bmatrix} 1 & 1 & 3 & 9 \\ 6 & 4 & 5 & 1 \\ 7 & 3 & 9 & 7 \\ 3 & 9 & 2 & 9 \end{bmatrix},$$

which contains 63 PRIMES, and

$$A(5,5) = \begin{bmatrix} 1 & 1 & 9 & 3 & 3 \\ 9 & 9 & 5 & 6 & 3 \\ 8 & 9 & 4 & 1 & 7 \\ 3 & 3 & 7 & 3 & 1 \\ 3 & 2 & 9 & 3 & 9 \end{bmatrix},$$

which contains 116 PRIMES. S. C. Root found the a 6×6 array containing 187 primes:

$$A(6,6) = \begin{bmatrix} 3 & 1 & 7 & 3 & 3 & 3 \\ 9 & 9 & 5 & 6 & 3 & 9 \\ 1 & 1 & 8 & 1 & 4 & 2 \\ 1 & 3 & 6 & 3 & 7 & 3 \\ 3 & 4 & 9 & 1 & 9 & 9 \\ 3 & 7 & 9 & 3 & 7 & 9 \end{bmatrix}.$$

In 1998, M. Oswald found five new 6×6 arrays with 187 primes:

$$\begin{bmatrix} 1 & 3 & 9 & 1 & 9 & 9 \\ 3 & 1 & 7 & 2 & 3 & 4 \\ 9 & 9 & 4 & 7 & 9 & 3 \\ 9 & 1 & 5 & 7 & 1 & 3 \\ 9 & 8 & 3 & 6 & 1 & 7 \\ 9 & 1 & 7 & 3 & 3 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 9 & 1 & 9 & 9 \\ 9 & 1 & 7 & 2 & 3 & 4 \\ 6 & 9 & 4 & 7 & 9 & 3 \\ 7 & 1 & 5 & 7 & 1 & 3 \\ 9 & 8 & 3 & 6 & 1 & 7 \\ 9 & 1 & 7 & 3 & 3 & 3 \end{bmatrix},$$
$$\begin{bmatrix} 3 & 1 & 7 & 3 & 3 & 3 \\ 9 & 9 & 5 & 6 & 3 & 9 \\ 1 & 1 & 8 & 1 & 4 & 2 \\ 1 & 3 & 6 & 3 & 7 & 3 \\ 3 & 4 & 9 & 1 & 9 & 9 \\ 3 & 7 & 9 & 9 & 3 & 9 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 7 & 3 & 3 & 3 \\ 9 & 9 & 5 & 6 & 3 & 9 \\ 1 & 1 & 8 & 1 & 4 & 2 \\ 1 & 3 & 6 & 3 & 7 & 3 \\ 3 & 4 & 9 & 1 & 9 & 9 \\ 9 & 7 & 9 & 3 & 7 & 9 \end{bmatrix},$$
$$\begin{bmatrix} 3 & 1 & 7 & 3 & 3 & 3 \\ 9 & 9 & 5 & 6 & 3 & 9 \\ 1 & 1 & 8 & 1 & 4 & 5 \\ 1 & 3 & 6 & 3 & 7 & 3 \\ 3 & 4 & 9 & 1 & 9 & 9 \\ 9 & 9 & 9 & 2 & 3 & 3 \end{bmatrix}.$$

Rivera and Ayala conjecture that the 30-prime solution for $A(3,3)$ is maximal and unique. The following intervals have been completely searched without finding another 30-prime or better 3×3 array: $[1, 67 \times 10^6]$, $[100 \times 10^6, 133 \times 10^6]$, $[200 \times 10^6, 228 \times 10^6]$, $[300 \times 10^6, 325 \times 10^6]$, and $[400 \times 10^6, 418 \times 10^6]$.

Heuristic arguments by Rivera and Ayala suggest that the maximum possible number of primes in 4×4, 5×5, and 6×6 arrays are 58–63, 112–121, and 205–218, respectively.

see also ARRAY, PRIME ARITHMETIC PROGRESSION, PRIME CONSTELLATION, PRIME STRING

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Weisstein, E. W. "Prime Arrays." `http://www.astro.virginia.edu/~eww6n/math/notebooks/PrimeArray.m`.

Prime Circle

A prime circle of order $2m$ is a CIRCULAR PERMUTATION of the numbers from 1 to $2m$ with adjacent PAIRS summing to a PRIME. The number of prime circles for $m = 1, 2, \ldots$, are 1, 1, 1, 2, 48, ...

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Prime Cluster

see PRIME CONSTELLATION

Prime Constellation

A prime constellation, also called a PRIME k-TUPLE or PRIME k-TUPLET, is a sequence of k consecutive numbers such that the difference between the first and last is, in some sense, the least possible. More precisely, a prime k-tuplet is a sequence of consecutive PRIMES $(p_1, p_2, \ldots, p_k)$ with $p_k - p_1 = s(k)$, where $s(k)$ is the smallest number s for which there exist k integers $b_1 < b_2 < \ldots < b_k$, $b_k - b_1 = s$ and, for every PRIME q, not all the residues modulo q are represented by b_1, b_2, $\ldots$, b_k (Forbes). For each k, this definition excludes a finite number of clusters at the beginning of the prime number sequence. For example, (97, 101, 103, 107, 109) satisfies the conditions of the definition of a prime 5-tuplet, but (3, 5, 7, 11, 13) does not because all three residues modulo 3 are represented (Forbes).

A prime double with $s(2) = 2$ is of the form $(p, p+2)$ and is called a pair of TWIN PRIMES. Prime doubles of the form $(p, p+6)$ are called SEXY PRIMES. A prime triplet has $s(3) = 6$. However, the constellation $(p, p+2, p+4)$ cannot exist, since both $p+2$ and $p+4$ cannot be PRIME. However, there are several types of prime triplets which can exist: $(p, p+2, p+6)$, $(p, p+4, p+6)$, $(p, p+6, p+12)$. A PRIME QUADRUPLET is a constellation of four successive PRIMES with minimal distance $s(4) = 8$, and is of the form $(p, p+2, p+6, p+8)$. The sequence $s(n)$ therefore begins 2, 6, 8, and continues 12, 16, 20, 26, 30, ... (Sloane's A008407). Another quadruplet constellation is $(p, p+6, p+12, p+18)$.

The first FIRST HARDY-LITTLEWOOD CONJECTURE states that the number of constellations $\leq x$ are asymptotically given by

$$P_x(p,p+2) \sim 2\prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2}\int_2^x \frac{dx'}{(\ln x')^2}$$
$$= 1.320323632\int_2^x \frac{dx'}{(\ln x')^2} \tag{1}$$
$$P_x(p,p+4) \sim 2\prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2}\int_2^x \frac{dx'}{(\ln x')^2}$$
$$= 1.320323632\int_2^x \frac{dx'}{(\ln x')^2} \tag{2}$$
$$P_x(p,p+6) \sim 4\prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2}\int_2^x \frac{dx'}{(\ln x')^2}$$
$$= 2.640647264\int_2^x \frac{dx'}{(\ln x')^2} \tag{3}$$
$$P_x(p,p+2,p+6) \sim \frac{9}{2}\prod_{p\geq 5}\frac{p^2(p-3)}{(p-1)^3}\int_2^x \frac{dx'}{(\ln x')^3}$$
$$= 2.858248596\int_2^x \frac{dx'}{(\ln x')^3} \tag{4}$$
$$P_x(p,p+4,p+6) \sim \frac{9}{2}\prod_{p\geq 5}\frac{p^2(p-3)}{(p-1)^3}\int_2^x \frac{dx'}{(\ln x')^3}$$
$$= 2.858248596\int_2^x \frac{dx'}{(\ln x')^3} \tag{5}$$
$$P_x(p,p+2,p+6,p+8) \sim \frac{27}{2}\prod_{p\geq 5}\frac{p^3(p-4)}{(p-1)^4}\int_2^x \frac{dx'}{(\ln x')^4}$$
$$= 4.151180864\int_2^x \frac{dx'}{(\ln x')^4} \tag{6}$$
$$P_x(p,p+4,p+6,p+10) \sim 27\prod_{p\geq 5}\frac{p^3(p-4)}{(p-1)^4}\int_2^x \frac{dx'}{(\ln x')^4}$$
$$= 8.302361728\int_2^x \frac{dx'}{(\ln x')^4}. \tag{7}$$

These numbers are sometimes called the HARDY-LITTLEWOOD CONSTANTS. (1) is sometimes called the extended TWIN PRIME CONJECTURE, and

$$C_{p,p+2} = 2\Pi_2, \tag{8}$$

where Π_2 is the TWIN PRIMES CONSTANT. Riesel (1994) remarks that the HARDY-LITTLEWOOD CONSTANTS can be computed to arbitrary accuracy without needing the infinite sequence of primes.

The integrals above have the analytic forms

$$\int_2^x \frac{dx'}{(\ln x')^2} = \mathrm{Li}(x) + \frac{2}{\ln 2} - \frac{n}{\ln n} \tag{9}$$
$$\int_2^x \frac{dx'}{(\ln x')^3} = \tfrac{1}{2}\,\mathrm{Li}(x) - \frac{x(1+\ln x)}{(\ln x)^2} + \frac{1}{\ln 2} + \frac{1}{(\ln 2)^2} \tag{10}$$
$$\int_2^x \frac{dx'}{(\ln x')^4} = \frac{1}{6}\left\{\mathrm{Li}(x) + \frac{2[2+\ln 2+(\ln 2)^2]}{(\ln 2)^3} - \frac{n[2+\ln n+(\ln n)^2]}{(\ln n)^3}\right\}, \tag{11}$$

where $\mathrm{Li}(x)$ is the LOGARITHMIC INTEGRAL.

The following table gives the number of prime constellations $\leq 10^8$, and the second table gives the values predicted by the Hardy-Littlewood formulas.

Count	10^5	10^6	10^7	10^8
$(p,p+2)$	1224	8169	58980	440312
$(p,p+4)$	1216	8144	58622	440258
$(p,p+6)$	2447	16386	117207	879908
$(p,p+2,p+6)$	259	1393	8543	55600
$(p,p+4,p+6)$	248	1444	8677	55556
$(p,p+2,p+6,p+8)$	38	166	899	4768
$(p,p+6,p+12,p+18)$	75	325	1695	9330

Hardy-Littlewood	10^5	10^6	10^7	10^8
$(p,p+2)$	1249	8248	58754	440368
$(p,p+4)$	1249	8248	58754	440368
$(p,p+6)$	2497	16496	117508	880736
$(p,p+2,p+6)$	279	1446	8591	55491
$(p,p+4,p+6)$	279	1446	8591	55491
$(p,p+2,p+6,p+8)$	53	184	863	4735
$(p,p+6,p+12,p+18)$				

Consider prime constellations in which each term is of the form n^2+1. Hardy and Littlewood showed that the number of prime constellations of this form $< x$ is given by

$$P(x) \sim C\sqrt{x}\,(\ln x)^{-1}, \tag{12}$$

where

$$C = \prod_{\substack{p>2\\ p\text{ prime}}}\left[1 - \frac{(-1)^{(p-1)/2}}{p-1}\right] = 1.3727\ldots \tag{13}$$

(Le Lionnais 1983).

Forbes gives a list of the "top ten" prime k-tuples for $2 \leq k \leq 17$. The largest known 14-constellations are (11319107721272355839 + 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50), (10756418345074847279 + 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50), (6808488664768715759 + 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50), (6120794469172998449 + 0, 2, 8, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 50), (5009128141636113611 + 0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50).

The largest known prime 15-constellations are (8424434363963335630 6067 + 0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56), (8985208997951457604337+0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56), (3594585413466972694697 + 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56), (3514383375461541232577+0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56), (3493864509985912609487 + 0, 2, 6, 12, 14, 20, 24, 26, 30, 36, 42, 44, 50, 54, 56).

The largest known prime 16-constellations are (3259125690557440336637+0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60), (1522014304823128379267+0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60), (47710850533373130107 + 0, 2, 6, 12, 14, 20, 26, 30, 32, 36, 42, 44, 50, 54, 56, 60), (13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73).

The largest known prime 17-constellations are (3259125690557440336631 + 0, 6, 8, 12, 18, 20, 26, 32, 36, 38, 42, 48, 50, 56, 60, 62, 66), (17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83) (13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79).

see also COMPOSITE RUNS, PRIME ARITHMETIC PROGRESSION, k-TUPLE CONJECTURE, PRIME k-TUPLES CONJECTURE, PRIME QUADRUPLET, SEXY PRIMES, TWIN PRIMES

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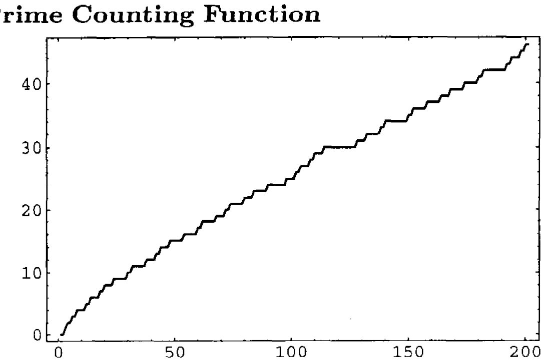
Prime Counting Function



The function $\pi(n)$ giving the number of PRIMES less than n (Shanks 1993, p. 15). The first few values are 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, ... (Sloane's A000720). The following table gives the values of $\pi(n)$ for powers of 10 (Sloane's A006880; Hardy and Wright 1979, p. 4; Shanks 1993, pp. 242–243; Ribenboim 1996, p. 237). Deleglise and Rivat (1996) have computed $\pi(10^{20})$.

$$\begin{aligned}
\pi(10^3) &= 168 \\
\pi(10^4) &= 1{,}229 \\
\pi(10^5) &= 9{,}592 \\
\pi(10^6) &= 78,498 \\
\pi(10^7) &= 664,579 \\
\pi(10^8) &= 5,761,455 \\
\pi(10^9) &= 50,847,534 \\
\pi(10^{10}) &= 455,052,511 \\
\pi(10^{11}) &= 4,118,054,813 \\
\pi(10^{12}) &= 37,607,912,018 \\
\pi(10^{13}) &= 346,065,536,839 \\
\pi(10^{14}) &= 3,204,941,750,802 \\
\pi(10^{15}) &= 29,844,570,422,669 \\
\pi(10^{16}) &= 279,238,341,033,925 \\
\pi(10^{17}) &= 2,623,557,157,654,233 \\
\pi(10^{18}) &= 24,739,954,287,740,860 \\
\pi(10^{19}) &= 234,057,667,276,344,607,
\end{aligned}$$

$\pi(10^9)$ is incorrectly given as 50,847,478 in Hardy and Wright (1979). The prime counting function can be expressed by LEGENDRE'S FORMULA, LEHMER'S FORMULA, MAPES' METHOD, or MEISSEL'S FORMULA. A brief history of attempts to calculate $\pi(n)$ is given by Berndt (1994). The following table is taken from Riesel (1994).

Method	Time	Storage
Legendre	$\mathcal{O}(x)$	$\mathcal{O}(x^{1/2})$
Meissel	$\mathcal{O}(x/(\ln x)^3)$	$\mathcal{O}(x^{1/2}/\ln x)$
Lehmer	$\mathcal{O}(x/(\ln x)^4)$	$\mathcal{O}(x^{1/3}/\ln x)$
Mapes'	$\mathcal{O}(x^{0.7})$	$\mathcal{O}(x^{0.7})$
Lagarias-Miller-Odlyzko	$\mathcal{O}(x^{2/3+\epsilon})$	$\mathcal{O}(x^{1/3+\epsilon})$
Lagarias-Odlyzko 1	$\mathcal{O}(x^{3/5+\epsilon})$	$\mathcal{O}(x^{\epsilon})$
Lagarias-Odlyzko 2	$\mathcal{O}(x^{1/2+\epsilon})$	$\mathcal{O}(x^{1/4+\epsilon})$

A modified version of the prime counting function is given by

$$\pi_0(p) \equiv \begin{cases} \pi(p) & \text{for } p \text{ composite} \\ \pi(p) - \frac{1}{2} & \text{for } p \text{ prime} \end{cases}$$

$$\pi_0(p) = \sum_{n=1}^{\infty} \frac{\mu(x) f(x^{1/n})}{n},$$

where $\mu(n)$ is the MÖBIUS FUNCTION and $f(x)$ is the RIEMANN-MANGOLDT FUNCTION.

The notation $\pi_{a,b}$ is also used to denote the number of PRIMES of the form $ak + b$ (Shanks 1993, pp. 21–22).

Groups of EQUINUMEROUS values of $\pi_{a,b}$ include ($\pi_{3,1}$, $\pi_{3,2}$), ($\pi_{4,1}$, $\pi_{4,3}$), ($\pi_{5,1}$, $\pi_{5,2}$, $\pi_{5,3}$, $\pi_{5,4}$), ($\pi_{6,1}$, $\pi_{6,5}$), ($\pi_{7,1}$, $\pi_{7,2}$, $\pi_{7,3}$, $\pi_{7,4}$, $\pi_{7,5}$, $\pi_{7,6}$), ($\pi_{8,1}$, $\pi_{8,3}$, $\pi_{8,5}$, $\pi_{8,7}$), ($\pi_{9,1}$, $\pi_{9,2}$, $\pi_{9,4}$, $\pi_{9,5}$, $\pi_{9,7}$, $\pi_{9,8}$), and so on. The values of $\pi_{n,k}$ for small n are given in the following table for the first few powers of ten (Shanks 1993).

n	$\pi_{3,1}(n)$	$\pi_{3,2}(n)$	$\pi_{4,1}(n)$	$\pi_{4,3}(n)$
10^1	1	2	1	2
10^2	11	13	11	13
10^3	80	87	80	87
10^4	611	617	609	619
10^5	4784	4807	4783	4808
10^6	39231	39266	39175	39322
10^7	332194	332384	332180	332398

n	$\pi_{5,1}(n)$	$\pi_{5,2}(n)$	$\pi_{5,3}(n)$	$\pi_{5,4}(n)$
10^1	0	2	1	0
10^2	5	7	7	5
10^3	40	47	42	38
10^4	306	309	310	303
10^5	2387	2412	2402	2390
10^6	19617	19622	19665	19593
10^7	166104	166212	166230	166032

n	$\pi_{6,1}(n)$	$\pi_{6,5}(n)$
10^1	1	1
10^2	11	12
10^3	80	86
10^4	611	616
10^5	4784	4806
10^6	39231	39265

n	$\pi_{7,1}$	$\pi_{7,2}$	$\pi_{7,3}$	$\pi_{7,4}$	$\pi_{7,5}$	$\pi_{7,6}$
10^1	0	1	1	0	1	0
10^2	3	4	5	3	5	4
10^3	28	27	30	26	29	27
10^4	203	203	209	202	211	200
10^5	1593	1584	1613	1601	1604	1596
10^6	13063	13065	13105	13069	13105	13090

n	$\pi_{8,1}(n)$	$\pi_{8,3}(n)$	$\pi_{8,5}(n)$	$\pi_{8,7}(n)$
10^1	0	1	1	1
10^2	5	7	6	6
10^3	37	44	43	43
10^4	295	311	314	308
10^5	2384	2409	2399	2399
10^6	19552	19653	19623	19669
10^7	165976	166161	166204	166237

Note that since $\pi_{8,1}(n)$, $\pi_{8,3}(n)$, $\pi_{8,5}(n)$, and $\pi_{8,7}(n)$ are EQUINUMEROUS,

$$\pi_{4,1}(n) = \pi_{8,1}(n) + \pi_{8,5}$$
$$\pi_{4,3}(n) = \pi_{8,3}(n) + \pi_{8,7}$$

are also equinumerous.

Erdős proved that there exist at least one PRIME of the form $4k+1$ and at least one PRIME of the form $4k+3$ between n and $2n$ for all $n > 6$.

The smallest x such that $x \geq n\pi(x)$ for $n = 2, 3, \ldots$ are 2, 27, 96, 330, 1008, ... (Sloane's A038625), and the corresponding $\pi(x)$ are 1, 9, 24, 24, 66, 168, ... (Sloane's A038626). The number of solutions of $x \geq n\pi(x)$ for $n = 2, 3, \ldots$ are 4, 3, 3, 6, 7, 6, ... (Sloane's A038627).

see also BERTELSEN'S NUMBER, EQUINUMEROUS, PRIME ARITHMETIC PROGRESSION, PRIME NUMBER THEOREM, RIEMANN WEIGHTED PRIME-POWER COUNTING FUNCTION

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Prime Cut

Find two numbers such that $x^2 \equiv y^2 \pmod{n}$. If you know the GREATEST COMMON DIVISOR of n and $x-y$, there exists a high probability of determining a PRIME factor. Taking small numbers x which additionally give small PRIMES $x^2 \equiv p \pmod{n}$ further increases the chances of finding a PRIME factor.

Prime Decomposition

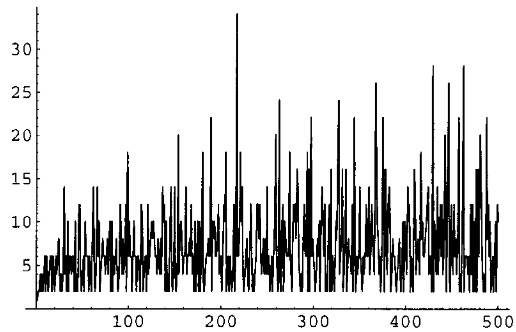
Given an INTEGER n, the prime decomposition is written

$$n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_n}^{\alpha_n},$$

where p_i are the n PRIME factors, each of order α_i. Each factor ${p_i}^{\alpha_i}$ is called a PRIMARY.

see also PRIMARY, PRIME FACTORIZATION ALGORITHMS, PRIME NUMBER

Prime Difference Function



$$d_n \equiv p_{n+1} - p_n.$$

The first few values are 1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, ... (Sloane's A001223). Rankin has shown that

$$d_n > \frac{c \ln n \ln \ln n \ln \ln \ln \ln n}{(\ln \ln \ln n)^2}$$

for infinitely many n and for some constant c (Guy 1994).

An integer n is called a JUMPING CHAMPION if n is the most frequently occurring difference between consecutive primes $n \le N$ for some N (Odlyzko *et al.*).

see also ANDRICA'S CONJECTURE, GOOD PRIME, JUMPING CHAMPION, PÓLYA CONJECTURE, PRIME GAPS, SHANKS' CONJECTURE, TWIN PEAKS

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Prime Diophantine Equations

$k+2$ is PRIME IFF the 14 DIOPHANTINE EQUATIONS in 26 variables

$$wz + h + j - q = 0 \tag{1}$$
$$(gk + 2g + k + 1)(h + j) + h - z = 0 \tag{2}$$
$$16(k+1)^3(k+2)(n+1)^2 + 1 - f^2 = 0 \tag{3}$$
$$2n + p + q + z - q = 0 \tag{4}$$
$$e^3(e+2)(a+1)^2 + 1 - o^2 = 0 \tag{5}$$
$$(a^2 - 1)y^2 + 1 - x^2 = 0 \tag{6}$$
$$16r^2y^4(a^2 - 1) + 1 - u^2 = 0 \tag{7}$$
$$n + l + v - y = 0 \tag{8}$$
$$(a^2 - 1)l^2 + 1 - m^2 = 0 \tag{9}$$
$$ai + k + 1 - l - i = 0 \tag{10}$$
$$\{[a + u^2(u^2 - a)]^2 - 1\}(n + 4dy)^2 + 1 - (x + cu)^2 = 0 \tag{11}$$
$$p + l(a - n - 1) + b(2an + 2a - n^2 - 2n - 2) - m = 0 \tag{12}$$
$$q + y(a - p - 1) + s(2ap + 2a - p^2 - 2p - 2) - x = 0 \tag{13}$$
$$z + pl(a - p) + t(2ap - p^2 - 1) - pm = 0 \tag{14}$$

have a POSITIVE integral solution.

References

Riesel, H. *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, p. 39, 1994.

Prime Factorization

see FACTORIZATION, PRIME DECOMPOSITION, PRIME FACTORIZATION ALGORITHMS, PRIME FACTORS

Prime Factorization Algorithms

Many ALGORITHMS have been devised for determining the PRIME factors of a given number. They vary quite a bit in sophistication and complexity. It is very difficult to build a general-purpose algorithm for this computationally "hard" problem, so any additional information which is known about the number in question or its factors can often be used to save a large amount of time.

The simplest method of finding factors is so-called "DIRECT SEARCH FACTORIZATION" (a.k.a. TRIAL DIVISION). In this method, all possible factors are systematically tested using trial division to see if they actually DIVIDE the given number. It is practical only for very small numbers.

see also BRENT'S FACTORIZATION METHOD, CONTINUED FRACTION FACTORIZATION ALGORITHM, DIRECT SEARCH FACTORIZATION, DIXON'S FACTORIZATION METHOD, ELLIPTIC CURVE FACTORIZATION METHOD, EULER'S FACTORIZATION METHOD, EXCLUDENT FACTORIZATION METHOD, FERMAT'S FACTORIZATION METHOD, LEGENDRE'S FACTORIZATION

METHOD, LENSTRA ELLIPTIC CURVE METHOD, NUMBER FIELD SIEVE FACTORIZATION METHOD, POLLARD $p-1$ FACTORIZATION METHOD, POLLARD ρ FACTORIZATION ALGORITHM, QUADRATIC SIEVE FACTORIZATION METHOD, TRIAL DIVISION, WILLIAMS $p+1$ FACTORIZATION METHOD

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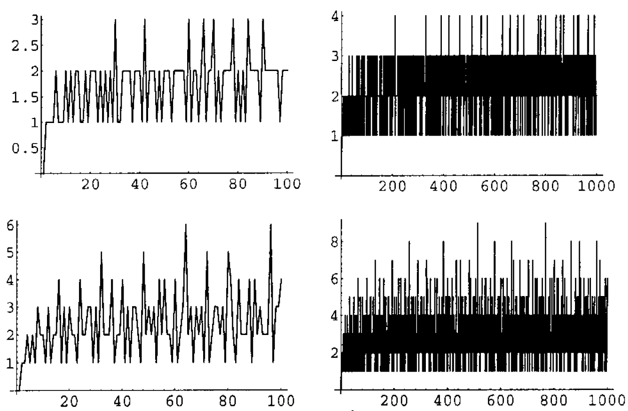
Prime Factors



The number of DISTINCT PRIME FACTORS of a number n is denoted $\omega(n)$. The first few values for $n = 1, 2, \ldots$ are 0, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 2, 1, 2, 2, 1, 1, 2, 1, 2, ... (Sloane's A001221; top figure). The number of not necessarily distinct prime factors of a number n is denoted $r(n)$. The first few values for $n = 1, 2, \ldots$ are 0, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 3, 1, 2, 2, 4, 1, 3, 1, 3, ... (Sloane's A001222; bottom figure).

see also DISTINCT PRIME FACTORS, DIVISOR FUNCTION, GREATEST PRIME FACTOR, LEAST PRIME FACTOR, LIOUVILLE FUNCTION, PÓLYA CONJECTURE, PRIME FACTORIZATION ALGORITHMS

References

Sloane, N. J. A. Sequences A001222/M0094 and A001221/M0056 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Prime Field

A GALOIS FIELD $GF(p)$ where p is PRIME.

Prime Gaps

Letting

$$d_n \equiv p_{n+1} - p_n$$

be the PRIME DIFFERENCE FUNCTION, Rankin has showed that

$$d_n > \frac{c \ln n \ln \ln n \ln \ln \ln \ln n}{(\ln \ln \ln n)^2}$$

for infinitely many n are for some constant c (Guy 1994).

Let $p(d)$ be the smallest PRIME following d or more consecutive COMPOSITE NUMBERS. The largest known is

$$p(804) = 90,874,329,412,297.$$

The largest known prime gap is of length 4247, occurring following $10^{314} - 1929$ (Baugh and O'Hara 1992), although this gap is almost certainly not maximal (i.e., there probably exists a smaller number having a gap of the same length following it).

Let $c(n)$ be the smallest starting INTEGER $c(n)$ for a run of n consecutive COMPOSITE NUMBERS, also called a COMPOSITE RUN. No general method other than exhaustive searching is known for determining the first occurrence for a maximal gap, although arbitrarily large gaps exist (Nicely 1998). Cramér (1937) and Shanks (1964) conjectured that a maximal gap of length n appears at approximately $\exp(\sqrt{n}\,)$. Wolf conjectures that the maximal gap of length n appears approximately at

$$\frac{n}{\pi(n)[2 \ln \pi(n) - \ln n + \ln(2C_2)]},$$

where $\pi(n)$ is the PRIME COUNTING FUNCTION and C_2 is the TWIN PRIMES CONSTANT.

The first few $c(n)$ for $n = 1, 2, \ldots$ are 4, 8, 8, 24, 24, 90, 90, 114, ... (Sloane's A030296). The following table gives the same sequence omitting degenerate runs which are part of a run with greater n, and is a complete list of smallest maximal runs up to 10^{15}. $c(n)$ in this table is given by Sloane's A008950, and n by Sloane's A008996. The ending integers for the run corresponding to $c(n)$ are given by Sloane's A008995. Young and Potler (1989) determined the first occurrences of prime gaps up to 72,635,119,999,997, with all first occurrences found between 1 and 673. Nicely (1998) extended the list of maximal prime gaps to a length of 915, denoting gap lengths by the difference of bounding PRIMES, $c(n) - 1$.

n	$c(n)$	n	$c(n)$
1	4	319	2,300,942,550
3	8	335	3,842,610,774
5	24	353	4,302,407,360
7	90	381	10,726,904,660
13	114	383	20,678,048,298
17	524	393	22,367,084,960
19	888	455	25,056,082,088
21	1,130	463	42,652,618,344
33	1,328	467	127,976,334,672
35	9,552	473	182,226,896,240
43	15,684	485	241,160,024,144
51	19,610	489	297,501,075,800
71	31,398	499	303,371,455,242
85	155,922	513	304,599,508,538
95	360,654	515	416,608,695,822
111	370,262	531	461,690,510,012
113	492,114	533	614,487,453,424
117	1,349,534	539	738,832,927,928
131	1,357,202	581	1,346,294,310,750
147	2,010,734	587	1,408,695,493,610
153	4,652,354	601	1,968,188,556,461
179	17,051,708	651	2,614,941,710,599
209	20,831,324	673	7,177,162,611,713
219	47,326,694	715	13,828,048,559,701
221	122,164,748	765	19,581,334,192,423
233	189,695,660	777	42,842,283,925,352
247	191,912,784	803	90,874,329,411,493
249	387,096,134	805	171,231,342,420,521
281	436,273,010	905	218,209,405,436,543
287	1,294,268,492	915	1,189,459,969,825,483
291	1,453,168,142		

see also JUMPING CHAMPION, PRIME CONSTELLATION, PRIME DIFFERENCE FUNCTION, SHANKS' CONJECTURE

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Prime-Generating Polynomial

Legendre showed that there is no RATIONAL algebraic function which always gives PRIMES. In 1752, Goldbach showed that no POLYNOMIAL with INTEGER COEFFICIENTS can give a PRIME for all integral values. However, there exists a POLYNOMIAL in 10 variables with INTEGER COEFFICIENTS such that the set of PRIMES equals the set of POSITIVE values of this POLYNOMIAL obtained as the variables run through all NONNEGATIVE INTEGERS, although it is really a set of DIOPHANTINE EQUATIONS in disguise (Ribenboim 1991).

$P(n)$	Range	#	Reference
$36n^2 - 810n + 2753$	[0, 44]	45	Fung and Ruby
$47n^2 - 1701n + 10181$	[0, 42]	43	Fung and Ruby
$n^2 - n + 41$	[0, 39]	40	Euler
$2n^2 + 29$	[0, 28]	29	Legendre
$n^2 + n + 17$	[0, 15]	16	Legendre
$2n^2 + 11$	[0, 10]	11	
$n^3 + n^2 + 17$	[0, 10]	11	

The above table gives some low-order polynomials which generate only PRIMES for the first few NONNEGATIVE values (Mollin and Williams 1990). The best-known of these formulas is that due to Euler (Euler 1772, Ball and Coxeter 1987). Le Lionnais (1983) has christened numbers p such that the Euler-like polynomial

$$n^2 - n + p \qquad (1)$$

is PRIME for $p = 0, 1, \ldots, p-2$ as LUCKY NUMBERS OF EULER (where the case $p = 41$ corresponds to Euler's formula). Rabinovitch (1913) showed that for a PRIME $p > 0$, Euler's polynomial represents a PRIME for $n \in [0, p-2]$ (excluding the trivial case $p = 3$) IFF the FIELD $\mathbb{Q}(\sqrt{1-4p})$ has CLASS NUMBER $h = 1$ (Rabinowitz 1913, Le Lionnais 1983, Conway and Guy 1996). As established by Stark (1967), there are only nine numbers $-d$ such that $h(-d) = 1$ (the HEEGNER NUMBERS -2, -3, -7, -11, -19, -43, -67, and -163), and of these, only 7, 11, 19, 43, 67, and 163 are of the required form. Therefore, the only LUCKY NUMBERS OF EULER are 2, 3, 5, 11, 17, and 41 (Le Lionnais 1983, Sloane's A014556), and there does not exist a better prime-generating polynomial of Euler's form.

Euler also considered quadratics of the form

$$2x^2 + p \qquad (2)$$

and showed this gives PRIMES for $x \in [0, p-1]$ for PRIME $p > 0$ IFF $\mathbb{Q}(\sqrt{-2p})$ has CLASS NUMBER 2, which permits only $p = 3$, 5, 11, and 29. Baker (1971) and Stark (1971) showed that there are so such FIELDS for $p > 29$. Similar results have been found for POLYNOMIALS of the form

$$px^2 + px + n \tag{3}$$

(Hendy 1974).

see also CLASS NUMBER, HEEGNER NUMBER, LUCKY NUMBER OF EULER, PRIME ARITHMETIC PROGRESSION, PRIME DIOPHANTINE EQUATIONS, SCHINZEL'S HYPOTHESIS

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Prime Group

When the ORDER h of a finite GROUP is a PRIME number, there is only one possible GROUP of ORDER h. Furthermore, the GROUP is CYCLIC.

see also p-GROUP

Prime Ideal

An IDEAL I such that if $ab \in I$, then either $a \in I$ or $b \in I$.

see also DEDEKIND RING, IDEAL, KRULL DIMENSION, MAXIMAL IDEAL, STICKELBERGER RELATION, STONE SPACE

Prime Knot

A KNOT other than the UNKNOT which cannot be expressed as a sum of two other KNOTS, neither of which is unknotted. A KNOT which is not prime is called a COMPOSITE KNOT. It is often possible to combine two prime knots to create two different COMPOSITE KNOTS, depending on the orientation of the two.

There is no known FORMULA for giving the number of distinct prime knots as a functions of number of crossings. For the first few n crossings, the numbers of prime knots are 0, 0, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, ... (Sloane's A002863).

see also COMPOSITE KNOT, KNOT

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Prime k-Tuple

see PRIME CONSTELLATION

Prime k-Tuples Conjecture

see also k-TUPLE CONJECTURE

Prime k-Tuplet

see PRIME CONSTELLATION

Prime Manifold

An n-MANIFOLD which cannot be "nontrivially" decomposed into other n-MANIFOLDS.

Prime Number

A prime number is a POSITIVE INTEGER p which has no DIVISORS other than 1 and p itself. Although the number 1 used to be considered a prime, it requires special treatment in so many definitions and applications involving primes greater than or equal to 2 that it is usually placed into a class of its own. Since 2 is the only EVEN prime, it is also somewhat special, so the set of all primes excluding 2 is called the "ODD PRIMES." The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ... (Sloane's A000040, Hardy and Wright 1979, p. 3). POSITIVE INTEGERS other than 1 which are not prime are called COMPOSITE.

The function which gives the number of primes less than a number n is denoted $\pi(n)$ and is called the PRIME COUNTING FUNCTION. The theorem giving an asymptotic form for $\pi(n)$ is called the PRIME NUMBER THEOREM.

Prime numbers can be generated by sieving processes (such as the ERATOSTHENES SIEVE), and LUCKY NUMBERS, which are also generated by sieving, appear to share some interesting asymptotic properties with the primes.

Many PRIME FACTORIZATION ALGORITHMS have been devised for determining the prime factors of a given INTEGER. They vary quite a bit in sophistication and complexity. It is very difficult to build a general-purpose algorithm for this computationally "hard" problem, so any additional information which is known about the number in question or its factors can often be used to save a large amount of time. The simplest method of finding factors is so-called "DIRECT SEARCH FACTORIZATION" (a.k.a. TRIAL DIVISION). In this method, all possible factors are systematically tested using trial division to see if they actually DIVIDE the given number. It is practical only for very small numbers.

Because of their importance in encryption algorithms such as RSA ENCRYPTION, prime numbers can be important commercial commodities. In fact, Roger Schlafly has obtained U.S. Patent 5,373,560 (12/13/94) on the following two primes (expressed in hexadecimal notation):

```
98A3DF52AEAE9799325CB258D767EBD1F4630E9B
9E21732A4AFB1624BA6DF911466AD8DA960586F4
A0D5E3C36AF099660BDDC1577E54A9F402334433
ACB14BCB
```

and

```
93E8965DAFD9DFECFD00B466B68F90EA68AF5DC9
FED915278D1B3A137471E65596C37FED0C7829FF
8F8331F81A2700438ECDCC09447DC397C685F397
294F722BCC484AEDF28BED25AAAB35D35A65DB1F
D62C9D7BA55844FEB1F9401E671340933EE43C54
E4DC459400D7AD61248B83A2624835B31FFF2D95
95A5B90B276E44F9.
```

The FUNDAMENTAL THEOREM OF ARITHMETIC states that any POSITIVE INTEGER can be represented in exactly one way as a PRODUCT of primes. EUCLID'S SECOND THEOREM demonstrated that there are an infinite number of primes. However, it is not known if there are an infinite number of primes of the form x^2+1, whether there are an INFINITE number of TWIN PRIMES, or if a prime can always be found between n^2 and $(n+1)^2$.

Prime numbers satisfy many strange and wonderful properties. For example, there exists a CONSTANT $\theta \approx 1.3064$ known as MILLS' CONSTANT such that

$$\left\lfloor \theta^{3^n} \right\rfloor, \tag{1}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, is prime for all $n \geq 1$. However, it is not known if θ is IRRATIONAL. There also exists a CONSTANT $\omega \approx 1.9287800$ such that

$$\left\lfloor \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^\omega}}}}}}_{n} \right\rfloor \tag{2}$$

(Ribenboim 1996, p. 186) is prime for every $n \geq 1$.

Explicit FORMULAS exist for the nth prime both as a function of n and in terms of the primes 2, ..., p_{n-1} (Hardy and Wright 1979, pp. 5–6; Guy 1994, pp. 36–41). Let

$$F(j) = \left\lfloor \cos^2\left[\pi \frac{(j-1)!+1}{j}\right]\right\rfloor \tag{3}$$

for integral $j > 1$, and define $F(1) = 1$, where $\lfloor x \rfloor$ is again the FLOOR FUNCTION. Then

$$p_n = 1 + \sum_{m=1}^{2^n} \left\lfloor \left\lfloor \frac{n}{\sum_{j=1}^m F(j)} \right\rfloor^{1/n} \right\rfloor \tag{4}$$

$$= 1 + \sum_{m=1}^{2^n} \left\lfloor \left\lfloor \frac{n}{1+\pi(m)} \right\rfloor^{1/n} \right\rfloor, \tag{5}$$

where $\pi(m)$ is the PRIME COUNTING FUNCTION. It is also true that

$$p_{n+1} = 1 + p_n + F(p_n+1) + F(p_n+1)F(p_n+2) + \prod_{j=1}^{p} F(p_n+j) \tag{6}$$

(Ribenboim 1996, pp. 180–182). Note that the number of terms in the summation to obtain the nth prime is 2^n, so these formulas turn out not to be practical in the study of primes. An interesting INFINITE PRODUCT formula due to Euler which relates π and the nth PRIME p_n is

$$\pi = \frac{2}{\prod_{i=n}^{\infty}\left[1 + \frac{\sin(\frac{1}{2}\pi p_n)}{p_n}\right]} \tag{7}$$

$$= \frac{2}{\prod_{i=n}^{\infty}\left[1 + \frac{(-1)^{(p_n-1)/2}}{p_n}\right]} \tag{8}$$

(Blatner 1997). Conway (Guy 1983, Conway and Guy 1996, p. 147) gives an algorithm for generating primes based on 14 fractions, but it is actually just a concealed version of a SIEVE.

Some curious identities satisfied by primes p are

$$\sum_{k=1}^{p-1} \left\lfloor \frac{k^3}{p} \right\rfloor = \frac{(p-2)(p-1)(p+1)}{4} \tag{9}$$

$$\sum_{k=1}^{(p-1)(p-2)} \left\lfloor (kp)^{1/3} \right\rfloor = \tfrac{1}{4}(3p-5)(p-2)(p-1) \tag{10}$$

(Doster 1993),

$$\prod_{p \text{ prime}} \frac{p^2+1}{p^2-1} = \frac{5}{2} \tag{11}$$

(Le Lionnais 1983, p. 46),

$$\sum_{k=1}^{\infty} x^k \ln k = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{x^{p^k}}{1-x^{p^k}}, \tag{12}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} e^{-kx} \ln k$$

$$= -\ln 2 \sum_{k=1}^{\infty} \frac{1}{e^{2^k x}-1} + \sum_{\substack{p \text{ an} \\ \text{odd prime}}} \ln p \sum_{k=1}^{\infty} \frac{1}{e^{p^k x}+1} \tag{13}$$

(Berndt 1994, p. 114).

It has been proven that the set of prime numbers is a DIOPHANTINE SET (Ribenboim 1991, pp. 106–107). Ramanujan also showed that

$$\frac{d\pi(x)}{dx} \sim \frac{1}{x \ln x} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} x^{1/n}, \tag{14}$$

where $\pi(x)$ is the PRIME COUNTING FUNCTION and $\mu(n)$ is the MÖBIUS FUNCTION (Berndt 1994, p. 117). B. M. Bredihin proved that

$$f(x,y) = x^2 + y^2 + 1 \tag{15}$$

takes prime values for infinitely many integral pairs (x, y) (Honsberger 1976, p. 30). In addition, the function

$$f(x,y) = \tfrac{1}{2}(y-1) \left\lfloor |B^2(x,y)-1| - (B^2(x,y)-1) \right\rfloor + 2, \tag{16}$$

where

$$B(x,y) = x(y+1) - (y!+1), \tag{17}$$

$y!$ is the FACTORIAL, and $\lfloor x \rfloor$ is the FLOOR FUNCTION, generates only prime numbers for POSITIVE integral arguments. It not only generates every prime number, but generates ODD primes exactly once each, with all other values being 2 (Honsberger 1976, p. 33). For example,

$$f(1,2) = 3 \tag{18}$$

$$f(5,4) = 5 \tag{19}$$

$$f(103,6) = 7, \tag{20}$$

with no new primes generated for $x, y \leq 1000$.

For n an INTEGER ≥ 2, n is prime IFF

$$\binom{n-1}{k} \equiv (-1)^k \pmod{n} \tag{21}$$

for $k = 0, 1, \ldots, n-1$ (Deutsch 1996).

Cheng (1979) showed that for x sufficiently large, there always exist at least two prime factors between $(x - x^\alpha)$ and x for $\alpha \geq 0.477\ldots$ (Le Lionnais 1983, p. 26). Let $f(n)$ be the number of decompositions of n into two or more consecutive primes. Then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n=1}^{x} f(n) = \ln 2 \tag{22}$$

(Moser 1963, Le Lionnais 1983, p. 30). Euler showed that the sum of the inverses of primes is infinite

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty \tag{23}$$

(Hardy and Wright 1979, p. 17), although it diverges very slowly. The sum exceeds 1, 2, 3, ... after 3, 59, 361139, ... (Sloane's A046024) primes, and its asymptotic equation is

$$\sum_{\substack{p=2 \\ p \text{ prime}}}^{x} \frac{1}{p} = \ln \ln x + B_1 + o(1), \tag{24}$$

where B_1 is MERTENS CONSTANT (Hardy and Wright 1979, p. 351). Dirichlet showed the even stronger result that

$$\sum_{\substack{\text{prime } p \equiv b \pmod{a} \\ (a,b)=1}} \frac{1}{p} = \infty \tag{25}$$

(Davenport 1980, p. 34).

Despite the fact that $\sum 1/p$ diverges, Brun showed that

$$\sum_{\substack{p \\ p+2 \text{ prime}}} \frac{1}{p} = B < \infty, \tag{26}$$

where B is BRUN'S CONSTANT. The function defined by

$$P(n) \equiv \sum_p \frac{1}{p^n} \tag{27}$$

taken over the primes converges for $n > 1$ and is a generalization of the RIEMANN ZETA FUNCTION known as the PRIME ZETA FUNCTION.

The probability that the largest prime factor of a RANDOM NUMBER x is less than $\sqrt{x}$ is $\ln 2$ (Beeler *et al.* 1972, Item 29). The probability that two INTEGERS picked at random are RELATIVELY PRIME is $[\zeta(2)]^{-1} = 6/\pi^2$, where $\zeta(x)$ is the RIEMANN ZETA FUNCTION (Cesaro and Sylvester 1883). Given three INTEGERS chosen at random, the probability that no common factor will divide them all is

$$[\zeta(3)]^{-1} \approx 1.202^{-1} = 0.832\ldots, \tag{28}$$

where $\zeta(3)$ is APÉRY'S CONSTANT. In general, the probability that n random numbers lack a pth POWER common divisor is $[\zeta(np)]^{-1}$ (Beeler *et al.* 1972, Item 53).

Large primes include the large MERSENNE PRIMES, FERRIER'S PRIME, and $391581(2^{216193}-1)$ (Cipra 1989). The largest known prime as of 1998, is the MERSENNE PRIME $2^{3021377} - 1$.

Primes consisting of consecutive DIGITS (counting 0 as coming after 9) include 2, 3, 5, 7, 23, 67, 89, 4567, 78901, ... (Sloane's A006510).

see also ADLEMAN-POMERANCE-RUMELY PRIMALITY TEST, ALMOST PRIME, ANDRICA'S CONJECTURE, BERTRAND'S POSTULATE, BROCARD'S CONJECTURE, BRUN'S CONSTANT, CARMICHAEL'S CONJECTURE, CARMICHAEL FUNCTION, CARMICHAEL NUMBER, CHEBYSHEV FUNCTION, CHEBYSHEV-SYLVESTER CONSTANT, CHEN'S THEOREM, CHINESE HYPOTHESIS, COMPOSITE NUMBER, COMPOSITE RUNS, COPELAND-ERDŐS CONSTANT, CRAMER CONJECTURE, CUNNINGHAM CHAIN, CYCLOTOMIC POLYNOMIAL, DE POLIGNAC'S CONJECTURE, DIRICHLET'S THEOREM, DIVISOR, ERDŐS-KAC THEOREM, EUCLID'S THEOREMS, FEIT-THOMPSON CONJECTURE, FERMAT NUMBER, FERMAT QUOTIENT, FERRIER'S PRIME, FORTUNATE PRIME, FUNDAMENTAL THEOREM OF ARITHMETIC, GIGANTIC PRIME, GIUGA'S CONJECTURE, GOLDBACH CONJECTURE, GOOD PRIME, GRIMM'S CONJECTURE, HARDY-RAMANUJAN THEOREM, IRREGULAR PRIME, KUMMER'S CONJECTURE, LEHMER'S PROBLEM, LINNIK'S THEOREM, LONG PRIME, MERSENNE NUMBER, MERTENS FUNCTION, MILLER'S PRIMALITY TEST, MIRIMANOFF'S CONGRUENCE, MÖBIUS FUNCTION, PALINDROMIC NUMBER, PÉPIN'S TEST, PILLAI'S CONJECTURE, POULET NUMBER, PRIMARY, PRIME ARRAY, PRIME CIRCLE, PRIME FACTORIZATION ALGORITHMS, PRIME NUMBER OF MEASUREMENT, PRIME NUMBER THEOREM, PRIME POWER SYMBOL, PRIME STRING, PRIME TRIANGLE, PRIME ZETA FUNCTION, PRIMITIVE PRIME FACTOR, PRIMORIAL, PROBABLE PRIME, PSEUDOPRIME, REGULAR PRIME, RIEMANN FUNCTION, ROTKIEWICZ THEOREM, SCHNIRELMANN'S THEOREM, SELFRIDGE'S CONJECTURE, SEMIPRIME, SHAH-WILSON CONSTANT, SIERPIŃSKI'S COMPOSITE NUMBER THEOREM, SIERPIŃSKI'S PRIME SEQUENCE THEOREM, SMOOTH NUMBER, SOLDNER'S CONSTANT, SOPHIE GERMAIN PRIME, TITANIC PRIME, TOTIENT FUNCTION, TOTIENT VALENCE FUNCTION, TWIN PRIMES, TWIN PRIMES CONSTANT, VINOGRADOV'S THEOREM, VON MANGOLDT FUNCTION, WARING'S CONJECTURE, WIEFERICH PRIME, WILSON PRIME, WILSON QUOTIENT, WILSON'S THEOREM, WITNESS, WOLSTENHOLME'S THEOREM, ZSIGMONDY THEOREM

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Prime Number of Measurement

The set of numbers generated by excluding the SUMS of two or more consecutive earlier members is called the prime numbers of measurement, or sometimes the SEGMENTED NUMBERS. The first few terms are 1, 2, 4, 5, 8, 10, 14, 15, 16, 21, ... (Sloane's A002048). Excluding two *and* three terms gives the sequence 1, 2, 4, 5, 8, 10, 14, 15, 16, 19, 20, 21, ... (Sloane's A005242).

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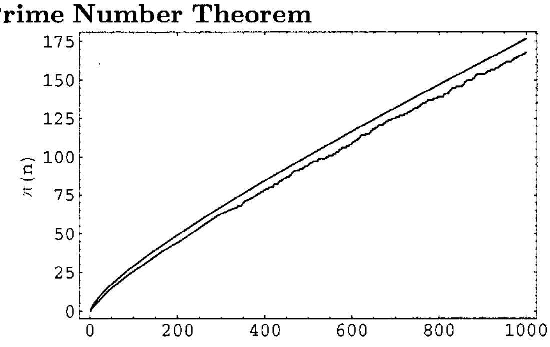
Prime Number Theorem



The theorem giving an asymptotic form for the PRIME COUNTING FUNCTION $\pi(n)$ for number of PRIMES less than some INTEGER n. Legendre (1808) suggested that, for large n,

$$\pi(n) \sim \frac{n}{A \ln n + B}, \tag{1}$$

with $A = 1$ and $B = -1.08366$ (where B is sometimes called LEGENDRE'S CONSTANT), a formula which is correct in the leading term only (Wagon 1991, pp. 28–29). In 1791, Gauss became the first to suggest instead

$$\pi(n) \sim \frac{n}{\ln n}. \tag{2}$$

Gauss later refined his estimate to

$$\pi(n) \sim \mathrm{Li}(n), \tag{3}$$

where $\mathrm{Li}(n)$ is the LOGARITHMIC INTEGRAL. This function has $n/\ln n$ as the leading term and has been shown to be a better estimate than $n/\ln n$ alone. The statement (3) is often known as "the" prime number theorem and was proved independently by Hadamard and Vallée Poussin in 1896. A plot of $\pi(n)$ (lower curve) and $\mathrm{Li}(n)$ is shown above for $n \leq 1000$.

For small n, it has been checked and always found that $\pi(n) < \mathrm{Li}(n)$. However, Skewes proved that the first crossing of $\pi(n) - \mathrm{Li}(n) = 0$ occurs before $10^{10^{10^{34}}}$ (the SKEWES NUMBER). The upper bound for the crossing has subsequently been reduced to 10^{371}. Littlewood (1914) proved that the INEQUALITY reverses infinitely often for sufficiently large n (Ball and Coxeter 1987). Lehman (1966) proved that at least 10^{500} reversals occur for numbers with 1166 or 1167 DECIMAL DIGITS.

Chebyshev (Rubinstein and Sarnak 1994) put limits on the RATIO

$$\frac{7}{8} < \frac{\pi(n)}{\frac{n}{\ln n}} < \frac{9}{8}, \tag{4}$$

and showed that if the LIMIT

$$\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\ln n}} \tag{5}$$

existed, then it would be 1. This is, in fact, the prime number theorem.

Hadamard and Vallée Poussin proved the prime number theorem by showing that the RIEMANN ZETA FUNCTION $\zeta(z)$ has no zeros of the form $1+it$ (Smith 1994, p. 128). In particular, Vallée Poussin showed that

$$\pi(x) = \mathrm{Li}(x) + \mathcal{O}\left(\frac{x}{\ln x} e^{-a\sqrt{\ln x}}\right) \tag{6}$$

for some constant a. A simplified proof was found by Selberg and Erdős (1949) (Ball and Coxeter 1987, p. 63).

Riemann estimated the PRIME COUNTING FUNCTION with

$$\pi(n) \sim \mathrm{Li}(n) - \tfrac{1}{2}\mathrm{Li}(n^{1/2}), \tag{7}$$

which is a better approximation than $\mathrm{Li}(n)$ for $n < 10^7$. Riemann (1859) also suggested the RIEMANN FUNCTION

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}), \tag{8}$$

where μ is the MÖBIUS FUNCTION (Wagon 1991, p. 29). An even better approximation for small n (by a factor of 10 for $n < 10^9$) is the GRAM SERIES.

The prime number theorem is equivalent to

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1, \tag{9}$$

where $\psi(x)$ is the SUMMATORY MANGOLDT FUNCTION.

The RIEMANN HYPOTHESIS is equivalent to the assertion that

$$|\operatorname{Li}(x) - \pi(x)| \leq c\sqrt{x}\ln x \tag{10}$$

for some value of c (Ingham 1932, Ball and Coxeter 1987). Some limits obtained without assuming the RIEMANN HYPOTHESIS are

$$\pi(x) = \operatorname{Li}(x) + \mathcal{O}[xe^{-(\ln x)^{1/2}/15}] \tag{11}$$

$$\pi(x) = \operatorname{Li}(x) + \mathcal{O}[xe^{-0.009(\ln x)^{3/5}/(\ln\ln x)^{1/5}}]. \tag{12}$$

Ramanujan showed that for sufficiently large x,

$$\pi^2(x) < \frac{ex}{\ln x}\pi\left(\frac{x}{e}\right). \tag{13}$$

The largest known PRIME for which the inequality fails is 38,358,837,677 (Berndt 1994, pp. 112–113). The related inequality

$$\operatorname{Li}^2(x) < \frac{ex}{\ln x}\operatorname{Li}\left(\frac{x}{e}\right) \tag{14}$$

is true for $x \geq 2418$ (Berndt 1994, p. 114).

see also BERTRAND'S POSTULATE, DIRICHLET'S THEOREM, GRAM SERIES, PRIME COUNTING FUNCTION, RIEMANN'S FORMULA, RIEMANN FUNCTION, RIEMANN-MANGOLDT FUNCTION, RIEMANN WEIGHTED PRIME-POWER COUNTING FUNCTION, SKEWES NUMBER

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Prime Pairs

see TWIN PRIMES

Prime Patterns Conjecture

see k-TUPLE CONJECTURE

Prime Polynomial

see PRIME-GENERATING POLYNOMIAL

Prime Power Conjecture

An Abelian planar DIFFERENCE SET of order n exists only for n a PRIME POWER. Gordon (1994) has verified it to be true for $n < 2,000,000$.

see also DIFFERENCE SET

References

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Prime Power Symbol

The symbol $p^e \| n$ means, for p a PRIME, that $p^e | n$, but $p^{e+1} \nmid n$.

Prime Quadratic Effect

Let $\pi_{m,n}(x)$ denote the number of PRIMES $\leq x$ which are congruent to n modulo m. Then one might expect that

$$\Delta(x) \equiv \pi_{4,3}(x) - \pi_{4,1}(x) \sim \tfrac{1}{2}\pi(x^{1/2}) > 0$$

(Berndt 1994). Although this is true for small numbers, Hardy and Littlewood showed that $\Delta(x)$ changes sign infinitely often. (The first number for which it is false is 26861.) The effect was first noted by Chebyshev in 1853, and is sometimes called the CHEBYSHEV PHENOMENON. It was subsequently studied by Shanks (1959), Hudson (1980), and Bays and Hudson (1977, 1978, 1979). The effect was also noted by Ramanujan, who incorrectly claimed that $\lim_{x\to\infty} \Delta(x) = \infty$ (Berndt 1994).

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Prime Quadruplet

A PRIME CONSTELLATION of four successive PRIMES with minimal distance $(p, p+2, p+6, p+8)$. The quadruplet (2, 3, 5, 7) has smaller minimal distance, but it is an exceptional special case. With the exception of (5, 7, 11, 13), a prime quadruple must be of the form $(30n+11, 30n+13, 30n+17, 30n+19)$. The first few values of n which give prime quadruples are $n = 0$, 3, 6, 27, 49, 62, 69, 108, 115, ... (Sloane's A014561), and the first few values of p are 5 (the exceptional case), 11, 101, 191, 821, 1481, 1871, 2081, 3251, 3461, ... The asymptotic FORMULA for the frequency of prime quadruples is analogous to that for other PRIME CONSTELLATIONS,

$$\begin{aligned} P_x(p,p+2,p+6,p+8) &\sim \frac{27}{2}\prod_{p\geq 5}\frac{p^3(p-4)}{(p-1)^4}\int_2^x \frac{dx}{(\ln x)^4} \\ &= 4.151180864\int_2^x \frac{dx}{(\ln x)^4}, \end{aligned}$$

where $c = 4.15118\ldots$ is the Hardy-Littlewood constant for prime quadruplets. Roonguthai found the large prime quadruplets with

$$\begin{aligned} p &= 10^{99} + 349781731 \\ p &= 10^{199} + 21156403891 \\ p &= 10^{299} + 140159459341 \\ p &= 10^{399} + 34993836001 \\ p &= 10^{499} + 883750143961 \\ p &= 10^{599} + 1394283756151 \\ p &= 10^{699} + 547634621251 \end{aligned}$$

(Roonguthai).

see also PRIME ARITHMETIC PROGRESSION, PRIME CONSTELLATION, PRIME k-TUPLES CONJECTURE, SEXY PRIMES, TWIN PRIMES

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Prime Representation

Let $a \neq b$, A, and B denote POSITIVE INTEGERS satisfying

$$(a,b) = 1 \qquad (A,B) = 1,$$

(i.e., both pairs are RELATIVELY PRIME), and suppose every PRIME $p \equiv B \pmod{A}$ with $(p, 2ab) = 1$ is expressible if the form $ax^2 - by^2$ for some INTEGERS x and y. Then every PRIME q such that $q \equiv -B \pmod{A}$ and $(q, 2ab) = 1$ is expressible in the form $bX^2 - aY^2$ for some INTEGERS X and Y (Halter-Koch 1993, Williams 1991).

Prime Form	Representation
$4n+1$	x^2+y^2
$8n+1, 8n+3$	x^2+2y^2
$8n\pm 1$	x^2-2y^2
$6n+1$	x^2+3y^2
$12n+1$	x^2-3y^2
$20n+1, 20n+9$	x^2+5y^2
$10n+1, 10n+9$	x^2-5y^2
$14n+1, 14n+9, 14n+25$	x^2+7y^2
$28n+1, 28n+9, 28n+25$	x^2-7y^2
$30n+1, 30n+49$	x^2+15y^2
$60n+1, 60n+49$	x^2-15y^2
$30n-7, 30n+17$	$5x^2+3y^2$
$60n-7, 60n+17$	$5x^2-3y^2$
$24n+1, 24n+7$	x^2+6y^2
$24n+1, 24n+19$	x^2-6y^2
$24n+5, 24n+11$	$2x^2+3y^2$
$24n+5, 24n-1$	$2x^2-3y^2$

References

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 70–73, 1994.

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Williams, K. S. "On an Assertion of Ramanujan Concerning Binary Quadratic Forms." *J. Number Th.* **38**, 118–133, 1991.

Prime Ring

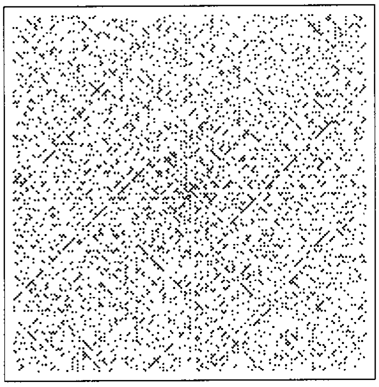
A RING for which the product of any pair of IDEALS is zero only if one of the two IDEALS is zero. All SIMPLE RINGS are prime.

see also IDEAL, RING, SEMIPRIME RING, SIMPLE RING

Prime Sequence

see PRIME ARITHMETIC PROGRESSION, PRIME ARRAY, PRIME-GENERATING POLYNOMIAL, SIERPIŃSKI'S PRIME SEQUENCE THEOREM

Prime Spiral



The numbers arranged in a SPIRAL

$$\begin{array}{ccc} 5 & 4 & 3 \\ 6 & 1 & 2 \\ 7 & 8 & 9 \end{array}$$

with PRIMES indicated in black, as first drawn by S. Ulam. Unexpected patterns of diagonal lines are apparent in such a plot, as illustrated in the above 199×199 grid.

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Prime String

Call a number n a prime string from the left if n and all numbers obtained by successively removing the right-most DIGIT are PRIME. There are 83 left prime strings in base 10. The first few are 2, 3, 5, 7, 23, 29, 31, 37, 53, 59, 71, 73, 79, 233, 239, 293, 311, 313, 317, 373, 379, 593, 599, ... (Sloane's A024770), the largest being 73,939,133. Similarly, call a number n a prime string from the right if n and all numbers obtained by successively removing the left-most DIGIT are PRIME. The first few are 2, 3, 5, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97, 103, 107, 113, 137, 167, 173, ... (Sloane's A033664). A large right prime string is 933,739,397.

see also PRIME ARRAY, PRIME NUMBER

References

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Sloane, N. J. A. Sequence A024770 in "An On-Line Version of the Encyclopedia of Integer Sequences."033664

Prime Sum

Let

$$\Sigma(n) \equiv \sum_{i=1}^{n} p_i$$

be the sum of the first n PRIMES. The first few terms are 2, 5, 10, 17, 28, 41, 58, 77, ... (Sloane's A007504). Bach and Shallit (1996) show that

$$\Sigma(n) \sim \frac{n^2}{2 \log n},$$

and provide a general technique for estimating such sums.

see also PRIMORIAL

References

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Sloane, N. J. A. Sequence A007504/M1370 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Prime Theta Function

The prime theta function is defined as

$$\theta(n) \equiv \sum_{i=1}^{n} \ln p_i,$$

where p_i is the ith PRIME. As shown by Bach and Shallit (1996),

$$\theta(n) \sim n.$$

References

Bach, E. and Shallit, J. *Algorithmic Number Theory, Vol. 1: Efficient Algorithms.* Cambridge, MA: MIT Press, pp. 206 and 233, 1996.

Prime Triangle

$$\begin{array}{c} * \\ 1\ \ 2 \\ 1\ \ 2\ \ 3 \\ 1\ \ 2\ \ 3\ \ 4 \\ 1\ \ 4\ \ 3\ \ 2\ \ 5 \\ 1\ \ 4\ \ 3\ \ 2\ \ 5\ \ 6 \end{array}$$

This triangle has rows beginning with 1 and ending with n, with the SUM of each two consecutive entries being a PRIME.

see also PASCAL'S TRIANGLE

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 106, 1994.

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Prime Unit

1 and −1 are the only INTEGERS which divide every INTEGER. They are therefore called the prime units.

see also INTEGER, PRIME NUMBER, UNIT

Prime Zeta Function

The prime zeta function

$$P(n) \equiv \sum_p \frac{1}{p^n}, \tag{1}$$

where the sum is taken over PRIMES is a generalization of the RIEMANN ZETA FUNCTION

$$\zeta(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n}, \tag{2}$$

where the sum is over *all* integers. The prime zeta function can be expressed in terms of the RIEMANN ZETA FUNCTION by

$$\begin{aligned}\ln \zeta(n) &= -\sum_{p\geq 2} \ln(1-p^{-n}) = \sum_{p\geq 2}\sum_{k=1}^{\infty} \frac{p^{-kn}}{k} \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p\geq 2} p^{-kn} = \sum_{k=1}^{\infty} \frac{P(kn)}{k}.\end{aligned} \tag{3}$$

Inverting then gives

$$P(n) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln \zeta(kn), \tag{4}$$

where $\mu(k)$ is the MÖBIUS FUNCTION. The values for the first few integers starting with two are

$$P(2) \approx 0.452247 \tag{5}$$

$$P(3) \approx 0.174763 \tag{6}$$

$$P(4) \approx 0.0769931 \tag{7}$$

$$P(5) \approx 0.035755. \tag{8}$$

see also MÖBIUS FUNCTION, RIEMANN ZETA FUNCTION, ZETA FUNCTION

References

Hardy, G. H. and Weight, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Oxford University Press, pp. 355–356, 1979.

Primequad

see PRIME QUADRUPLET

Primitive Abundant Number

An ABUNDANT NUMBER for which all PROPER DIVISORS are DEFICIENT is called a primitive abundant number (Guy 1994, p. 46). The first few ODD primitive abundant numbers are 945, 1575, 2205, 3465, ... (Sloane's A006038).

see also ABUNDANT NUMBER, DEFICIENT NUMBER, HIGHLY ABUNDANT NUMBER, SUPERABUNDANT NUMBER, WEIRD NUMBER

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 46, 1994.

Sloane, N. J. A. Sequence A006038/M5486 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Primitive Function

see INTEGRAL

Primitive Irreducible Polynomial

An IRREDUCIBLE POLYNOMIAL which generates all elements of an extension field from a base field. For any PRIME or PRIME POWER q and any POSITIVE INTEGER n, there exists a primitive irreducible POLYNOMIAL of degree n over GF(q).

see also GALOIS FIELD, IRREDUCIBLE POLYNOMIAL

Primitive Polynomial Modulo 2

A special type of POLYNOMIAL of which a subclass has COEFFICIENTS of only 0 or 1. Such POLYNOMIALS define a RECURRENCE RELATION which can be used to obtain a new RANDOM bit from the n preceding ones.

Primitive Prime Factor

If $n \geq 1$ is the smallest INTEGER such that $p | a^n - b^n$ (or $a^n + b^n$), then p is a primitive prime factor.

Primitive Pseudoperfect Number

see PRIMITIVE SEMIPERFECT NUMBER

Primitive Recursive Function

For-loops (which have a fixed iteration limit) are a special case of while-loops. A function which can be implemented using only for-loops is called primitive recursive. (In contrast, a COMPUTABLE FUNCTION can be coded using a combination of for- and while-loops, or while-loops only.)

The ACKERMANN FUNCTION is the simplest example of a well-defined TOTAL FUNCTION which is COMPUTABLE but not primitive recursive, providing a counterexample to the belief in the early 1900s that every COMPUTABLE FUNCTION was also primitive recursive (Dötzel 1991).

see also ACKERMANN FUNCTION, COMPUTABLE FUNCTION, TOTAL FUNCTION

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Primitive Root

A number g is a primitive root of m if

$$g^k \not\equiv 1 \pmod{m} \tag{1}$$

for $1 \leq k < m$ and

$$g^m \equiv 1 \pmod{m}. \tag{2}$$

Only $m = 2$, 4, p^a, and $2p^a$ have primitive roots (where $p > 2$ and a is an INTEGER). For composite m, there may be more than one primitive root (both 3 and 7 are primitive roots mod 10), but for prime p, there is only one primitive root. It is the INTEGER g satisfying $1 \leq g \leq p-1$ such that $g \pmod{p}$ has ORDER $p-1$.

The primitive root of m can also be defined as a cyclic generator of the multiplicative group (mod m) when m is a prime POWER or twice a PRIME POWER. Let p be any ODD PRIME $k \geq 1$, and let

$$s \equiv \sum_{j=1}^{p-1} j^k. \tag{3}$$

Then

$$s = \begin{cases} -1 \pmod{p} & \text{for } p-1 | k \\ 0 \pmod{p} & \text{for } p-1 \nmid k. \end{cases} \tag{4}$$

For numbers m with primitive roots, all y satisfying $(p, y) = 1$ are representable as

$$y \equiv g^t \pmod{m}, \tag{5}$$

where $t = 0, 1, \ldots, \phi(m)-1$, t is known as the index, and y is an INTEGER. Kearnes showed that for any POSITIVE INTEGER m, there exist infinitely many PRIMES p such that

$$m < g_p < p - m. \tag{6}$$

Call the least primitive root g_p. Burgess (1962) proved that

$$g_p \leq Cp^{1/4+\epsilon} \tag{7}$$

for C and ϵ POSITIVE constants and p sufficiently large.

The table below gives the primitive roots (for prime $m = p$; Sloane's A001918) and least primitive roots (for composite m) for the first few INTEGERS

m	g	m	g	m	g
2	1	53	2	134	7
3	2	54	5	137	3
4	3	58	3	139	2
5	2	59	2	142	7
6	5	61	2	146	5
7	3	62	3	149	2
9	2	67	2	151	6
10	3	71	7	157	5
11	2	73	5	158	3
13	2	74	5	162	5
14	3	79	3	163	2
17	3	81	2	166	5
18	5	82	7	167	5
19	2	83	2	169	2
22	7	86	3	173	2
23	5	89	3	178	3
25	2	94	5	179	2
26	7	97	5	181	2
27	2	98	3	191	19
29	2	101	2	193	5
31	3	103	5	194	5
34	3	106	3	197	2
37	2	107	2	199	3
38	3	109	6	202	3
41	6	113	3	206	5
43	3	118	11	211	2
46	5	121	2	214	5
47	5	122	7	218	11
49	3	125	2	223	3
50	3	127	3	226	3
		131	2	227	2

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Primitive Roots." §24.3.4 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 827, 1972.

Guy, R. K. "Primitive Roots." §F9 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 248–249, 1994.

Sloane, N. J. A. Sequence A001918/M0242 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Primitive Root of Unity

A number r is an nth ROOT OF UNITY if $r^n = 1$ and a primitive nth root of unity if, in addition, n is the smallest INTEGER of $k = 1, \ldots, n$ for which $r^k = 1$.

see also ROOT OF UNITY

Primitive Semiperfect Number

A SEMIPERFECT NUMBER for which none of its PROPER DIVISORS are pseudoperfect (Guy 1994, p. 46). The first few are 6, 20, 28, 88, 104, 272 ... (Sloane's A006036). Primitive pseudoperfect numbers are also called IRREDUCIBLE SEMIPERFECT NUMBERS. There are infinitely many primitive pseudoperfect numbers which are not HARMONIC DIVISOR NUMBERS, and infinitely many ODD primitive semiperfect numbers.

see also HARMONIC DIVISOR NUMBER, SEMIPERFECT NUMBER

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 46, 1994.
Sloane, N. J. A. Sequence A006036/M4133 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Primitive Sequence

A SEQUENCE in which no term DIVIDES any other.

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 202, 1994.

Primorial

For a PRIME p,

$$\text{primorial}(p_i) = p_i\# \equiv \prod_{j=1}^{i} p_j,$$

where p_i is the ith PRIME. The first few values for $p_i\#$, are 2, 6, 30, 210, 2310, 30030, 510510, ... (Sloane's A002110).

$p\# - 1$ is PRIME for PRIMES $p = 3$, 5, 11, 41, 89, 317, 337, 991, 1873, 2053, 2377, 4093, 4297, ... (Sloane's A014563; Guy 1994), or p_n for $n = 2$, 3, 5, 13, 24, 66, 68, 167, 287, 310, 352, 564, 590, ... $p\# + 1$ is known to be PRIME for the PRIMES $p = 2$, 3, 5, 7, 11, 31, 379, 1019, 1021, 2657, 3229, 4547, 4787, 11549, ... (Sloane's A005234; Guy 1994, Mudge 1997), or p_n for $n = 1$, 2, 3, 4, 5, 11, 75, 171, 172, 384, 457, 616, 643, 1391, ... Both forms have been tested to $p = 25000$ (Caldwell 1995). It is not known if there are an infinite number of PRIMES for which $p\# + 1$ is PRIME or COMPOSITE (Ribenboim 1989).

see also FACTORIAL, FORTUNATE PRIME, PRIME SUM SMARANDACHE NEAR-TO-PRIMORIAL FUNCTION, TWIN PEAKS

References
Borning, A. "Some Results for $k! + 1$ and $2 \cdot 3 \cdot 5 \cdot p + 1$." *Math. Comput.* **26**, 567–570, 1972.
Buhler, J. P.; Crandall, R. E.; and Penk, M. A. "Primes of the form $M! + 1$ and $\cdot 3 \cdot 5 \cdot p + 1$." *Math. Comput.* **38**, 639–643, 1982.
Caldwell, C. "On The Primality of $n! \pm 1$ and $2 \cdot 3 \cdot 5 \cdots p \pm 1$." *Math. Comput.* **64**, 889–890, 1995.
Dubner, H. "Factorial and Primorial Primes." *J. Rec. Math.* **19**, 197–203, 1987.
Dubner, H. "A New Primorial Prime." *J. Rec. Math.* **21**, 276, 1989.
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 7–8, 1994.
Leyland, P. `ftp://sable.ox.ac.uk/pub/math/factors/primorial-.Z` and `primorial+.Z`.
Mudge, M. "Not Numerology but Numeralogy!" *Personal Computer World*, 279–280, 1997.
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, p. 4, 1989.
Sloane, N. J. A. Sequences A014563, A002110/M1691, and A005234/M0669 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Temper, M. "On the Primality of $k! + 1$ and $\cdot 3 \cdot 5 \cdots p + 1$." *Math. Comput.* **34**, 303–304, 1980.

Prince Rupert's Cube

The largest CUBE which can be made to pass through a given CUBE. (In other words, the CUBE having a side length equal to the side length of the largest HOLE of a SQUARE CROSS-SECTION which can be cut through a unit CUBE without splitting it into two pieces.) The Prince Rupert's cube has side length $3\sqrt{2}/4 = 1.06065\ldots$, and any CUBE this size or smaller can be made to pass through the original CUBE.

see also CUBE, SQUARE

References
Cundy, H. and Rollett, A. "Prince Rupert's Cubes." §3.15.2 in *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 157–158, 1989.
Schrek, D. J. E. "Prince Rupert's Problem and Its Extension by Pieter Nieuwland." *Scripta Math.* **16**, 73–80 and 261–267, 1950.

Principal

The original amount borrowed or lent on which INTEREST is then paid or given.

see also INTEREST

Principal Curvatures

The MAXIMUM and MINIMUM of the NORMAL CURVATURE κ_1 and κ_2 at a given point on a surface are called the principal curvatures. The principal curvatures measure the MAXIMUM and MINIMUM bending of a REGULAR SURFACE at each point. The GAUSSIAN CURVATURE K and MEAN CURVATURE H are related to κ_1 and κ_2 by

$$K = \kappa_1\kappa_2 \tag{1}$$

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2). \tag{2}$$

This can be written as a QUADRATIC EQUATION

$$\kappa^2 - 2H\kappa + K = 0, \tag{3}$$

which has solutions

$$\kappa_1 = H + \sqrt{H^2 - K} \tag{4}$$

$$\kappa_2 = H - \sqrt{H^2 - K}. \tag{5}$$

see also GAUSSIAN CURVATURE, MEAN CURVATURE, NORMAL CURVATURE, NORMAL SECTION, PRINCIPAL DIRECTION, PRINCIPAL RADIUS OF CURVATURE, RODRIGUES'S CURVATURE FORMULA

References
Geometry Center. "Principal Curvatures." `http://www.geom.umn.edu/zoo/diffgeom/surfspace/concepts/curvatures/prin-curv.html`.
Gray, A. "Normal Curvature." §14.2 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 270–273, 277, and 283, 1993.

Principal Curve

A curve $\boldsymbol{\alpha}$ on a REGULAR SURFACE M is a principal curve IFF the velocity $\boldsymbol{\alpha}'$ always points in a PRINCIPAL DIRECTION, i.e.,

$$S(\boldsymbol{\alpha}') = \kappa_i \boldsymbol{\alpha}',$$

where S is the SHAPE OPERATOR and κ_i is a PRINCIPAL CURVATURE. If a SURFACE OF REVOLUTION generated by a plane curve is a REGULAR SURFACE, then the MERIDIANS and PARALLELS are principal curves.

References

Gray, A. "Principal Curves" and "The Differential Equation for the Principal Curves." §18.1 and 21.1 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 410–413, 1993.

Principal Direction

The directions in which the PRINCIPAL CURVATURES occur.

see also PRINCIPAL DIRECTION

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 270, 1993.

Principal Ideal

An IDEAL I of a RING R is called principal if there is an element a of R such that

$$I = aR = \{ar : r \in R\}.$$

In other words, the IDEAL is generated by the element a. For example, the IDEALS $n\mathbb{Z}$ of the RING of INTEGERS $\mathbb{Z}$ are all principal, and in fact all IDEALS of $\mathbb{Z}$ are principal.

see also IDEAL, RING

Principal Normal Vector

see NORMAL VECTOR

Principal Quintic Form

A general QUINTIC EQUATION

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad (1)$$

can be reduced to one of the form

$$y^5 + b_2y^2 + b_1y + b_0 = 0, \qquad (2)$$

called the principal quintic form.

NEWTON'S RELATIONS for the ROOTS y_j in terms of the b_js is a linear system in the b_j, and solving for the b_js expresses them in terms of the POWER sums $s_n(y_j)$. These POWER sums can be expressed in terms of the a_js, so the b_js can be expressed in terms of the a_js. For a quintic to have no quartic or cubic term, the sums of the ROOTS and the sums of the SQUARES of the ROOTS vanish, so

$$s_1(y_j) = 0 \qquad (3)$$

$$s_2(y_j) = 0. \qquad (4)$$

Assume that the ROOTS y_j of the new quintic are related to the ROOTS x_j of the original quintic by

$$y_j = {x_j}^2 + \alpha x_j + \beta. \qquad (5)$$

Substituting this into (1) then yields two equations for α and β which can be multiplied out, simplified by using NEWTON'S RELATIONS for the POWER sums in the x_j, and finally solved. Therefore, α and β can be expressed using RADICALS in terms of the COEFFICIENTS a_j. Again by substitution into (4), we can calculate $s_3(y_j)$, $s_4(y_j)$ and $s_5(y_j)$ in terms of α and β and the x_j. By the previous solution for α and β and again by using NEWTON'S RELATIONS for the POWER sums in the x_j, we can ultimately express these POWER sums in terms of the a_j.

see also BRING QUINTIC FORM, NEWTON'S RELATIONS, QUINTIC EQUATION

Principal Radius of Curvature

Given a 2-D SURFACE, there are two "principal" RADII OF CURVATURE. The larger is denoted R_1, and the smaller R_2. These are PERPENDICULAR to each other, and both PERPENDICULAR to the tangent PLANE of the surface.

see also GAUSSIAN CURVATURE, MEAN CURVATURE, RADIUS OF CURVATURE

Principal Value

see CAUCHY PRINCIPAL VALUE

Principal Vector

A tangent vector $\mathbf{v_p} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$ is a principal vector IFF

$$\det \begin{bmatrix} {v_2}^2 & -v_1v_2 & {v_1}^2 \\ E & F & G \\ e & f & g \end{bmatrix} = 0,$$

where e, f, and g are coefficients of the first FUNDAMENTAL FORM and E, F, G of the second FUNDAMENTAL FORM.

see also FUNDAMENTAL FORMS, PRINCIPAL CURVE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 410, 1993.

Principal Vertex

A VERTEX x_i of a SIMPLE POLYGON P is a principal VERTEX if the diagonal $[x_{i-1}, x_{i+1}]$ intersects the boundary of P only at x_{i-1} and x_{i+1}.

see also EAR, MOUTH

References

Meisters, G. H. "Polygons Have Ears." *Amer. Math. Monthly* **82**, 648–751, 1975.
Meisters, G. H. "Principal Vertices, Exposed Points, and Ears." *Amer. Math. Monthly* **87**, 284–285, 1980.
Toussaint, G. "Anthropomorphic Polygons." *Amer. Math. Monthly* **98**, 31–35, 1991.

Principle

A loose term for a true statement which may be a POSTULATE, THEOREM, etc.

see also AREA PRINCIPLE, ARGUMENT PRINCIPLE, AXIOM, CAVALIERI'S PRINCIPLE, CONJECTURE, CONTINUITY PRINCIPLE, COUNTING GENERALIZED PRINCIPLE, DIRICHLET'S BOX PRINCIPLE, DUALITY PRINCIPLE, DUHAMEL'S CONVOLUTION PRINCIPLE, EUCLID'S PRINCIPLE, FUBINI PRINCIPLE, HASSE PRINCIPLE, INCLUSION-EXCLUSION PRINCIPLE, INDIFFERENCE PRINCIPLE, INDUCTION PRINCIPLE, INSUFFICIENT REASON PRINCIPLE, LEMMA, LOCAL-GLOBAL PRINCIPLE, MULTIPLICATION PRINCIPLE, PERMANENCE OF MATHEMATICAL RELATIONS PRINCIPLE, PONCELET'S CONTINUITY PRINCIPLE, PONTRYAGIN MAXIMUM PRINCIPLE, PORISM, POSTULATE, SCHWARZ REFLECTION PRINCIPLE, SUPERPOSITION PRINCIPLE, SYMMETRY PRINCIPLE, THEOREM, THOMSON'S PRINCIPLE, TRIANGLE TRANSFORMATION PRINCIPLE, WELL-ORDERING PRINCIPLE

Pringsheim's Theorem

Let $C^\omega(I)$ be the set of real ANALYTIC FUNCTIONS on I. Then $C^\omega(I)$ is a SUBALGEBRA of $C^\infty(I)$. A NECESSARY and SUFFICIENT condition for a function $f \in C^\infty(I)$ to belong to $C^\omega(I)$ is that

$$|f^{(n)}(x)| \leq k^n n!$$

for $n = 0, 1, \ldots$ for a suitable constant k.

see also ANALYTIC FUNCTION, SUBALGEBRA

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 207, 1980.

Printer's Errors

Typesetting "errors" in which exponents or multiplication signs are omitted but the resulting expression is equivalent to the original one. Examples include

$$2^5 9^2 = 2592$$

$$3^4 425 = 34425$$

$$31^2 325 = 312325$$

$$2^5 \cdot \tfrac{25}{31} = 25\tfrac{25}{31},$$

where a whole number followed by a fraction is interpreted as addition (e.g., $1\frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$).

see also ANOMALOUS CANCELLATION

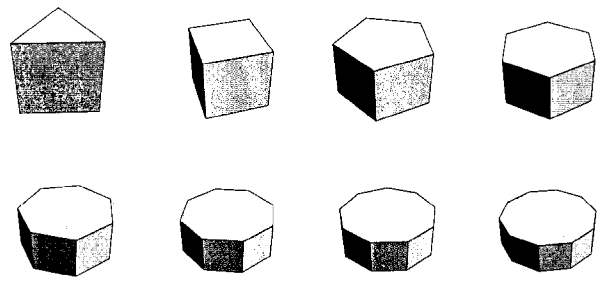
References

Dudeney, H. E. *Amusements in Mathematics.* New York: Dover, 1970.
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 174–175, 1979.

Prior Distribution

see BAYESIAN ANALYSIS

Prism



A POLYHEDRON with two congruent POLYGONAL faces and all remaining faces PARALLELOGRAMS. The 3-prism is simply the CUBE. The simple prisms and antiprisms include: decagonal antiprism, decagonal prism, hexagonal antiprism, hexagonal prism, octagonal antiprism, octagonal prism, pentagonal antiprism, pentagonal prism, square antiprism, and triangular prism. The DUAL POLYHEDRON of a simple (Archimedean) prism is a BIPYRAMID.

The triangular prism, square prism (cube), and hexagonal prism are all SPACE-FILLING POLYHEDRA.

see also ANTIPRISM, AUGMENTED HEXAGONAL PRISM, AUGMENTED PENTAGONAL PRISM, AUGMENTED TRIANGULAR PRISM, BIAUGMENTED PENTAGONAL PRISM, BIAUGMENTED TRIANGULAR PRISM, CUBE, METABIAUGMENTED HEXAGONAL PRISM, PARABIAUGMENTED HEXAGONAL PRISM, PRISMATOID, PRISMOID, TRAPEZOHEDRON, TRIAUGMENTED HEXAGONAL PRISM, TRIAUGMENTED TRIANGULAR PRISM

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 127, 1987.
Cromwell, P. R. *Polyhedra.* New York: Cambridge University Press, pp. 85–86, 1997.
Weisstein, E. W. "Prisms and Antiprisms." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Prism.m`.

Prismatic Ring

A MÖBIUS STRIP with finite width.

see also MÖBIUS STRIP

References

Gardner, M. "Twisted Prismatic Rings." Ch. 5 in *Fractal Music, Hypercards, and More Mathematical Recreations from Scientific American Magazine.* New York: W. H. Freeman, 1992.

Prismatoid

A POLYHEDRON having two POLYGONS in PARALLEL planes as bases and TRIANGULAR or TRAPEZOIDAL lateral faces with one side lying in one base and the opposite VERTEX or side lying in the other base. Examples include the CUBE, PYRAMIDAL FRUSTUM, RECTANGULAR PARALLELEPIPED, PRISM, and PYRAMID. Let A_1 be the AREA of the lower base, A_2 the AREA of the upper base, M the AREA of the midsection, and h the ALTITUDE. Then

$$V = \tfrac{1}{6}h(A_1 + 4M + A_2).$$

see also GENERAL PRISMATOID, PRISMOID

References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 128 and 132, 1987.

Prismoid

A PRISMATOID having planar sides and the same number of vertices in both of its parallel planes. The faces of a prismoid are therefore either TRAPEZOIDS or PARALLELOGRAMS. Ball and Coxeter (1987) use the term to describe an ANTIPRISM.

see also ANTIPRISM, PRISM, PRISMATOID

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 130, 1987.

Prisoner's Dilemma

A problem in GAME THEORY first discussed by A. Tucker. Suppose each of two prisoners A and B, who are not allowed to communicate with each other, is offered to be set free if he implicates the other. If neither implicates the other, both will receive the usual sentence. However, if the prisoners implicate each other, then both are presumed guilty and granted harsh sentences.

A DILEMMA arises in deciding the best course of action in the absence of knowledge of the other prisoner's decision. Each prisoner's best strategy would appear to be to turn the other in (since if A makes the worst-case assumption that B will turn him in, then B will walk free and A will be stuck in jail if he remains silent). However, if the prisoners turn each other in, they obtain the worst possible outcome for both.

see also DILEMMA, TIT-FOR-TAT

References
Axelrod, R. *The Evolution of Cooperation* New York: BasicBooks, 1985.
Goetz, P. "Phil's Good Enough Complexity Dictionary." `http://www.cs.buffalo.edu/~goetz/dict.html`.

Prizes

see MATHEMATICS PRIZES

Probability

Probability is the branch of mathematics which studies the possible outcomes of given events together with their relative likelihoods and distributions. In common usage, the word "probability" is used to mean the chance that a particular event (or set of events) will occur expressed on a linear scale from 0 (impossibility) to 1 (certainty), also expressed as a PERCENTAGE between 0 and 100%. The analysis of events governed by probability is called STATISTICS.

There are several competing interpretations of the actual "meaning" of probabilities. Frequentists view probability simply as a measure of the frequency of outcomes (the more conventional interpretation), while BAYESIANS treat probability more subjectively as a statistical procedure which endeavors to estimate parameters of an underlying distribution based on the observed distribution.

A properly normalized function which assigns a probability "density" to each possible outcome within some interval is called a PROBABILITY FUNCTION, and its cumulative value (integral for a continuous distribution or sum for a discrete distribution) is called a DISTRIBUTION FUNCTION.

Probabilities are defined to obey certain assumptions, called the PROBABILITY AXIOMS. Let a SAMPLE SPACE contain the UNION ($\cup$) of all possible events E_i, so

$$S \equiv \left(\bigcup_{i=1}^{N} E_i\right), \tag{1}$$

and let E and F denote subsets of S. Further, let $F' =$ not-F be the complement of F, so that

$$F \cup F' = S. \tag{2}$$

Then the set E can be written as

$$E = E \cap S = E \cap (F \cup F') = (E \cap F) \cup (E \cap F'), \tag{3}$$

where $\cap$ denotes the intersection. Then

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F') - P[(E \cap F) \cap (E \cap F')] \\ &= P(E \cap F) + P(E \cap F') - P[(F \cap F') \cap (E \cap E)] \\ &= P(E \cap F) + P(E \cap F') - P(\varnothing \cap E) \\ &= P(E \cap F) + P(E \cap F') - P(\varnothing) \\ &= P(E \cap F) + P(E \cap F'), \end{aligned} \tag{4}$$

where $\varnothing$ is the EMPTY SET.

Let $P(E|F)$ denote the CONDITIONAL PROBABILITY of E given that F has already occurred, then

$$P(E) = P(E|F)P(F) + P(E|F')P(F') \quad (5)$$
$$= P(E|F)P(F) + P(E|F')[1 - P(F)] \quad (6)$$
$$P(A \cap B) = P(A)P(B|A) \quad (7)$$
$$= P(B)P(A|B) \quad (8)$$
$$P(A' \cap B) = P(A')P(B|A') \quad (9)$$
$$P(E|F) = \frac{P(E \cap F)}{P(F)} \quad (10)$$

A very important result states that

$$P(E \cup F) = P(E) + P(F) - P(E \cap F), \quad (11)$$

which can be generalized to

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_i P(A_i) - \sum_{ij}{}' P(A_i \cup A_j)$$
$$+ \sum_{ijk}{}'' P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right). \quad (12)$$

see also BAYES' FORMULA, CONDITIONAL PROBABILITY, DISTRIBUTION, DISTRIBUTION FUNCTION, LIKELIHOOD, PROBABILITY AXIOMS, PROBABILITY FUNCTION, PROBABILITY INEQUALITY, STATISTICS

Probability Axioms

Given an event E in a SAMPLE SPACE S which is either finite with N elements or countably infinite with $N = \infty$ elements, then we can write

$$S \equiv \left(\bigcup_{i=1}^N E_i\right),$$

and a quantity $P(E_i)$, called the PROBABILITY of event E_i, is defined such that

1. $0 \le P(E_i) < 1$.
2. $P(S) = 1$.
3. Additivity: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, where E_1 and E_2 are mutually exclusive.
4. Countable additivity: $P\left(\cup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$ for $n = 1, 2, \ldots, N$ where $E_1, E_2, \ldots$ are mutually exclusive (i.e., $E_1 \cap E_2 = \emptyset$).

see also SAMPLE SPACE, UNION

Probability Density Function

see PROBABILITY FUNCTION

Probability Distribution Function

see PROBABILITY FUNCTION

Probability Function

The probability density function $P(x)$ (also called the PROBABILITY DENSITY FUNCTION) of a continuous distribution is defined as the derivative of the (cumulative) DISTRIBUTION FUNCTION $D(x)$,

$$D'(x) = [P(x)]_{-\infty}^x = P(x) - P(-\infty) = P(x), \quad (1)$$

so

$$D(x) = P(X \le x) \equiv \int_{-\infty}^x P(y)\,dy. \quad (2)$$

A probability density function satisfies

$$P(x \in B) = \int_B P(x)\,dx \quad (3)$$

and is constrained by the normalization condition,

$$P(-\infty < x < \infty) = \int_{-\infty}^{\infty} P(x)\,dx \equiv 1. \quad (4)$$

Special cases are

$$P(a \le x \le b) = \int_a^b P(x)\,dx \quad (5)$$
$$P(a \le x \le a + da) = \int_a^{a+da} P(x)\,dx \approx P(a)\,da \quad (6)$$
$$P(x = a) = \int_a^a P(x)\,dx = 0. \quad (7)$$

If $u = u(x,y)$ and $v = v(x,y)$, then

$$P_{u,v}(u,v) = P_{x,y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|. \quad (8)$$

Given the MOMENTS of a distribution (μ, σ, and the GAMMA STATISTICS γ_r), the asymptotic probability function is given by

$$P(x) = Z(x)$$
$$-[\tfrac{1}{6}\gamma_1 Z^{(3)}(x)] + [\tfrac{1}{24}\gamma_2 Z^{(4)}(x) + \tfrac{1}{72}{\gamma_1}^2 Z^{(6)}(x)]$$
$$-[\tfrac{1}{120}\gamma_3 Z^{(5)}(x) + \tfrac{1}{144}\gamma_1\gamma_2 Z^{(7)}(x) + \tfrac{1}{1296}{\gamma_1}^3 Z^{(9)}(x)]$$
$$+[\tfrac{1}{720}\gamma_4 Z^{(6)}(x) + (\tfrac{1}{1152}{\gamma_2}^2 + \tfrac{1}{720}\gamma_1\gamma_3) Z^{(8)}(x)$$
$$+\tfrac{1}{1728}{\gamma_1}^2\gamma_2 Z^{(10)}(x) + \tfrac{1}{31104}{\gamma_1}^4 Z^{(12)}(x)] + \ldots, \quad (9)$$

where

$$Z(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad (10)$$

is the NORMAL DISTRIBUTION, and

$$\gamma_r = \frac{\kappa_r}{\sigma^{r+2}} \quad (11)$$

for $r \geq 1$ (with κ_r CUMULANTS and σ the STANDARD DEVIATION; Abramowitz and Stegun 1972, p. 935).

see also CONTINUOUS DISTRIBUTION, CORNISH-FISHER ASYMPTOTIC EXPANSION, DISCRETE DISTRIBUTION, DISTRIBUTION FUNCTION, JOINT DISTRIBUTION FUNCTION

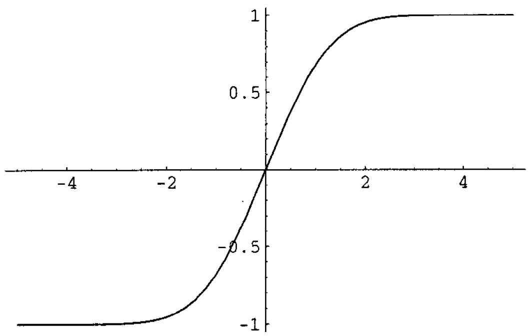
References

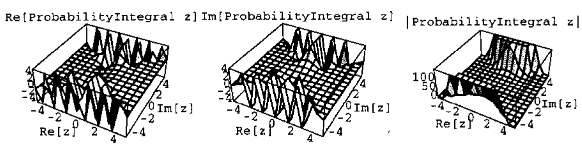
Abramowitz, M. and Stegun, C. A. (Eds.). "Probability Functions." Ch. 26 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 925–964, 1972.

Probability Inequality

If $B \supset A$ (B is a superset of A), then $P(A) \leq P(B)$.

Probability Integral





$$\alpha(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2}\, dt \tag{1}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2}\, dt \tag{2}$$

$$= 2\Phi(x) \tag{3}$$

$$= \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right), \tag{4}$$

where $\Phi(x)$ is the NORMAL DISTRIBUTION FUNCTION and ERF is the error function.

see also ERF, NORMAL DISTRIBUTION FUNCTION

Probability Measure

Consider a PROBABILITY SPACE $(S, \mathbb{S}, P)$ where $(S, \mathbb{S})$ is a MEASURABLE SPACE and P is a MEASURE on $\mathbb{S}$ with $P(S) = 1$. Then the MEASURE P is said to be a probability measure. Equivalently, P is said to be normalized.

see also MEASURABLE SPACE, MEASURE, PROBABILITY, PROBABILITY SPACE, STATE SPACE

Probability Space

A triple $(S, \mathbb{S}, P)$, where $(S, \mathbb{S})$ is a measurable space and P is a MEASURE on $\mathbb{S}$ with $P(S) = 1$.

see also MEASURABLE SPACE, MEASURE, PROBABILITY, PROBABILITY MEASURE, RANDOM VARIABLE, STATE SPACE

Probable Error

The first QUARTILE of a standard NORMAL DISTRIBUTION occurs when

$$\int_0^t \Phi(z)\, dz = \tfrac{1}{4}.$$

The solution is $t = 0.6745\ldots$. The value of t giving $1/4$ is known as the probable error of a NORMALLY DISTRIBUTED variate. However, the number δ corresponding to the 50% CONFIDENCE INTERVAL,

$$P(\delta) \equiv 1 - 2\int_0^{|\delta|} \phi(t)\, dt = \tfrac{1}{2},$$

is sometimes also called the probable error.

see also SIGNIFICANCE

Probable Prime

A number satisfying FERMAT'S LITTLE THEOREM (or some other primality test) for some nontrivial base. A probable prime which is shown to be COMPOSITE is called a PSEUDOPRIME (otherwise, of course, it is a PRIME).

see also PRIME NUMBER, PSEUDOPRIME

Problem

An exercise whose solution is desired.

see also ALHAZEN'S BILLIARD PROBLEM, ALHAZEN'S PROBLEM, ANDRÉ'S PROBLEM, APOLLONIUS' PROBLEM, APOLLONIUS PURSUIT PROBLEM, ARCHIMEDES' CATTLE PROBLEM, ARCHIMEDES' PROBLEM, BALLOT PROBLEM, BASLER PROBLEM, BERTRAND'S PROBLEM, BILLIARD TABLE PROBLEM, BIRTHDAY PROBLEM, BISHOPS PROBLEM, BOLZA PROBLEM, BOOK STACKING PROBLEM, BOUNDARY VALUE PROBLEM, BOVINUM PROBLEMA, BRACHISTOCHRONE PROBLEM, BRAHMAGUPTA'S PROBLEM, BROCARD'S PROBLEM, BUFFON-LAPLACE NEEDLE PROBLEM, BUFFON'S NEEDLE PROBLEM, BURNSIDE PROBLEM, BUSEMANN-PETTY PROBLEM, CANNONBALL PROBLEM, CASTILLON'S PROBLEM, CATALAN'S DIOPHANTINE PROBLEM, CATALAN'S PROBLEM, CATTLE PROBLEM OF ARCHIMEDES, CAUCHY PROBLEM, CHECKER-JUMPING PROBLEM, CLOSED CURVE PROBLEM, COIN PROBLEM, COLLATZ PROBLEM, CONDOM PROBLEM, CONGRUUM PROBLEM, CONSTANT PROBLEM, COUPON COLLECTOR'S PROBLEM, CROSSED LADDERS PROBLEM, CUBE DOVETAILING PROBLEM, DECISION PROBLEM, DEDEKIND'S PROBLEM, DELIAN PROBLEM, DE

MERE'S PROBLEM, DIAGONALS PROBLEM, DIDO'S PROBLEM, DILEMMA, DINITZ PROBLEM, DIRICHLET DIVISOR PROBLEM, DISK COVERING PROBLEM, EQUICHORDAL PROBLEM, EXTENSION PROBLEM, FAGNANO'S PROBLEM, FEJÉS TÓTH'S PROBLEM, FERMAT'S PROBLEM, FERMAT'S SIGMA PROBLEM, FISHER-BEHRENS PROBLEM, FIVE DISKS PROBLEM, FOUR COINS PROBLEM, FOUR TRAVELERS PROBLEM, FUSS'S PROBLEM, GAUSS'S CIRCLE PROBLEM, GAUSS'S CLASS NUMBER PROBLEM, GLOVE PROBLEM, GUTHRIE'S PROBLEM, HABERDASHER'S PROBLEM, HADWIGER PROBLEM, HALTING PROBLEM, HANSEN'S PROBLEM, HEESCH'S PROBLEM, HEILBRONN TRIANGLE PROBLEM, HILBERT'S PROBLEMS, ILLUMINATION PROBLEM, INDETERMINATE PROBLEMS, INITIAL VALUE PROBLEM, INTERNAL BISECTORS PROBLEM, ISOPERIMETRIC PROBLEM, ISOVOLUME PROBLEM, JEEP PROBLEM, JOSEPHUS PROBLEM, KAKEYA NEEDLE PROBLEM, KAKUTANI'S PROBLEM, KATONA'S PROBLEM, KEPLER PROBLEM, KINGS PROBLEM, KIRKMAN'S SCHOOLGIRL PROBLEM, KISSING CIRCLES PROBLEM, KNAPSACK PROBLEM, KNOT PROBLEM, KÖNIGSBERG BRIDGE PROBLEM, KURATOWSKI'S CLOSURE-COMPONENT PROBLEM, LAM'S PROBLEM, LANGFORD'S PROBLEM, LEBESGUE MEASURABILITY PROBLEM, LEBESGUE MINIMAL PROBLEM, LEHMER'S PROBLEM, LEMOINE'S PROBLEM, LIFTING PROBLEM, LUCAS' MARRIED COUPLES PROBLEM, MALFATTI'S RIGHT TRIANGLE PROBLEM, MALFATTI'S TANGENT TRIANGLE PROBLEM, MARRIED COUPLES PROBLEM, MATCH PROBLEM, MAXIMUM CLIQUE PROBLEM, MÉNAGE PROBLEM, METRIC EQUIVALENCE PROBLEM, MICE PROBLEM, MIKUSIŃSKI'S PROBLEM, MÖBIUS PROBLEM, MONEY-CHANGING PROBLEM, MONKEY AND COCONUT PROBLEM, MONTY HALL PROBLEM, MORTALITY PROBLEM, MOSER'S CIRCLE PROBLEM, NAPOLEON'S PROBLEM, NAVIGATION PROBLEM, NEAREST NEIGHBOR PROBLEM, NP-COMPLETE PROBLEM, NP-PROBLEM, ORCHARD-PLANTING PROBLEM, ORCHARD VISIBILITY PROBLEM, P-PROBLEM, PARTY PROBLEM, PIANO MOVER'S PROBLEM, PLANAR BUBBLE PROBLEM, PLATEAU'S PROBLEM, POINTS PROBLEM, POSTAGE STAMP PROBLEM, POTHENOT PROBLEM, PROUHET'S PROBLEM, QUEENS PROBLEM, RAILROAD TRACK PROBLEM, RIEMANN'S MODULI PROBLEM, SATISFIABILITY PROBLEM, SCHOOLGIRL PROBLEM, SCHUR'S PROBLEM, SCHWARZ'S TRIANGLE PROBLEM, SHARING PROBLEM, SHEPHARD'S PROBLEM, SINCLAIR'S SOAP FILM PROBLEM, SMALL WORLD PROBLEM, SNELLIUS-POTHENOT PROBLEM, STEENROD'S REALIZATION PROBLEM, STEINER'S PROBLEM, STEINER'S SEGMENT PROBLEM, SURVEYING PROBLEMS, SYLVESTER'S FOUR-POINT PROBLEM, SYLVESTER'S LINE PROBLEM, SYLVESTER'S TRIANGLE PROBLEM, SYRACUSE PROBLEM, SYZYGIES PROBLEM, TARRY-ESCOTT PROBLEM, TAUTOCHRONE PROBLEM, THOMSON PROBLEM, THREE JUG PROBLEM, TRAVELING SALESMAN PROBLEM, TRAWLER PROBLEM, ULAM'S PROBLEM, UTILITY PROBLEM, VIBRATION PROBLEM, WALLIS'S PROBLEM, WARING'S PROBLEM

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Procedure

A specific prescription for carrying out a task or solving a problem. Also called an ALGORITHM, METHOD, or TECHNIQUE

see also BISECTION PROCEDURE, MAEHLY'S PROCEDURE

Proclus' Axiom

If a line intersects one of two parallel lines, it must intersect the other also. This AXIOM is equivalent to the PARALLEL AXIOM.

References

Dunham, W. "Hippocrates' Quadrature of the Lune." Ch. 1 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, p. 54, 1990.

Procrustian Stretch

see HYPERBOLIC ROTATION

Product

The term "product" refers to the result of one or more MULTIPLICATIONS. For example, the mathematical statement $a \times b = c$ would be read "a TIMES b EQUALS c," where c is the product.

The product symbol is defined by

$$\prod_{i=1}^{n} f_i \equiv f_1 \cdot f_2 \cdots f_n.$$

Useful product identities include

$$\ln\left(\prod_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} \ln f_i$$

$$\prod_{i=1}^{\infty} f_i = \exp\left(\sum_{i=1}^{\infty} \ln f_i\right).$$

For $0 \leq a_i < 1$, then the products $\prod_{i=1}^{\infty}(1+a_i)$ and $\prod_{i=1}^{\infty}(1-a_i)$ converge and diverge as $\prod_{i=1}^{\infty} a_i$.

see also CROSS PRODUCT, DOT PRODUCT, INNER PRODUCT, MATRIX PRODUCT, MULTIPLICATION, NONASSOCIATIVE PRODUCT, OUTER PRODUCT, SUM, TENSOR PRODUCT, TIMES, VECTOR TRIPLE PRODUCT

References

Guy, R. K. "Products Taken over Primes." §B87 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 102–103, 1994.

Product Formula

Let α be a NONZERO RATIONAL NUMBER $\alpha = \pm p_1{}^{\alpha_1} p_2{}^{\alpha_2} \cdots p_L{}^{\alpha_L}$, where $p_1, \ldots, p_L$ are distinct PRIMES, $\alpha_l \in \mathbb{Z}$ and $\alpha_l \neq 0$. Then

$$|a| \prod_{p \text{ prime}} |\alpha|_p = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \cdots p_L{}^{\alpha_L} \times p_1{}^{-\alpha_1} p_2{}^{-\alpha_2} \cdots p_L{}^{-\alpha_L} = 1.$$

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Product-Moment Coefficient of Correlation

see CORRELATION COEFFICIENT

Product Neighborhood

see TUBULAR NEIGHBORHOOD

Product Rule

The DERIVATIVE identity

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right] \\
&= \lim_{h\to 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right] = f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

see also CHAIN RULE, EXPONENT LAWS, QUOTIENT RULE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 11, 1972.

Product Space

A Cartesian product equipped with a "product topology" is called a product space (or product topological space, or direct product).

References

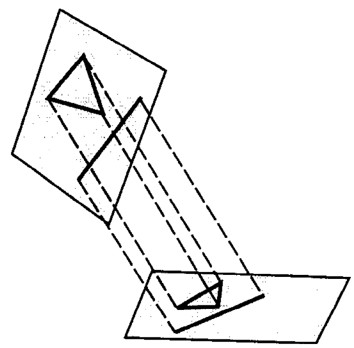
Iyanaga, S. and Kawada, Y. (Eds.). "Product Spaces." §408L *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 1281–1282, 1980.

Program

A precise sequence of instructions designed to accomplish a given task. The implementation of an ALGORITHM on a computer using a programming language is an example of a program.

see also ALGORITHM

Projection



A projection is the transformation of POINTS and LINES in one PLANE onto another PLANE by connecting corresponding points on the two planes with PARALLEL lines. This can be visualized as shining a (point) light source (located at infinity) through a translucent sheet of paper and making an image of whatever is drawn on it on a second sheet of paper. The branch of geometry dealing with the properties and invariants of geometric figures under projection is called PROJECTIVE GEOMETRY.

The projection of a VECTOR $\mathbf{a}$ onto a VECTOR $\mathbf{u}$ is given by

$$\mathrm{proj}_{\mathbf{u}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u},$$

and the length of this projection is

$$|\mathrm{proj}_{\mathbf{u}}\mathbf{a}| = \frac{|\mathbf{a}\cdot\mathbf{u}|}{|\mathbf{u}|}.$$

General projections are considered by Foley and Van-Dam (1983).

see also MAP PROJECTION, POINT-PLANE DISTANCE, PROJECTIVE GEOMETRY, REFLECTION

References

Casey, J. "Theory of Projections." Ch. 11 in *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples, 2nd ed., rev. enl.* Dublin: Hodges, Figgis, & Co., pp. 349–367, 1893.

Foley, J. D. and VanDam, A. *Fundamentals of Interactive Computer Graphics, 2nd ed.* Reading, MA: Addison-Wesley, 1990.

Projection Operator

$$\tilde{p} \equiv |\phi_i(x)\rangle\langle\phi_i(t)|$$

$$\tilde{p} \sum_j c_j |\phi_j(t)\rangle = c_i |\phi_i(x)\rangle$$

$$\sum_i |\phi_i(x)\rangle\langle\phi_i(x)| = 1.$$

see also BRA, KET

Projective Collineation

A COLLINEATION which transforms every 1-D form projectively. Any COLLINEATION which transforms one range into a projectively related range is a projective collineation. Every PERSPECTIVE COLLINEATION is a projective collineation.

see also COLLINEATION, ELATION, HOMOLOGY (GEOMETRY), PERSPECTIVE COLLINEATION

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Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, pp. 247–248, 1969.

Projective General Linear Group

The projective general linear group $PGL_n(q)$ is the GROUP obtained from the GENERAL LINEAR GROUP $GL_n(q)$ on factoring the scalar MATRICES contained in that group.

see also GENERAL LINEAR GROUP, PROJECTIVE GENERAL ORTHOGONAL GROUP, PROJECTIVE GENERAL UNITARY GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$, and $PSL_n(q) = L_n(q)$." §2.1 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Projective General Orthogonal Group

The projective general orthogonal group $PGO_n(q)$ is the GROUP obtained from the GENERAL ORTHOGONAL GROUP $GO_n(q)$ on factoring the scalar MATRICES contained in that group.

see also GENERAL ORTHOGONAL GROUP, PROJECTIVE GENERAL LINEAR GROUP, PROJECTIVE GENERAL UNITARY GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GO_n(q)$, $SO_n(q)$, $PGO_n(q)$, and $PSO_n(q)$, and $O_n(q)$." §2.4 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. xi–xii, 1985.

Projective General Unitary Group

The projective general unitary group $PGU_n(q)$ is the GROUP obtained from the GENERAL UNITARY GROUP $GU_n(q)$ on factoring the scalar MATRICES contained in that group.

see also GENERAL UNITARY GROUP, PROJECTIVE GENERAL LINEAR GROUP, PROJECTIVE GENERAL ORTHOGONAL GROUP, PROJECTIVE GENERAL UNITARY GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GU_n(q)$, $SU_n(q)$, $PGU_n(q)$, and $PSU_n(q) = U_n(q)$." §2.2 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Projective Geometry

The branch of geometry dealing with the properties and invariants of geometric figures under PROJECTION. The most amazing result arising in projective geometry is the DUALITY PRINCIPLE, which states that a duality exists between theorems such as PASCAL'S THEOREM and BRIANCHON'S THEOREM which allows one to be instantly transformed into the other. More generally, *all* the propositions in projective geometry occur in dual pairs, which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "POINT" and "LINE."

The AXIOMS of projective geometry are:

1. If A and B are distinct points on a PLANE, there is at least one LINE containing both A and B.
2. If A and B are distinct points on a PLANE, there is not more than one LINE containing both A and B.
3. Any two LINES on a PLANE have at least one point of the PLANE in common.
4. There is at least one LINE on a PLANE.
5. Every LINE contains at least three points of the PLANE.
6. All the points of the PLANE do not belong to the same LINE

(Veblin and Young 1910–18, Kasner and Newman 1989).

see also COLLINEATION, DESARGUES' THEOREM, FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY, INVOLUTION (LINE), PENCIL, PERSPECTIVITY, PROJECTIVITY, RANGE (LINE SEGMENT), SECTION (PENCIL)

References
Birkhoff, G. and Mac Lane, S. "Projective Geometry." §9.14 in *A Survey of Modern Algebra, 3rd ed.* New York: Macmillan, pp. 275–279, 1965.
Casey, J. "Theory of Projections." Ch. 11 in *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples, 2nd ed., rev. enl.* Dublin: Hodges, Figgis, & Co., pp. 349–367, 1893.
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Struik, D. *Lectures on Projected Geometry.* Reading, MA: Addison-Wesley, 1998.
Veblen, O. and Young, J. W. *Projective Geometry, 2 vols.* Boston, MA: Ginn, 1910–18.
Whitehead, A. N. *The Axioms of Projective Geometry.* New York: Hafner, 1960.

Projective Plane

A projective plane is derived from a usual PLANE by addition of a LINE AT INFINITY. A projective plane of order n is a set of n^2+n+1 POINTS with the properties that:

1. Any two POINTS determine a LINE,
2. Any two LINES determine a POINT,
3. Every POINT has $n+1$ LINES on it, and
4. Every LINE contains $n+1$ POINTS.

(Note that some of these properties are redundant.) A projective plane is therefore a SYMMETRIC $(n^2+n+1, n+1, 1)$ BLOCK DESIGN. An AFFINE PLANE of order n exists IFF a projective plane of order n exists.

A finite projective plane exists when the order n is a POWER of a PRIME, i.e., $n = p^a$ for $a \geq 1$. It is conjectured that these are the *only* possible projective planes, but proving this remains one of the most important unsolved problems in COMBINATORICS. The first few orders which are not of this form are 6, 10, 12, 14, 15, ...

It has been proven analytically that there are no projective planes of order 6. By answering LAM'S PROBLEM in the negative using massive computer calculations on top of some mathematics, it has been proved that there are no finite projective planes of order 10 (Lam 1991). The status of the order 12 projective plane remains open. The remarkable BRUCK-RYSER-CHOWLA THEOREM says that if a projective plane of order n exists, and $n = 1$ or 2 (mod 4), then n is the sum of two SQUARES. This rules out $n = 6$.

The projective plane of order 2, also known as the FANO PLANE, is denoted PG(2, 2). It has INCIDENCE MATRIX

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Every row and column contains 3 1s, and any pair of rows/columns has a single 1 in common.

The projective plane has EULER CHARACTERISTIC 1, and the HEAWOOD CONJECTURE therefore shows that any set of regions on it can be colored using six colors only (Saaty 1986).

see also AFFINE PLANE, BRUCK-RYSER-CHOWLA THEOREM, FANO PLANE, LAM'S PROBLEM, MAP COLORING, MOUFANG PLANE, PROJECTIVE PLANE PK^2, REAL PROJECTIVE PLANE

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 281–287, 1987.
Lam, C. W. H. "The Search for a Finite Projective Plane of Order 10." *Amer. Math. Monthly* **98**, 305–318, 1991.
Lindner, C. C. and Rodger, C. A. *Design Theory.* Boca Raton, FL: CRC Press, 1997.
Pinkall, U. "Models of the Real Projective Plane." Ch. 6 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 63–67, 1986.
Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, p. 45, 1986.

Projective Plane PK^2

The 2-D SPACE consisting of the set of TRIPLES

$$\{(a,b,c) : a,b,c \in K, \text{ not all zero}\},$$

where triples which are SCALAR multiples of each other are identified.

Projective Space

A SPACE which is invariant under the GROUP G of all general LINEAR homogeneous transformation in the SPACE concerned, but not under all the transformations of any GROUP containing G as a SUBGROUP.

A projective space is the space of 1-D VECTOR SUBSPACES of a given VECTOR SPACE. For REAL VECTOR SPACES, the NOTATION $\mathbb{RP}^n$ or $\mathbb{P}^n$ denotes the REAL projective space of dimension n (i.e., the SPACE of 1-D VECTOR SUBSPACES of $\mathbb{R}^{n+1}$) and $\mathbb{CP}^n$ denotes the COMPLEX projective space of COMPLEX dimension n (i.e., the space of 1-D COMPLEX VECTOR SUBSPACES of $\mathbb{C}^{n+1}$). $\mathbb{P}^n$ can also be viewed as the set consisting of $\mathbb{R}^n$ together with its POINTS AT INFINITY.

Projective Special Linear Group

The projective special linear group $PSL_n(q)$ is the GROUP obtained from the SPECIAL LINEAR GROUP $SL_n(q)$ on factoring by the SCALAR MATRICES contained in that GROUP. It is SIMPLE for $n \geq 2$ except for

$$PSL_2(2) = S_3$$
$$PSL_2(3) = A_4,$$

and is therefore also denoted $L_n(Q)$.

see also PROJECTIVE SPECIAL ORTHOGONAL GROUP, PROJECTIVE SPECIAL UNITARY GROUP, SPECIAL LINEAR GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$, and $PSL_n(q) = L_n(q)$." §2.1 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Projective Special Orthogonal Group

The projective special orthogonal group $PSO_n(q)$ is the GROUP obtained from the SPECIAL ORTHOGONAL GROUP $SO_n(q)$ on factoring by the SCALAR MATRICES contained in that GROUP. In general, this GROUP is not SIMPLE.

see also PROJECTIVE SPECIAL LINEAR GROUP, PROJECTIVE SPECIAL UNITARY GROUP, SPECIAL ORTHOGONAL GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GO_n(q)$, $SO_n(q)$, $PGO_n(q)$, and $PSO_n(q)$, and $O_n(q)$." §2.4 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. xi–xii, 1985.

Projective Special Unitary Group

The projective special unitary group $PSU_n(q)$ is the GROUP obtained from the SPECIAL UNITARY GROUP $SU_n(q)$ on factoring by the SCALAR MATRICES contained in that GROUP. $PSU_n(q)$ is SIMPLE except for

$$\begin{aligned} PSU_2(2) &= S_3 \\ PSU_2(3) &= A_4 \\ PSU_3(2) &= 3^2 : Q_8, \end{aligned}$$

so it is given the simpler name $U_n(q)$, with $U_2(q) = L_2(q)$.

see also PROJECTIVE SPECIAL LINEAR GROUP, PROJECTIVE SPECIAL ORTHOGONAL GROUP, SPECIAL UNITARY GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GU_n(q)$, $SU_n(q)$, $PGU_n(q)$, and $PSU_n(q) = U_n(q)$." §2.2 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Projective Symplectic Group

The projective symplectic group $PSp_n(q)$ is the GROUP obtained from the SYMPLECTIC GROUP $Sp_n(q)$ on factoring by the SCALAR MATRICES contained in that GROUP. $PSp_{2m}(q)$ is SIMPLE except for

$$\begin{aligned} PSp_2(2) &= S_3 \\ PSp_2(3) &= A_4 \\ PSp_4(2) &= S_6, \end{aligned}$$

so it is given the simpler name $S_{2m}(q)$, with $S_2(q) = L_2(q)$.

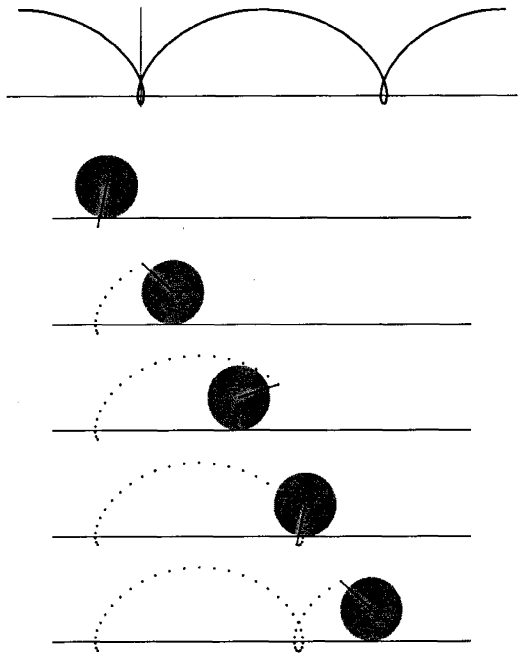
References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $Sp_n(q)$ and $PSp_n(q) = S_n(q)$." §2.3 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. x–xi, 1985.

Projectivity

The product of any number of PERSPECTIVITIES.

see also INVOLUTION (TRANSFORMATION), PERSPECTIVITY

Prolate Cycloid



The path traced out by a fixed point at a RADIUS $b > a$, where a is the RADIUS of a rolling CIRCLE, also sometimes called an EXTENDED CYCLOID. The prolate cycloid contains loops, and has parametric equations

$$x = a\phi - b\sin\phi \tag{1}$$
$$y = a - b\cos\phi. \tag{2}$$

The ARC LENGTH from $\phi = 0$ is

$$s = 2(a+b)E(u), \tag{3}$$

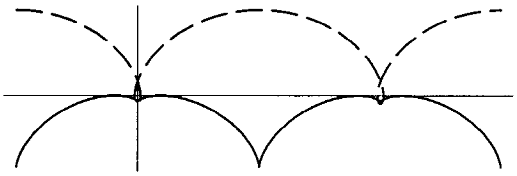
where

$$\sin(\tfrac{1}{2}\phi) = \operatorname{sn} u \tag{4}$$
$$k^2 = \frac{4ab}{(a+c)^2}. \tag{5}$$

see also CURTATE CYCLOID, CYCLOID

References
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 46–50, 1991.

Prolate Cycloid Evolute

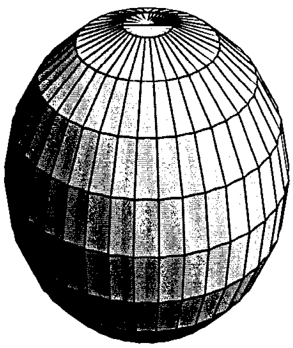


The EVOLUTE of the PROLATE CYCLOID is given by

$$x = \frac{a[-2b\phi + 2a\phi\cos\phi - 2a\sin\phi + b\sin(2\phi)]}{2(a\cos\phi - b)}$$

$$y = \frac{a(a - b\cos\phi)^2}{b(a\cos\phi - b)}.$$

Prolate Spheroid



A SPHEROID which is "pointy" instead of "squashed," i.e., one for which the polar radius c is greater than the equatorial radius a, so $c > a$. A prolate spheroid has Cartesian equations

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

The ELLIPTICITY of the prolate spheroid is defined by

$$e \equiv \sqrt{\frac{c^2 - a^2}{c^2}} = \frac{\sqrt{c^2 - a^2}}{c} = \sqrt{1 - \frac{a^2}{c^2}}, \tag{2}$$

so that

$$1 - e^2 = \frac{a^2}{c^2}. \tag{3}$$

Then

$$r = a\left(1 + \frac{e^2}{1 - e^2}\sin^2\delta\right)^{-1/2}. \tag{4}$$

The SURFACE AREA and VOLUME are

$$S = 2\pi a^2 + 2\pi\frac{ac}{e}\sin^{-1} e \tag{5}$$

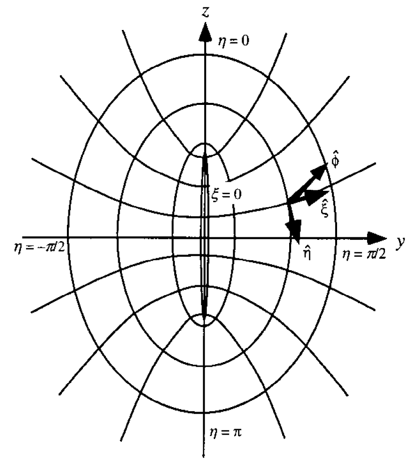
$$V = \tfrac{4}{3}\pi a^2 c. \tag{6}$$

see also DARWIN-DE SITTER SPHEROID, ELLIPSOID, OBLATE SPHEROID, PROLATE SPHEROIDAL COORDINATES, SPHERE, SPHEROID

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 131, 1987.

Prolate Spheroidal Coordinates



A system of CURVILINEAR COORDINATES in which two sets of coordinate surfaces are obtained by revolving the curves of the ELLIPTIC CYLINDRICAL COORDINATES about the x-AXIS, which is relabeled the z-AXIS. The third set of coordinates consists of planes passing through this axis.

$$x = a\sinh\xi\sin\eta\cos\phi \tag{1}$$

$$y = a\sinh\xi\sin\eta\sin\phi \tag{2}$$

$$z = a\cosh\xi\cos\eta, \tag{3}$$

where $\xi \in [0, \infty)$, $\eta \in [0, \pi)$, and $\phi \in [0, 2\pi)$. Arfken (1970) uses (u, v, φ) instead of (ξ, η, z). The SCALE FACTORS are

$$h_\xi = a\sqrt{\sinh^2\xi + \sin^2\eta} \tag{4}$$

$$h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta} \tag{5}$$

$$h_\phi = a\sinh\xi\sin\eta. \tag{6}$$

The LAPLACIAN is

$$\nabla^2 f = \frac{1}{a^2(\sinh^2\xi + \sin^2\eta)}\left\{\frac{1}{\sinh\xi}\frac{\partial}{\partial\xi}\left(\sinh\xi\frac{\partial f}{\partial\xi}\right) + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial f}{\partial\eta}\right) + \frac{\partial^2 f}{\partial\phi^2}\right\} \tag{7}$$

$$= \frac{1}{a^2(\sin^2\eta + \sinh^2\xi)}\left[(\csc^2\eta + \operatorname{csch}^2\xi)\frac{\partial^2}{\partial\phi^2} + \cot\eta\frac{\partial}{\partial\eta} + \left[\frac{\partial^2}{\partial\eta^2} + \coth\xi\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\xi^2}\right]\right]. \tag{8}$$

An alternate form useful for "two-center" problems is defined by

$$\xi_1 = \cosh\xi \tag{9}$$

$$\xi_2 = \cos\eta \tag{10}$$

$$\xi_3 = \phi, \tag{11}$$

where $\xi_1 \in [1,\infty]$, $\xi_2 \in [-1,1]$, and $\xi_3 \in [0,2\pi)$ (Abramowitz and Stegun 1972). In these coordinates,

$$z = a\xi_1\xi_2 \tag{12}$$
$$x = a\sqrt{({\xi_1}^2-1)(1-{\xi_2}^2)}\cos\xi_3 \tag{13}$$
$$y = a\sqrt{({\xi_1}^2-1)(1-{\xi_2}^2)}\sin\xi_3. \tag{14}$$

In terms of the distances from the two FOCI,

$$\xi_1 = \frac{r_1+r_2}{2a} \tag{15}$$
$$\xi_2 = \frac{r_1-r_2}{2a} \tag{16}$$
$$2a = r_{12}. \tag{17}$$

The SCALE FACTORS are

$$h_{\xi_1} = a\sqrt{\frac{{\xi_1}^2-{\xi_2}^2}{{\xi_1}^2-1}} \tag{18}$$
$$h_{\xi_2} = a\sqrt{\frac{{\xi_1}^2-{\xi_2}^2}{1-{\xi_2}^2}} \tag{19}$$
$$h_{\xi_3} = a\sqrt{({\xi_1}^2-1)(1-{\xi_2}^2)}, \tag{20}$$

and the LAPLACIAN is

$$\nabla^2 f = \frac{1}{a^2}\left\{\frac{1}{{\xi_1}^2-{\xi_2}^2}\frac{\partial}{\partial\xi_1}\left[({\xi_1}^2-1)\frac{\partial f}{\partial\xi_1}\right] + \frac{1}{{\xi_1}^2-{\xi_2}^2}\frac{\partial}{\partial\xi_2}\left[(1-{\xi_2}^2)\frac{\partial f}{\partial\xi_2}\right] + \frac{1}{({\xi_1}^2-1)(1-{\xi_2}^2)}\frac{\partial^2 f}{\partial{\xi_3}^2}\right\}. \tag{21}$$

The HELMHOLTZ DIFFERENTIAL EQUATION is separable in prolate spheroidal coordinates.

see also HELMHOLTZ DIFFERENTIAL EQUATION—PROLATE SPHEROIDAL COORDINATES, LATITUDE, LONGITUDE, OBLATE SPHEROIDAL COORDINATES, SPHERICAL COORDINATES

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Definition of Prolate Spheroidal Coordinates." §21.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 752, 1972.

Arfken, G. "Prolate Spheroidal Coordinates (u, v, ϕ)." §2.10 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, pp. 103–107, 1970.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 661, 1953.

Prolate Spheroidal Wave Function

The WAVE EQUATION in PROLATE SPHEROIDAL COORDINATES is

$$\nabla^2\Phi + k^2\Phi = \frac{\partial}{\partial\xi_1}\left[({\xi_1}^2-1)\frac{\partial\Phi}{\partial\xi_1}\right] + \frac{\partial}{\partial\xi_2}\left[(1-{\xi_2}^2)\frac{\partial\Phi}{\partial\xi_2}\right] + \frac{{\xi_1}^2-{\xi_2}^2}{({\xi_1}^2-1)(1-{x_2}^2)}\frac{\partial^2\Phi}{\partial\phi^2} + c^2({\xi_1}^2-{\xi_2}^2)\Phi = 0, \tag{1}$$

where

$$c \equiv \tfrac{1}{2}ak. \tag{2}$$

Substitute in a trial solution

$$\Phi = R_{mn}(c,\xi_1)S_{mn}(c,\xi_2)\frac{\cos}{\sin}(m\phi) \tag{3}$$

$$\frac{d}{d\xi_1}\left[({\xi_1}^2-1)\frac{d}{d\xi_1}R_{mn}(c,\xi_1)\right] - \left(\lambda_{mn} - c^2{\xi_1}^2 + \frac{m^2}{{\xi_1}^2-1}\right)R_{mn}(c,\xi_1) = 0. \tag{4}$$

The radial differential equation is

$$\frac{d}{d\xi_2}\left[({\xi_2}^2-1)\frac{d}{d\xi_2}S_{mn}(c,\xi_2)\right] - \left(\lambda_{mn} - c^2{\xi_2}^2 + \frac{m^2}{{\xi_2}^2-1}\right)R_{mn}(c,\xi_2) = 0, \tag{5}$$

and the angular differential equation is

$$\frac{d}{d\xi_2}\left[(1-{\xi_2}^2)\frac{d}{d\xi_2}S_{mn}(c,\xi_2)\right] - \left(\lambda_{mn} - c^2{\xi_2}^2 + \frac{m^2}{1-{\xi_2}^2}\right)R_{mn}(c,\xi_2) = 0. \tag{6}$$

Note that these are identical (except for a sign change). The prolate angular function of the first kind is given by

$$S_{mn}^{(1)} = \begin{cases} \sum_{r=1,3,\ldots}^{\infty} d_r(c)P_{m+r}^m(\eta) & \text{for } n-m \text{ odd} \\ \sum_{r=0,2,\ldots}^{\infty} d_r(c)P_{m+r}^m(\eta) & \text{for } n-m \text{ even}, \end{cases} \tag{7}$$

where $P_m^k(\eta)$ is an associated LEGENDRE POLYNOMIAL. The prolate angular function of the second kind is given by

$$S_{mn}^{(2)} = \begin{cases} \sum\limits_{r=\ldots,-1,1,3,\ldots} d_r(c)Q_{m+r}^m(\eta) & \text{for } n-m \text{ odd} \\ \sum\limits_{r=\ldots,-2,0,2,\ldots} d_r(c)Q_{m+r}^m(\eta) & \text{for } n-m \text{ even}, \end{cases} \tag{8}$$

where $Q_k^m(\eta)$ is an associated LEGENDRE FUNCTION OF THE SECOND KIND and the COEFFICIENTS d_r satisfy the RECURRENCE RELATION

$$\alpha_k d_{k+2} + (\beta_k - \lambda_{mn})d_k + \gamma_k d_{k-2} = 0, \tag{9}$$

with

$$\alpha_k = \frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)} \tag{10}$$
$$\beta_k = (m+k)(m+k+1) + \frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}c^2 \tag{11}$$
$$\gamma_k = \frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)}. \tag{12}$$

Various normalization schemes are used for the ds (Abramowitz and Stegun 1972, p. 758). Meixner and Schäfke (1954) use

$$\int_{-1}^{1} [S_{mn}(c,\eta)]^2 \, d\eta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}. \tag{13}$$

Stratton *et al.* (1956) use

$$\frac{(n+m)!}{(n-m)!} = \begin{cases} \sum_{r=1,3,\ldots}^{\infty} \frac{(r+2m)!}{r!} d_r & \text{for } n-m \text{ odd} \\ \sum_{r=0,2,\ldots}^{\infty} \frac{(r+2m)!}{r!} d_r & \text{for } n-m \text{ even.} \end{cases} \tag{14}$$

Flammer (1957) uses

$$S_{mn}(c,0) = \begin{cases} P_n^{m+1}(0) & \text{for } n-m \text{ odd} \\ P_n^m(0) & \text{for } n-m \text{ even.} \end{cases} \tag{15}$$

see also OBLATE SPHEROIDAL WAVE FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spheroidal Wave Functions." Ch. 21 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 751–759, 1972.

Flammer, C. *Spheroidal Wave Functions.* Stanford, CA: Stanford University Press, 1957.

Meixner, J. and Schäfke, F. W. *Mathieusche Funktionen und Sphäroidfunktionen.* Berlin: Springer-Verlag, 1954.

Stratton, J. A.; Morse, P. M.; Chu, L. J.; Little, J. D. C.; and Corbató, F. J. *Spheroidal Wave Functions.* New York: Wiley, 1956.

Pronic Number

A FIGURATE NUMBER of the form $P_n = 2T_n = n(n+1)$, where T_n is the nth TRIANGULAR NUMBER. The first few are 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, ... (Sloane's A002378). The GENERATING FUNCTION of the pronic numbers is

$$\frac{2x}{(1-x)^3} = 2x + 6x^2 + 12x^3 + 20x^4 + \ldots.$$

The first few n for which P_n are PALINDROMIC are 1, 2, 16, 77, 538, 1621, ... (Sloane's A028336), and the first few PALINDROMIC NUMBERS which are pronic are 2, 6, 272, 6006, 289982, ... (Sloane's A028337).

References

De Geest, P. "Palindromic Products of Two Consecutive Integers." `http://www.ping.be/~ping6758/consec.htm`.

Sloane, N. J. A. Sequences A028336, A028337, and A002378/M1581 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Proof

A rigorous mathematical argument which unequivocally demonstrates the truth of a given PROPOSITION. A mathematical statement which has been proven is called a THEOREM.

There is some debate among mathematicians as to just what constitutes a proof. The FOUR-COLOR THEOREM is an example of this debate, since its "proof" relies on an exhaustive computer testing of many individual cases which cannot be verified "by hand." While many mathematicians regard computer-assisted proofs as valid, some purists do not.

see also PARADOX, PROPOSITION, THEOREM

References

Garnier, R. and Taylor, J. *100% Mathematical Proof.* New York: Wiley, 1996.

Solow, D. *How to Read and Do Proofs: An Introduction to Mathematical Thought Process.* New York: Wiley, 1982.

Proofreading Mistakes

If proofreader A finds a mistakes and proofreader B finds b mistakes, c of which were also found by A, how many mistakes were missed by both A and B? Assume there are a total of m mistakes, so proofreader A finds a FRACTION a/m of all mistakes, and also a FRACTION c/b of the mistakes found by B. Assuming these fractions are the same, then solving for m gives

$$m = \frac{ab}{c}.$$

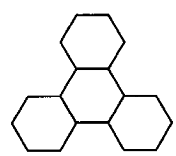
The number of mistakes missed by both is therefore approximately

$$N = m - a - b + c = \frac{(a-c)(b-c)}{c}.$$

References

Pólya, G. "Probabilities in Proofreading." *Amer. Math. Monthly*, **83**, 42, 1976.

Propeller



A 4-POLYHEX.

References

Gardner, M. *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions and Other Mathematical Sleight-of-Mind from Scientific American.* New York: Vintage, p. 147, 1978.

Proper Cover

see COVER

Proper Divisor

A DIVISOR of a number n excluding n itself.

see also ALIQUANT DIVISOR, ALIQUOT DIVISOR, DIVISOR

Proper Fraction

A FRACTION $p/q < 1$.

see also FRACTION, REDUCED FRACTION

Proper Integral

An INTEGRAL which has neither limit INFINITE and from which the INTEGRAND does not approach INFINITY at any point in the range of integration.

see also IMPROPER INTEGRAL, INTEGRAL

Proper k-Coloring

see k-COLORING

Proper Subset

A SUBSET which is not the entire SET. For example, consider a SET $\{1, 2, 3, 4, 5\}$. Then $\{1, 2, 4\}$ and $\{1\}$ are proper subsets, while $\{1, 2, 6\}$ and $\{1, 2, 3, 4, 5\}$ are not.

see also SET, SUBSET

Proper Superset

A SUPERSET which is not the entire SET.

see also SET, SUPERSET

Proportional

If a is proportional to b, then a/b is a constant. The relationship is written $a \propto b$, which implies

$$a = cb,$$

for some constant c.

Proposition

A statement which is to be proved.

Propositional Calculus

The formal basis of LOGIC dealing with the notion and usage of words such as "NOT," "OR," "AND," and "IMPLIES." Many systems of propositional calculus have been devised which attempt to achieve consistency, completeness, and independence of AXIOMS.

see also LOGIC, P-SYMBOL

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 254–255, 1989.

Nidditch, P. H. *Propositional Calculus.* New York: Free Press of Glencoe, 1962.

Prosthaphaeresis Formulas

TRIGONOMETRY formulas which convert a product of functions into a sum or difference.

Proth's Theorem

For $N = h \cdot 2^n + 1$ with ODD h and $2^n > h$, if there exists an INTEGER a such that

$$a^{(N-1)/2} \equiv -1 \pmod{N},$$

then N is PRIME.

Protractor

A ruled SEMICIRCLE used for measuring and drawing ANGLES.

Prouhet's Problem

A generalization of the TARRY-ESCOTT PROBLEM to three or more sets of INTEGERS.

see also TARRY-ESCOTT PROBLEM

References

Wright, E. M. "Prouhet's 1851 Solution of the Tarry-Escott Problem of 1910." *Amer. Math. Monthly* **102**, 199–210, 1959.

Prüfer Ring

A metric space $\hat{\mathbb{Z}}$ in which the closure of a congruence class $B(j, m)$ is the corresponding congruence class $\{x \in \hat{\mathbb{Z}} | x \equiv j \pmod{m}\}$.

References

Fried, M. D. and Jarden, M. *Field Arithmetic.* New York: Springer-Verlag, pp. 7–11, 1986.

Postnikov, A. G. *Introduction to Analytic Number Theory.* Providence, RI: Amer. Math. Soc., 1988.

Prussian Hat

A device used in the Cornwell smoothness stabilized modification of the CLEAN ALGORITHM.

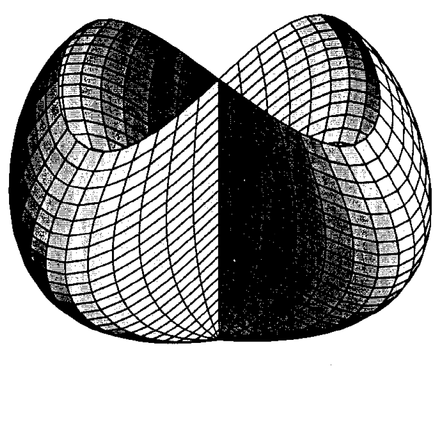
see also CLEAN ALGORITHM

Pseudoanalytic Function

A pseudoanalytic function is a function defined using generalized CAUCHY-RIEMANN EQUATIONS. Pseudoanalytic functions come as close as possible to having COMPLEX derivatives and are nonsingular "quasiregular" functions.

see also ANALYTIC FUNCTION, SEMIANALYTIC, SUBANALYTIC

Pseudocrosscap



A surface constructed by placing a family of figure-eight curves into $\mathbb{R}^3$ such that the first and last curves reduce to points. The surface has parametric equations

$$x(u,v) = (1-u^2)\sin v$$
$$y(u,v) = (1-u^2)\sin(2v)$$
$$z(u,v) = u.$$

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 247–248, 1993.

Pseudocylindrical Projection

A projection in which latitude lines are parallel but meridians are curves.

see also CYLINDRICAL PROJECTION, ECKERT IV PROJECTION, ECKERT VI PROJECTION, MOLLWEIDE PROJECTION, ROBINSON PROJECTION, SINUSOIDAL PROJECTION

References

Dana, P. H. "Map Projections." http://www.utexas.edu/depts/grg/gcraft/notes/mapproj/mapproj.html.

Pseudogroup

An algebraic structure whose elements consist of selected HOMEOMORPHISMS between open subsets of a SPACE, with the composition of two transformations defined on the largest possible domain. The "germs" of the elements of a pseudogroup form a GROUPOID (Weinstein 1996).

see also GROUP, GROUPOID, INVERSE SEMIGROUP

References

Weinstein, A. "Groupoids: Unifying Internal and External Symmetry." *Not. Amer. Math. Soc.* **43**, 744–752, 1996.

Pseudolemniscate Case

The case of the WEIERSTRAß ELLIPTIC FUNCTION with invariants $g_2 = -1$ and $g_3 = 0$.

see also EQUIANHARMONIC CASE, LEMNISCATE CASE

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Pseudo-Lemniscate Case ($g_2 = -1$, $g_3 = 0$)." §18.15 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 662–663, 1972.

Pseudoperfect Number

see SEMIPERFECT NUMBER

Pseudoprime

A pseudoprime is a COMPOSITE number which passes a test or sequence of tests which fail for most COMPOSITE numbers. Unfortunately, some authors drop the "COMPOSITE" requirement, calling any number which passes the specified tests a pseudoprime even if it is PRIME. Pomerance, Selfridge, and Wagstaff (1980) restrict their use of "pseudoprime" to ODD COMPOSITE numbers. "Pseudoprime" used without qualification means FERMAT PSEUDOPRIME.

CARMICHAEL NUMBERS are ODD COMPOSITE numbers which are pseudoprimes to every base; they are sometimes called ABSOLUTE PSEUDOPRIMES. The following table gives the number of FERMAT PSEUDOPRIMES psp, EULER PSEUDOPRIMES epsp, and STRONG PSEUDOPRIMES spsp to the base 2, as well as CARMICHAEL NUMBERS CN which are less the first few powers of 10 (Guy 1994).

	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}
psp(2)	3	22	78	245	750	2057	5597	14884
epsp(2)	1	12	36	114	375	1071	2939	7706
spsp(2)	0	5	16	46	162	488	1282	3291
CN	1	7	16	43	105	255	646	1547

see also CARMICHAEL NUMBER, ELLIPTIC PSEUDOPRIME, EULER PSEUDOPRIME, EULER-JACOBI PSEUDOPRIME, EXTRA STRONG LUCAS PSEUDOPRIME, FERMAT PSEUDOPRIME, FIBONACCI PSEUDOPRIME, FROBENIUS PSEUDOPRIME, LUCAS PSEUDOPRIME, PERRIN PSEUDOPRIME, PROBABLE PRIME, SOMER-LUCAS PSEUDOPRIME, STRONG ELLIPTIC PSEUDOPRIME, STRONG FROBENIUS PSEUDOPRIME, STRONG LUCAS PSEUDOPRIME, STRONG PSEUDOPRIME

References

Grantham, J. "Frobenius Pseudoprimes." http://www.clark.net/pub/grantham/pseudo/pseudo.ps

Grantham, J. "Pseudoprimes/Probable Primes." http://www.clark.net/pub/grantham/pseudo.

Guy, R. K. "Pseudoprimes. Euler Pseudoprimes. Strong Pseudoprimes." §A12 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 27–30, 1994.

Pomerance, C.; Selfridge, J. L.; and Wagstaff, S. S. "The Pseudoprimes to $25 \cdot 10^9$." *Math. Comput.* **35**, 1003–1026, 1980. Available electronically from ftp://sable.ox.ac.uk/pub/math/primes/ps2.Z.

Pseudorandom Number

A slightly archaic term for a computer-generated RANDOM NUMBER. The prefix pseudo- is used to distinguish this type of number from a "truly" RANDOM NUMBER generated by a random physical process such as radioactive decay.

see also RANDOM NUMBER

References

Luby, M. *Pseudorandomness and Cryptographic Applications.* Princeton, NJ: Princeton University Press, 1996.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of*

Scientific Computing, 2nd ed. Cambridge, England: Cambridge University Press, p. 266, 1992.

Pseudorhombicuboctahedron

see ELONGATED SQUARE GYROBICUPOLA

Pseudoscalar

A SCALAR which reverses sign under inversion is called a pseudoscalar. The SCALAR TRIPLE PRODUCT

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

is a pseudoscalar. Given a transformation MATRIX A,

$$S' = \det |\mathsf{A}| S,$$

where det is the DETERMINANT.

see also PSEUDOTENSOR, PSEUDOVECTOR, SCALAR

References

Arfken, G. "Pseudotensors, Dual Tensors." §3.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 128–137, 1985.

Pseudosmarandache Function

The pseudosmarandache function $Z(n)$ is the smallest integer such that

$$\sum_{k=1}^{Z(n)} k = \tfrac{1}{2} Z(n)[Z(n)+1]$$

is divisible by n. The values for $n = 1, 2, \ldots$ are 1, 3, 2, 7, 4, 3, 6, 15, 8, 4, ... (Sloane's A011772).

see also SMARANDACHE FUNCTION

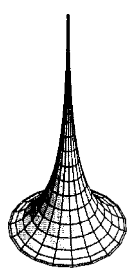
References

Ashbacher, C. "Problem 514." *Pentagon* **57**, 36, 1997.

Kashihara, K. "Comments and Topics on Smarandache Notions and Problems." Vail: Erhus University Press, 1996.

Sloane, N. J. A. Sequence A011772 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Pseudosphere



Half the SURFACE OF REVOLUTION generated by a TRACTRIX about its ASYMPTOTE to form a TRACTROID. The Cartesian parametric equations are

$$x = \operatorname{sech} u \cos v \quad (1)$$

$$y = \operatorname{sech} u \sin v \quad (2)$$

$$z = u - \tanh u \quad (3)$$

for $u \geq 0$.

It has constant NEGATIVE CURVATURE, and so is called a pseudosphere by analogy with the SPHERE, which has constant POSITIVE curvature. An equation for the GEODESICS is

$$\cosh^2 u + (v + c)^2 = k^2. \quad (4)$$

see also FUNNEL, GABRIEL'S HORN, TRACTRIX

References

Fischer, G. (Ed.). Plate 82 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 77, 1986.

Geometry Center. "The Pseudosphere." `http://www.geom.umn.edu/zoo/diffgeom/pseudosphere/`.

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 383–384, 1993.

Pseudosquare

Given an ODD PRIME p, a SQUARE NUMBER n satisfies $(n/p) = 0$ or 1 for all $p < n$, where (n/p) is the LEGENDRE SYMBOL. A number $n > 2$ which satisfies this relationship but is not a SQUARE NUMBER is called a pseudosquare. The only pseudoprimes less than 10^8 are 3 and 6.

see also SQUARE NUMBER

Pseudotensor

A TENSOR-like object which reverses sign under inversion. Given a transformation MATRIX A,

$$A_{ij}' = \det |\mathsf{A}| a_{ik} a_{jl} A_{kl},$$

where det is the DETERMINANT. A pseudotensor is sometimes also called a TENSOR DENSITY.

see also PSEUDOSCALAR, PSEUDOVECTOR, SCALAR, TENSOR DENSITY

References

Arfken, G. "Pseudotensors, Dual Tensors." §3.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 128–137, 1985.

Pseudovector

A typical VECTOR is transformed to its NEGATIVE under inversion. A VECTOR which is invariant under inversion is called a pseudovector, also called an AXIAL VECTOR in older literature (Morse and Feshbach 1953). The CROSS PRODUCT

$$\mathbf{A} \times \mathbf{B} \quad (1)$$

is a pseudovector, whereas the VECTOR TRIPLE PRODUCT

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \quad (2)$$

is a VECTOR.

$$[\text{pseudovector}] \times [\text{pseudovector}] = [\text{pseudovector}] \quad (3)$$

$$[\text{vector}] \times [\text{pseudovector}] = [\text{vector}]. \qquad (4)$$

Given a transformation MATRIX A,

$$C_i' = \det|\mathsf{A}| a_{ij} C_j. \qquad (5)$$

see also PSEUDOSCALAR, TENSOR, VECTOR

References
Arfken, G. "Pseudotensors, Dual Tensors." §3.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 128–137, 1985.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 46–47, 1953.

Psi Function

$$\Psi(z, s, v) \equiv \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s}$$

for $|z| < 1$ and $v \neq 0, -1, \ldots$ (Gradshteyn and Ryzhik 1980, pp. 1075–1076).

see also HURWITZ ZETA FUNCTION, RAMANUJAN PSI SUM, THETA FUNCTION

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, 1979.

PSLQ Algorithm

An ALGORITHM which finds INTEGER RELATIONS between real numbers $x_1, \ldots, x_n$ such that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0,$$

with not all $a_i = 0$. This algorithm terminates after a number of iterations bounded by a polynomial in n and uses a numerically stable matrix reduction procedure (Ferguson and Bailey 1992), thus improving upon the FERGUSON-FORCADE ALGORITHM. It is based on a partial sum of squares scheme (like the PSOS ALGORITHM) implemented using LQ decomposition. A much simplified version of the algorithm was developed by Ferguson *et al.* and extended to complex numbers.

see also FERGUSON-FORCADE ALGORITHM, INTEGER RELATION, LLL ALGORITHM, PSOS ALGORITHM

References
Bailey, D. H.; Borwein, J. M.; and Girgensohn, R. "Experimental Evaluation of Euler Sums." *Exper. Math.* **3**, 17–30, 1994.
Bailey, D. and Plouffe, S. "Recognizing Numerical Constants." `http://www.cecm.sfu.ca/organics/papers/bailey`.
Ferguson, H. R. P. and Bailey, D. H. "A Polynomial Time, Numerically Stable Integer Relation Algorithm." RNR Techn. Rept. RNR-91-032, Jul. 14, 1992.
Ferguson, H. R. P.; Bailey, D. H.; and Arno, S. "Analysis of PSLQ, An Integer Relation Finding Algorithm." Unpublished manuscript.

PSOS Algorithm

An INTEGER-RELATION algorithm which is based on a partial sum of squares approach, from which the algorithm takes its name.

see also FERGUSON-FORCADE ALGORITHM, HJLS ALGORITHM, INTEGER RELATION, LLL ALGORITHM, PSLQ ALGORITHM

References
Bailey, D. H. and Ferguson, H. R. P. "Numerical Results on Relations Between Numerical Constants Using a New Algorithm." *Math. Comput.* **53**, 649–656, 1989.

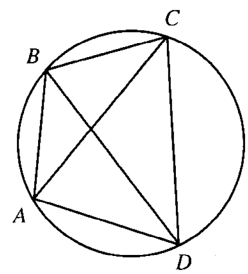
Ptolemy Inequality

For a QUADRILATERAL which is not CYCLIC, PTOLEMY'S THEOREM becomes an INEQUALITY:

$$AB \times CD + BC \times DA > AC \times BD.$$

see also PTOLEMY'S THEOREM, QUADRILATERAL

Ptolemy's Theorem



If a QUADRILATERAL is inscribed in a circle (i.e., for a cyclic quadrilateral), the sum of the products of the two pairs of opposite sides equals the product of the diagonals

$$AB \times CD + BC \times DA = AC \times BD.$$

This fact can be used to derive the TRIGONOMETRY addition formulas.

see also FUHRMANN'S THEOREM, PTOLEMY INEQUALITY

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 42–43, 1967.

Public-Key Cryptography

A type of CRYPTOGRAPHY in which the encoding key is revealed without compromising the encoded message. The two best-known methods are the KNAPSACK PROBLEM and RSA ENCRYPTION.

see also KNAPSACK PROBLEM, RSA ENCRYPTION

References
Diffie, W. and Hellman, M. "New Directions in Cryptography." *IEEE Trans. Info. Th.* **22**, 644–654, 1976.
Hellman, M. E. "The Mathematics of Public-Key Cryptography." *Sci. Amer.* **241**, 130–139, Aug. 1979.
Rivest, R.; Shamir, A.; and Adleman, L. "A Method for Obtaining Digital Signatures and Public-Key Cryptosystems." MIT Memo MIT/LCS/TM-82, 1982.
Wagon, S. "Public-Key Encryption." §1.2 in *Mathematica in Action.* New York: W. H. Freeman, pp. 20–22, 1991.

Puiseaux's Theorem

The whole neighborhood of any point y_i of an algebraic PLANE CURVE may be uniformly represented by a certain finite number of convergent developments in POWER SERIES,

$$x_i = \rho_\nu y_i + a_{\nu i1} t_\nu + a_{\nu i2} {t_\nu}^2 + \ldots.$$

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 207, 1959.

Pullback Map

A pullback is a general CATEGORICAL operation appearing in a number of mathematical contexts, sometimes going under a different name. If $T : V \to W$ is a linear transformation between VECTOR SPACES, then $T^* : W^* \to V^*$ (usually called TRANSPOSE MAP or DUAL MAP because its associated matrix is the MATRIX TRANSPOSE of T) is an example of a pullback map.

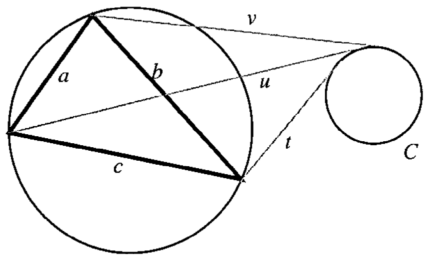
In the case of a DIFFEOMORPHISM and DIFFERENTIABLE MANIFOLD, a very explicit definition can be formulated. Given an r-form α on a MANIFOLD M_2, define the r-form $T^*(\alpha)$ on M_1 by its action on an r-tuple of tangent vectors $(X_1, \ldots, X_r)$ as the number $T^*(\alpha)(X_1, \ldots, X_r) = \alpha(T_* X_1, \ldots, T_* X_r)$. This defines a map on r-forms and is the pullback map.

see also CATEGORY

Pulse Function

see RECTANGLE FUNCTION

Purser's Theorem



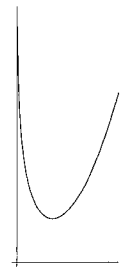
Let t, u, and v be the lengths of the tangents to a CIRCLE C from the vertices of a TRIANGLE with sides of lengths a, b, and c. Then the condition that C is tangent to the CIRCUMCIRCLE of the TRIANGLE is that

$$\pm at \pm bu \pm cv = 0.$$

The theorem was discovered by Casey prior to Purser's independent discovery.

see also CASEY'S THEOREM, CIRCUMCIRCLE

Pursuit Curve



If A moves along a known curve, then P describes a pursuit curve if P is always directed toward A and A and P move with uniform velocities. These were considered in general by the French scientist Pierre Bouguer in 1732. The case restricting A to a straight line was studied by Arthur Bernhart (MacTutor Archive). It has CARTESIAN COORDINATES equation

$$y = cx - \ln x.$$

see also APOLLONIUS PURSUIT PROBLEM, MICE PROBLEM

References
Bernhart, A. "Curves of Pursuit." *Scripta Math.* **20**, 125–141, 1954.
Bernhart, A. "Curves of Pursuit–II." *Scripta Math.* **23**, 49–65, 1957.
Bernhart, A. "Polygons of Pursuit." *Scripta Math.* **24**, 23–50, 1959.
Bernhart, A. "Curves of General Pursuit." *Scripta Math.* **24**, 189–206, 1959.
MacTutor History of Mathematics Archive. "Pursuit Curve." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Pursuit.html`.
Yates, R. C. "Pursuit Curve." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 170–171, 1952.

Push

An action which adds a single element to the top of a STACK, turning the STACK $(a_1, a_2, \ldots, a_n)$ into $(a_0, a_1, a_2, \ldots, a_n)$.

see also POKE MOVE, POP, STACK

Puzzle

A mathematical PROBLEM, usually not requiring advanced mathematics, to which a solution is desired. Puzzles frequently require the rearrangement of existing pieces (e.g., 15 PUZZLE) or the filling in of blanks (e.g., crossword puzzle).

see also 15 PUZZLE, BAGUENAUDIER, CALIBAN PUZZLE, CONWAY PUZZLE, CRYPTARITHMETIC, DISSECTION PUZZLES, ICOSIAN GAME, PYTHAGOREAN SQUARE PUZZLE, RUBIK'S CUBE, SLOTHOUBER-GRAATSMA PUZZLE, T-PUZZLE

References
Bogomolny, A. "Interactive Mathematics Miscellany and Puzzles." `http://www.cut-the-knot.com/`.

Dudeney, H. E. *Amusements in Mathematics.* New York: Dover, 1917.
Dudeney, H. E. *The Canterbury Puzzles and Other Curious Problems, 7th ed.* London: Thomas Nelson and Sons, 1949.
Dudeney, H. E. *536 Puzzles & Curious Problems.* New York: Scribner, 1967.
Fujii, J. N. *Puzzles and Graphs.* Washington, DC: National Council of Teachers, 1966.

Pyramid

A POLYHEDRON with one face a POLYGON and all the other faces TRIANGLES with a common VERTEX. An n-gonal regular pyramid (denoted Y_n) has EQUILATERAL TRIANGLES, and is possible only for $n = 3$, 4, 5. These correspond to the TETRAHEDRON, SQUARE PYRAMID, and PENTAGONAL PYRAMID, respectively. A pyramid therefore has a single cross-sectional shape in which the length scale of the CROSS-SECTION scales linearly with height. The AREA at a height z is given by

$$A(z) = A_b \left(\frac{z^2}{h^2} \right), \tag{1}$$

where A_b is the base AREA and h is the pyramid height. The VOLUME is therefore given by

$$V = \int_0^h A(z)\, dz = A_b \int_0^h \frac{z^2}{h^2}\, dz = \frac{A_b}{h^2}(\tfrac{1}{3}h^3) = \tfrac{1}{3}A_b h. \tag{2}$$

These results also hold for the CONE, TETRAHEDRON (triangular pyramid), SQUARE PYRAMID, etc.

The CENTROID is the same as for the CONE, given by

$$\bar{z} = \tfrac{1}{4}h. \tag{3}$$

The SURFACE AREA of a pyramid is

$$S = \tfrac{1}{2}ps, \tag{4}$$

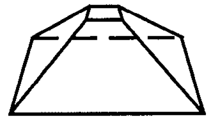
where s is the SLANT HEIGHT and p is the base PERIMETER. Joining two PYRAMIDS together at their bases gives a BIPYRAMID, also called a DIPYRAMID.

see also BIPYRAMID, ELONGATED PYRAMID, GYROELONGATED PYRAMID, PENTAGONAL PYRAMID, PYRAMID, PYRAMIDAL FRUSTUM, SQUARE PYRAMID, TETRAHEDRON, TRUNCATED SQUARE PYRAMID

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 128, 1987.
Hart, G. W. "Pyramids, Dipyramids, and Trapezohedra." `http://www.li.net/~george/virtual-polyhedra/pyramids-info.html`.

Pyramidal Frustum



Let s be the slant height, p_i the top and bottom base PERIMETERS, and A_i the top and bottom AREAS. Then the SURFACE AREA and VOLUME of the pyramidal frustum are given by

$$S = \tfrac{1}{2}(p_1 + p_2)s$$
$$V = \tfrac{1}{3}h(A_1 + A_2 + \sqrt{A_1 A_2}\,).$$

see also CONICAL FRUSTUM, FRUSTUM, PYRAMID, SPHERICAL SEGMENT, TRUNCATED SQUARE PYRAMID

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 128, 1987.
Dunham, W. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 3–4, 1990.

Pyramidal Number

A FIGURATE NUMBER corresponding to a configuration of points which form a pyramid with r-sided REGULAR POLYGON bases can be thought of as a generalized pyramidal number, and has the form

$$P_n^r = \tfrac{1}{6}(n+1)(2p_n^r + n) = \tfrac{1}{6}n(n+1)[(r-2)n + (5-r)]. \tag{1}$$

The first few cases are therefore

$$P_n^3 = \tfrac{1}{6}n(n+1)(n+2) \tag{2}$$
$$P_n^4 = \tfrac{1}{6}n(n+1)(2n+1) \tag{3}$$
$$P_n^5 = \tfrac{1}{2}n^2(n+1), \tag{4}$$

so $r = 3$ corresponds to a TETRAHEDRAL NUMBER Te_n, and $r = 4$ to a SQUARE PYRAMIDAL NUMBER P_n.

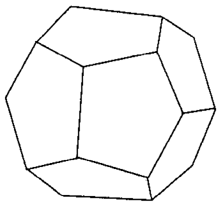
The pyramidal numbers can also be generalized to 4-D and higher dimensions (Sloane and Plouffe 1995).

see also HEPTAGONAL PYRAMIDAL NUMBER, HEXAGONAL PYRAMIDAL NUMBER, PENTAGONAL PYRAMIDAL NUMBER, SQUARE PYRAMIDAL NUMBER, TETRAHEDRAL NUMBER

References

Conway, J. H. and Guy, R. K. "Tetrahedral Numbers" and "Square Pyramidal Numbers" *The Book of Numbers.* New York: Springer-Verlag, pp. 44–49, 1996.
Sloane, N. J. A. and Plouffe, S. "Pyramidal Numbers." Extended entry for sequence M3382 in *The Encyclopedia of Integer Sequences.* San Diego, CA: Academic Press, 1995.

Pyritohedron



An irregular DODECAHEDRON composed of identical irregular PENTAGONS.

see also DODECAHEDRON, RHOMBIC DODECAHEDRON, TRIGONAL DODECAHEDRON

References
Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 63, 1990.

Pythagoras's Constant

The number

$$\sqrt{2} = 1.4142135623\ldots,$$

which the Pythagoreans proved to be IRRATIONAL. The Babylonians gave the impressive approximation

$$\sqrt{2} \approx 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.41421296296296\ldots$$

(Guy 1990, Conway and Guy 1996, pp. 181–182).

see also IRRATIONAL NUMBER, OCTAGON, PYTHAGORAS'S THEOREM, SQUARE

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 25 and 181–182, 1996.
Guy, R. K. "Review: The Mathematics of Plato's Academy." *Amer. Math. Monthly* **97**, 440–443, 1990.
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 126, 1993.

Pythagoras's Theorem

Proves that the DIAGONAL d of a SQUARE with sides of integral length s cannot be RATIONAL. Assume d/s is rational and equal to p/q where p and q are INTEGERS with no common factors. Then

$$d^2 = s^2 + s^2 = 2s^2,$$

so

$$\left(\frac{d}{s}\right)^2 = \left(\frac{p}{q}\right)^2 = 2,$$

and $p^2 = 2q^2$, so p^2 is even. But if p^2 is EVEN, then p is EVEN. Since p/q is defined to be expressed in lowest terms, q must be ODD; otherwise p and q would have the common factor 2. Since p is EVEN, we can let $p \equiv 2r$, then $4r^2 = 2q^2$. Therefore, $q^2 = 2r^2$, and q^2, so q must be EVEN. But q cannot be both EVEN and ODD, so there are no d and s such that d/s is RATIONAL, and d/s must be IRRATIONAL.

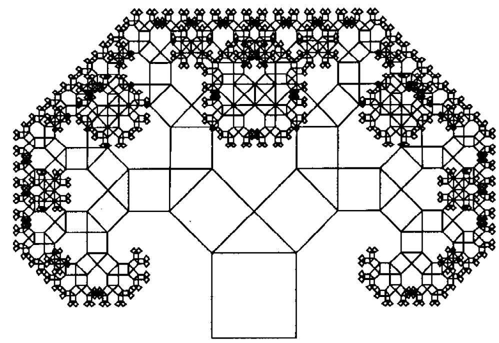
In particular, PYTHAGORAS'S CONSTANT $\sqrt{2}$ is IRRATIONAL. Conway and Guy (1996) give a proof of this fact using paper folding, as well as similar proofs for ϕ (the GOLDEN RATIO) and $\sqrt{3}$ using a PENTAGON and HEXAGON.

see also IRRATIONAL NUMBER, PYTHAGORAS'S CONSTANT, PYTHAGOREAN THEOREM

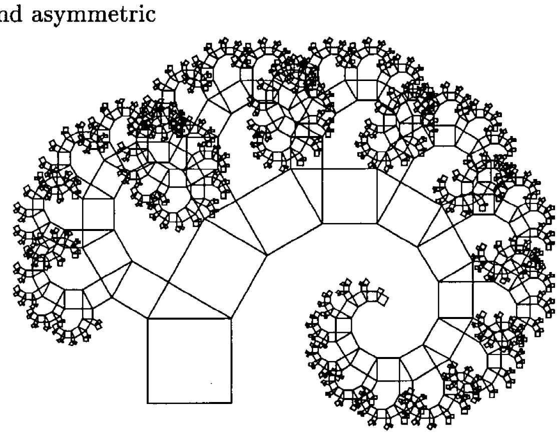
References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 183–186, 1996.
Pappas, T. "Irrational Numbers & the Pythagoras Theorem." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 98–99, 1989.

Pythagoras Tree

A FRACTAL with symmetric



and asymmetric



forms.

References
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 67–77 and 111–113, 1991.
Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Pythagorean Fraction

Given a PYTHAGOREAN TRIPLE (a, b, c), the fractions a/b and b/a are called Pythagorean fractions. Diophantus showed that the Pythagorean fractions consist precisely of fractions of the form $(p^2 - q^2)/(2pq)$.

References
Conway, J. H. and Guy, R. K. "Pythagorean Fractions." In *The Book of Numbers.* New York: Springer-Verlag, pp. 171–173, 1996.

Pythagorean Quadruple

POSITIVE INTEGERS a, b, c, and d which satisfy

$$a^2 + b^2 + c^2 = d^2. \tag{1}$$

For POSITIVE EVEN a and b, there exist such INTEGERS c and d; for POSITIVE ODD a and b, no such INTEGERS exist (Oliverio 1996). Oliverio (1996) gives the following generalization of this result. Let $S = (a_1, \ldots, a_{n-2})$, where a_i are INTEGERS, and let T be the number of ODD INTEGERS in S. Then IFF $T \not\equiv 2 \pmod 4$, there exist INTEGERS a_{n-1} and a_n such that

$${a_1}^2 + {a_2}^2 + \ldots + {a_{n-1}}^2 = {a_n}^2. \tag{2}$$

A set of Pythagorean quadruples is given by

$$a = 2mp \tag{3}$$
$$b = 2np \tag{4}$$
$$c = p^2 - (m^2 + n^2) \tag{5}$$
$$d = p^2 + (m^2 + n^2), \tag{6}$$

where m, n, and p are INTEGERS,

$$m + n + p \equiv 1 \pmod 2, \tag{7}$$

and

$$(m, n, p) = 1 \tag{8}$$

(Mordell 1969). This does not, however, generate all solutions. For instance, it excludes (36, 8, 3, 37). Another set of solutions can be obtained from

$$a = 2mp + 2nq \tag{9}$$
$$b = 2np - 2mq \tag{10}$$
$$c = p^2 + q^2 - (m^2 + n^2) \tag{11}$$
$$d = p^2 + q^2 + (m^2 + n^2) \tag{12}$$

(Carmichael 1915).

see also EULER BRICK, PYTHAGOREAN TRIPLE

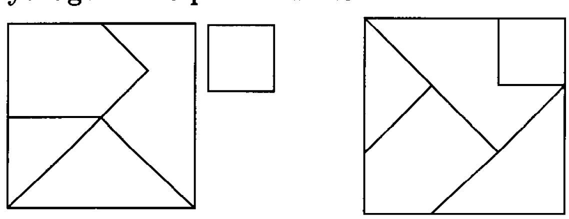
References

Carmichael, R. D. *Diophantine Analysis.* New York: Wiley, 1915.

Mordell, L. J. *Diophantine Equations.* London: Academic Press, 1969.

Oliverio, P. "Self-Generating Pythagorean Quadruples and N-tuples." *Fib. Quart.* **34**, 98–101, 1996.

Pythagorean Square Puzzle



Combine the two above squares on the left into the single large square on the right.

see also DISSECTION, T-PUZZLE

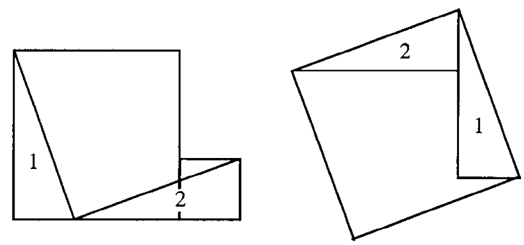
Pythagorean Theorem

For a RIGHT TRIANGLE with legs a and b and HYPOTENUSE c,

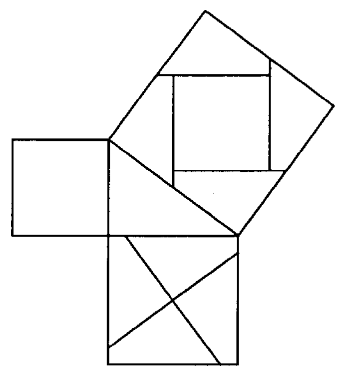
$$a^2 + b^2 = c^2. \tag{1}$$

Many different proofs exist for this most fundamental of all geometric theorems.

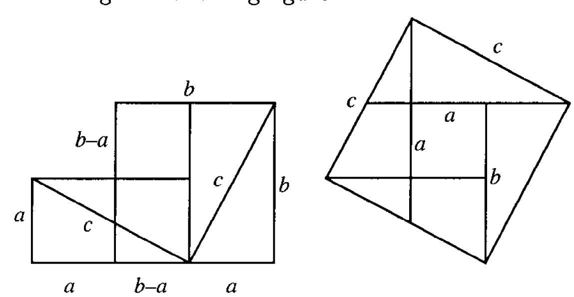
A clever proof by DISSECTION which reassembles two small squares into one larger one was given by the Arabian mathematician Thabit Ibn Qurra (Ogilvy 1994, Frederickson 1997).



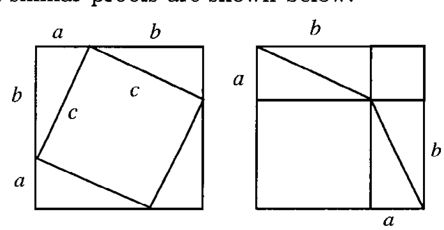
Another proof by DISSECTION is due to Perigal (Pergial 1873, Dudeney 1970, Madachy 1979, Ball and Coxeter 1987).



The Indian mathematician Bhaskara constructed a proof using the following figure.



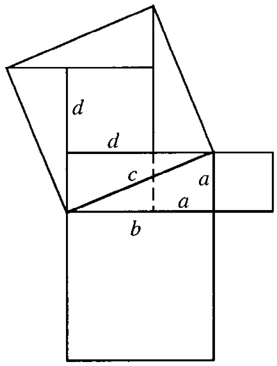
Several similar proofs are shown below.



$$c^2 + 4(\tfrac{1}{2}ab) = (a+b)^2 \tag{2}$$

$$c^2 + 2ab = a^2 + 2ab + b^2 \tag{3}$$

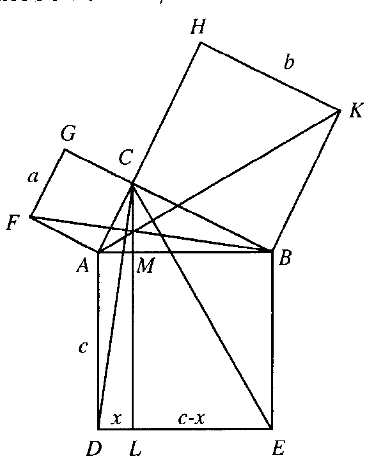
$$c^2 = a^2 + b^2. \tag{4}$$



In the above figure, the AREA of the large SQUARE is four times the AREA of one of the TRIANGLES plus the AREA of the interior SQUARE. From the figure, $d = b - a$, so

$$\begin{aligned} A &= 4(\tfrac{1}{2}ab) + d^2 = 2ab + (b-a)^2 = 2ab + b^2 - 2ab + a^2 \\ &= a^2 + b^2 = c^2. \end{aligned} \tag{5}$$

Perhaps the most famous proof of all times is Euclid's geometric proof. Euclid's proof used the figure below, which is sometimes known variously as the BRIDE'S CHAIR, PEACOCK'S TAIL, or WINDMILL.



Let ΔABC be a RIGHT TRIANGLE, $\square CAFG$, $\square CBKH$, and $\square ABED$ be squares, and $CL \| BD$. The TRIANGLES ΔFAB and ΔCAD are equivalent except for rotation, so

$$2\Delta FAB = 2\Delta CAD. \tag{6}$$

Shearing these TRIANGLES gives two more equivalent TRIANGLES

$$2\Delta CAD = \square ADLM. \tag{7}$$

Therefore,

$$\square ACGF = \square ADLM. \tag{8}$$

Similarly,

$$\square BC = 2\Delta ABK = 2\Delta BCE = \square BL \tag{9}$$

so

$$a^2 + b^2 = cx + c(c - x) = c^2. \tag{10}$$

Heron proved that AK, CL, and BF intersect in a point (Dunham 1990, pp. 48–53).

HERON'S FORMULA for the AREA of the TRIANGLE, contains the Pythagorean theorem implicitly. Using the form

$$K = \tfrac{1}{4}\sqrt{2a^2b^2 + 2a^2c^2 + ab^2c^2 - (a^4 + b^4 + c^4)} \tag{11}$$

and equating to the AREA

$$K = \tfrac{1}{2}ab \tag{12}$$

gives

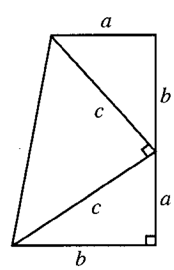
$$\tfrac{1}{4}a^2b^2 = 2a^2b^2 + 2a^2c^2 + ab^2c^2 - (a^4 + b^4 + c^4). \tag{13}$$

Rearranging and simplifying gives

$$a^2 + b^2 = c^2, \tag{14}$$

the Pythagorean theorem, where K is the AREA of a TRIANGLE with sides a, b, and c (Dunham 1990, pp. 128–129).

A novel proof using a TRAPEZOID was discovered by James Garfield (1876), later president of the United States, while serving in the House of Representatives (Pappas 1989, pp. 200–201; Bogomolny).



$$\begin{aligned} A_{\text{trapezoid}} &= \tfrac{1}{2}\sum [\text{bases}] \cdot [\text{altitude}] \\ &= \tfrac{1}{2}(a+b)(a+b) \\ &= \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2. \end{aligned} \tag{15}$$

Rearranging,

$$\tfrac{1}{2}(a^2 + 2ab + b^2) = ab + \tfrac{1}{2}c^2 \tag{16}$$

$$a^2 + 2ab + b^2 = 2ab + c^2 \tag{17}$$

$$a^2 + b^2 = c^2. \tag{18}$$

An algebraic proof (which would not have been accepted by the Greeks) uses the EULER FORMULA. Let the sides of a TRIANGLE be a, b, and c, and the PERPENDICULAR legs of RIGHT TRIANGLE be aligned along the real and imaginary axes. Then

$$a + bi = ce^{i\theta}. \tag{19}$$

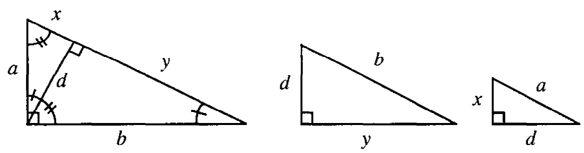
Taking the COMPLEX CONJUGATE gives

$$a - bi = ce^{-i\theta}. \tag{20}$$

Multiplying (19) by (20) gives

$$a^2 + b^2 = c^2. \tag{21}$$

Another algebraic proof proceeds by similarity.



It is a property of RIGHT TRIANGLES, such as the one shown in the above left figure, that the RIGHT TRIANGLE with sides x, a, and d (small triangle in the left figure; reproduced in the right figure) is similar to the RIGHT TRIANGLE with sides d, b, and y (large triangle in the left figure; reproduced in the middle figure), giving

$$\frac{x}{a} = \frac{a}{c} \qquad \frac{y}{b} = \frac{b}{c} \tag{22}$$

$$x = \frac{a^2}{c} \qquad y = \frac{b^2}{c}, \tag{23}$$

so

$$c \equiv x + y = \frac{a^2}{c} + \frac{b^2}{c} = \frac{a^2 + b^2}{c} \tag{24}$$

$$c^2 = a^2 + b^2. \tag{25}$$

Because this proof depends on proportions of potentially IRRATIONAL NUMBERS and cannot be translated directly into a GEOMETRIC CONSTRUCTION, it was not considered valid by Euclid.

see also BRIDE'S CHAIR, COSINES LAW, PEACOCK'S TAIL, PYTHAGORAS'S THEOREM, WINDMILL

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Pythagorean Triad

see PYTHAGOREAN TRIPLE

Pythagorean Triangle

see PYTHAGOREAN TRIPLE, RIGHT TRIANGLE

Pythagorean Triple

A Pythagorean triple is a TRIPLE of POSITIVE INTEGERS a, b, and c such that a RIGHT TRIANGLE exists with legs a, b and HYPOTENUSE c. By the PYTHAGOREAN THEOREM, this is equivalent to finding POSITIVE INTEGERS a, b, and c satisfying

$$a^2 + b^2 = c^2. \tag{1}$$

The smallest and best-known Pythagorean triple is $(a, b, c) = (3, 4, 5)$.

It is usual to consider only "reduced" (or "primitive") solutions in which a and b are RELATIVELY PRIME, since other solutions can be generated trivially from the primitive ones. For primitive solutions, one of a or b must be EVEN, and the other ODD (Shanks 1993, p. 141), with c always ODD. In addition, in every primitive Pythagorean triple, one side is always DIVISIBLE by 3 and one by 5.

Given a primitive triple (a_0, b_0, c_0), three new primitive triples are obtained from

$$(a_1, b_1, c_1) = (a_0, b_0, c_0)\mathsf{U} \quad (2)$$
$$(a_2, b_2, c_2) = (a_0, b_0, c_0)\mathsf{A} \quad (3)$$
$$(a_3, b_3, c_3) = (a_0, b_0, c_0)\mathsf{D}, \quad (4)$$

where

$$\mathsf{U} \equiv \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix} \quad (5)$$
$$\mathsf{A} \equiv \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \quad (6)$$
$$\mathsf{D} \equiv \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}. \quad (7)$$

Roberts (1977) proves that (a, b, c) is a primitive Pythagorean triple IFF

$$(a, b, c) = (3, 4, 5)\mathsf{M}, \quad (8)$$

where M is a FINITE PRODUCT of the MATRICES U, A, D. It therefore follows that every primitive Pythagorean triple must be a member of the INFINITE array

$$\begin{array}{ccc} & & \begin{matrix}(\ 7, & 24, & 25)\\(\ 55, & 48, & 73)\\(\ 45, & 28, & 53)\end{matrix} \\ & (\ 5,\ 12,\ 13) & \\ & & \begin{matrix}(\ 39, & 80, & 89)\\(119, & 120, & 169)\\(\ 77, & 36, & 85)\end{matrix} \\ (3,\ 4,\ 5) & (21,\ 20,\ 29) & \\ & & \begin{matrix}(\ 33, & 56, & 65)\\(\ 65, & 72, & 97)\\(\ 35, & 12, & 37)\end{matrix} \\ & (15,\ 8,\ 17) & \end{array} \quad (9)$$

For any Pythagorean triple, the PRODUCT of the two nonhypotenuse LEGS (i.e., the two smaller numbers) is always DIVISIBLE by 12, and the PRODUCT of all three sides is DIVISIBLE by 60. It is not known if there are two distinct triples having the same PRODUCT. The existence of two such triples corresponds to a NONZERO solution to the DIOPHANTINE EQUATION

$$xy(x^4 - y^4) = zw(z^4 - w^4) \quad (10)$$

(Guy 1994, p. 188).

Pythagoras and the Babylonians gave a formula for generating (not necessarily primitive) triples:

$$(2m, (m^2 - 1), (m^2 + 1)), \quad (11)$$

and Plato gave

$$(2m^2, (m^2 - 1)^2, (m^2 + 1)^2). \quad (12)$$

A general reduced solution (known to the early Greeks) is

$$(v^2 - u^2, 2uv, u^2 + v^2), \quad (13)$$

where u and v are RELATIVELY PRIME (Shanks 1993, p. 141). Let F_n be a FIBONACCI NUMBER. Then

$$(F_n F_{n+3}, 2F_{n+1}F_{n+2}, {F_{n+1}}^2 + {F_{n+2}}^2) \quad (14)$$

is also a Pythagorean triple.

For a Pythagorean triple (a, b, c),

$$P_3(a) + P_3(b) = P_3(c), \quad (15)$$

where P_3 is the PARTITION FUNCTION P (Garfunkel 1981, Honsberger 1985). Every three-term progression of SQUARES r^2, s^2, t^2 can be associated with a Pythagorean triple (X, Y, Z) by

$$r = X - Y \quad (16)$$
$$s = Z \quad (17)$$
$$t = X + Y \quad (18)$$

(Robertson 1996).

The AREA of a TRIANGLE corresponding to the Pythagorean triple $(u^2 - v^2, 2uv, u^2 + v^2)$ is

$$A = \tfrac{1}{2}(u^2 - v^2)(2uv) = uv(u^2 - v^2). \quad (19)$$

Fermat proved that a number of this form can never be a SQUARE NUMBER.

To find the number $L_p(s)$ of possible *primitive* TRIANGLES which may have a LEG (other than the HYPOTENUSE) of length s, factor s into the form

$$s = p_1^{\alpha_1} \cdots p_n^{\alpha_n}. \quad (20)$$

The number of such TRIANGLES is then

$$L_p(s) = \begin{cases} 0 & \text{for } s \equiv 2 \pmod 4 \\ 2^{n-1} & \text{otherwise,} \end{cases} \quad (21)$$

i.e., 0 for SINGLY EVEN s and 2 to the power one less than the number of distinct prime factors of s otherwise (Beiler 1966, pp. 115–116). The first few numbers for $s = 1, 2, \ldots$, are 0, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1, 2, 1, 0, 2, ... (Sloane's A024361). To find the number of ways $L(s)$ in which a number s can be the LEG (other than the HYPOTENUSE) of a *primitive or nonprimitive* RIGHT TRIANGLE, write the factorization of s as

$$s = 2^{a_0} {p_1}^{\alpha_1} \cdots {p_n}^{\alpha_n}. \quad (22)$$

Then

$$L(s) = \begin{cases} \frac{1}{2}[(2a_1+1)(2a_2+1)\cdots(2a_n+1)-1] \\ \quad \text{for } a_0 = 0 \\ \frac{1}{2}[(2a_0-1)(2a_1+1)(2a_2+1)\cdots(2a_n+1)-1] \\ \quad \text{for } a_0 \geq 2 \end{cases} \tag{23}$$

(Beiler 1966, p. 116). The first few numbers for $s = 1$, 2, ... are 0, 0, 1, 1, 1, 1, 1, 2, 2, 1, 1, 4, 1, ... (Sloane's A046079).

To find the number of ways $H_p(s)$ in which a number s can be the HYPOTENUSE of a *primitive* RIGHT TRIANGLE, write its factorization as

$$s = 2^{a_0}({p_1}^{a_1}\cdots{p_n}^{a_n})({q_1}^{b_1}\cdots{q_r}^{b_r}), \tag{24}$$

where the ps are of the form $4x-1$ and the qs are of the form $4x+1$. The number of possible *primitive* RIGHT TRIANGLES is then

$$H_p(s) = \begin{cases} 2^{r-1} & \text{for } n = 0 \text{ and } a_0 = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

The first few PRIMES of the form $4x+1$ are 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137, ... (Sloane's A002144), so the smallest side lengths which are the hypotenuses of 1, 2, 4, 8, 16, ... primitive right triangles are 5, 65, 1105, 32045, 1185665, 48612265, ... (Sloane's A006278). The number of possible *primitive or nonprimitive* RIGHT TRIANGLES having s as a HYPOTENUSE is

$$H(s) = \tfrac{1}{2}[(2b_1+1)(2b_2+1)\cdots(2b_r+1)-1] \tag{26}$$

(Beiler 1966, p. 117). The first few numbers for $s = 1$, 2, ... are 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, ... (Sloane's A046080).

Therefore, the total number of ways in which s may be either a LEG or HYPOTENUSE of a RIGHT TRIANGLE is given by

$$T(s) = L(s) + H(s). \tag{27}$$

The values for $s = 1$, 2, ... are 0, 0, 1, 1, 2, 1, 1, 2, 2, 2, 1, 4, 2, 1, 5, 3, ... (Sloane's A046081). The smallest numbers s which may be the sides of T general RIGHT TRIANGLES for $T = 1$, 2, ... are 3, 5, 16, 12, 15, 125, 24, 40, ... (Sloane's A006593; Beiler 1966, p. 114).

There are 50 Pythagorean triples with HYPOTENUSE less than 100, the first few of which, sorted by increasing c, are (3,4,5), (6,8,10), (5,12,13), (9,12,15), (8,15,17), (12,16,20), (15,20,25), (7,24,25), (10,24,26), (20,21,29), (18,24,30), (16,30,34), (21,28,35), ... (Sloane's A046083, A046084, and A046085). Of these, only 16 are primitive triplets with HYPOTENUSE less than 100: (3,4,5), (5,12,13), (8,15,17), (7,24,25), (20,21,29), (12,35,37), (9,40,41), (28,45,53), (11,60,61), (33,56,65), (16,63,65), (48,55,73), (36,77,85), (13,84,85), (39,80,89), and (65,72,97) (Sloane's A046086, A046087, and A046088). Of these 16 primitive triplets, seven are twin triplets (defined as triplets for which two members are consecutive integers). The first few twin triplets, sorted by increasing c, are (3,4,5), (5,12,13), (7,24,25), (20,21,29), (9,40,41), (11,60,61), (13,84,85), (15,112,113), ...

Let the number of triples with HYPOTENUSE less than N be denoted $\Delta(N)$, and the number of twin triplets with HYPOTENUSE less than N be denoted $\Delta_2(N)$. Then, as the following table suggests and Lehmer (1900) proved, the number of primitive solutions with HYPOTENUSE less than N satisfies

$$\lim_{N\to\infty} \frac{\Delta(N)}{N} = \frac{1}{2\pi} = 0.159155\ldots. \tag{28}$$

N	$\Delta(N)$	$\Delta(N)/N$	$\Delta_2(N)$
100	16	0.1600	7
500	80	0.1600	17
1000	158	0.1580	24
2000	319	0.1595	34
3000	477	0.1590	41
4000	639	0.1598	47
5000	792	0.1584	52
10000	1593	0.1593	74

Considering twin triplets in which the LEGS are consecutive, a closed form is available for the rth such pair. Consider the general reduced solution $(u^2-v^2, 2uv, u^2+v^2)$, then the requirement that the LEGS be consecutive integers is

$$u^2 - v^2 = 2uv \pm 1. \tag{29}$$

Rearranging gives

$$(u-v)^2 - 2v^2 = \pm 1. \tag{30}$$

Defining

$$u = x + y \tag{31}$$

$$v = y \tag{32}$$

then gives the PELL EQUATION

$$x^2 - 2y^2 = 1. \tag{33}$$

Solutions to the PELL EQUATION are given by

$$x = \frac{(1+\sqrt{2})^r + (1-\sqrt{2})^r}{2} \tag{34}$$

$$y = \frac{(1+\sqrt{2})^r - (1-\sqrt{2})^r}{2\sqrt{2}}, \tag{35}$$

so the lengths of the legs X_r and Y_r and the HYPOTENUSE Z_r are

$$\begin{aligned} X_r &= u^2 - v^2 = x^2 + 2xy \\ &= \frac{\sqrt{2}+1)^{2r+1} - (\sqrt{2}-1)^{2r+1}}{4} + \tfrac{1}{2}(-1)^r \quad (36) \\ Y_r &= 2uv = 2xy + 2y^2 \\ &= \frac{\sqrt{2}+1)^{2r+1} - (\sqrt{2}-1)^{2r+1}}{4} - \tfrac{1}{2}(-1)^r \quad (37) \\ Z_r &= u^2 + v^2 = x^2 + 2xy + 2y^2 \\ &= \frac{(\sqrt{2}+1)^{2r+1} + (\sqrt{2}-1)^{2r+1}}{2\sqrt{2}}. \quad (38) \end{aligned}$$

Denoting the length of the shortest LEG by A_r then gives

$$A_r = \frac{(\sqrt{2}+1)^{2r+1} - (\sqrt{2}-1)^{2r+1}}{4} - \frac{1}{2} \quad (39)$$

$$Z_r = \frac{(\sqrt{2}+1)^{2r+1} + (\sqrt{2}-1)^{2r+1}}{2\sqrt{2}} \quad (40)$$

(Beiler 1966, pp. 124–125 and 256–257), which cannot be solved exactly to give r as a function of Z_r. However, the approximate number of leg-leg twin triplets $\Delta_2^L(N) = r$ less than a given value of $Z_r = N$ can be found by noting that the second term in the DENOMINATOR of Z_r is a small number to the power $1 + 2r$ and can therefore be dropped, leaving

$$N = Z_r > \frac{(\sqrt{2}+1)^{1+2r}}{2\sqrt{2}} \quad (41)$$

$$N > (1+2r)\ln(\sqrt{2}+1) - \ln(2\sqrt{2}). \quad (42)$$

Solving for $r = \Delta_2^L(n)$ gives

$$\begin{aligned} \Delta_2^L(N) &< \frac{\ln N + \ln(2\sqrt{2}) - \ln(\sqrt{2}+1)}{2\ln(\sqrt{2}+1)} \\ &< \left\lfloor \frac{\ln N}{2\ln(1+\sqrt{2})} \right\rfloor \quad (43) \\ &\approx 0.567 \ln N. \quad (44) \end{aligned}$$

The first few LEG-LEG triplets are (3, 4, 5), (20, 21, 29), (119, 120, 169), (696, 697, 985), ... (Sloane's A046089, A046090, and A046091).

LEG-HYPOTENUSE twin triples $(a,b,c) = (v^2 - u^2, 2uv, u^2+v^2)$ occur whenever

$$u^2 + v^2 = 2uv + 1 \quad (45)$$

$$(u-v)^2 = 1, \quad (46)$$

that is to say when $v = u + 1$, in which case the HYPOTENUSE exceeds the EVEN LEG by unity and the twin triplet is given by $(1+2u, 2u(1+u), 1+2u(1+u))$. The number of leg-hypotenuse triplets with hypotenuse less than N is therefore given by

$$\Delta_2^L(N) = \left\lfloor \tfrac{1}{2}(\sqrt{2N-1} - 1) \right\rfloor, \quad (47)$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. The first few LEG-HYPOTENUSE triples are (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), ... (Sloane's A005408, A046092, and A046093).

The total number of twin triples $\Delta_2(N)$ less than N is therefore approximately given by

$$\begin{aligned} \Delta_2(N) &= \Delta_2^H(N) + \Delta_2^L(N) - 1 \quad (48) \\ &\approx \left\lfloor \tfrac{1}{2}\sqrt{2N-1} + 0.567\ln N - 1.5 \right\rfloor, \quad (49) \end{aligned}$$

where one has been subtracted to avoid double counting of the leg-leg-hypotenuse double-twin (3,4,5).

There is a general method for obtaining triplets of Pythagorean triangles with equal AREAS. Take the three sets of generators as

$$m_1 = r^2 + rs + s^2 \quad (50)$$

$$n_1 = r^2 - s^2 \quad (51)$$

$$m_2 = r^2 + rs + s^2 \quad (52)$$

$$n_2 = 2rs + s^2 \quad (53)$$

$$m_3 = r^2 + 2rs \quad (54)$$

$$n_3 = r^2 + rs + s^2. \quad (55)$$

Then the RIGHT TRIANGLE generated by each triple $({m_i}^2 - {n_i}^2, 2m_in_i, {m_i}^2 + {n_i}^2)$ has common AREA

$$A = rs(2r+s)(r+2s)(r+s)(r-s)(r^2+rs+s^2) \quad (56)$$

(Beiler 1966, pp. 126–127). The only EXTREMUM of this function occurs at $(r,s) = (0,0)$. Since $A(r,s) = 0$ for $r = s$, the smallest AREA shared by three nonprimitive RIGHT TRIANGLES is given by $(r,s) = (1,2)$, which results in an area of 840 and corresponds to the triplets (24, 70, 74), (40, 42, 58), and (15, 112, 113) (Beiler 1966, p. 126). The smallest AREA shared by three *primitive* RIGHT TRIANGLES is 13123110, corresponding to the triples (4485, 5852, 7373), (1380, 19019, 19069), and (3059, 8580, 9109) (Beiler 1966, p. 127).

One can also find quartets of RIGHT TRIANGLES with the same AREA. The QUARTET having smallest known area is (111, 6160, 6161), (231, 2960, 2969), (518, 1320, 1418), (280, 2442, 2458), with AREA 341,880 (Beiler 1966, p. 127). Guy (1994) gives additional information.

It is also possible to find sets of three and four Pythagorean triplets having the same PERIMETER (Beiler 1966, pp. 131–132). Lehmer (1900) showed that the number of primitive triples $N(p)$ with PERIMETER less than p is

$$\lim_{p\to\infty} N(p) = \frac{p\ln 2}{\pi^2} = 0.070230\ldots. \quad (57)$$

In 1643, Fermat challenged Mersenne to find a Pythagorean triplet whose HYPOTENUSE and SUM of the LEGS were SQUARES. Fermat found the smallest such solution:

$$X = 4565486027761 \quad (58)$$

$$Y = 1061652293520 \quad (59)$$

$$Z = 4687298610289, \quad (60)$$

with

$$Z = 2165017^2 \quad (61)$$

$$X + Y = 2372159^2. \quad (62)$$

A related problem is to determine if a specified INTEGER N can be the AREA of a RIGHT TRIANGLE with rational sides. 1, 2, 3, and 4 are not the AREAS of any RATIONAL-sided RIGHT TRIANGLES, but 5 is (3/2, 20/3, 41/6), as is 6 (3, 4, 5). The solution to the problem involves the ELLIPTIC CURVE

$$y^2 = x^3 - N^2x. \quad (63)$$

A solution (a, b, c) exists if (63) has a RATIONAL solution, in which case

$$x = \tfrac{1}{4}c^2 \quad (64)$$

$$y = \tfrac{1}{8}(a^2 - b^2)c \quad (65)$$

(Koblitz 1993). There is no known general method for determining if there is a solution for arbitrary N, but a technique devised by J. Tunnell in 1983 allows certain values to be ruled out (Cipra 1996).

see also HERONIAN TRIANGLE, PYTHAGOREAN QUADRUPLE, RIGHT TRIANGLE

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Q

$\mathbb{Q}$

The FIELD of RATIONAL NUMBERS.

see also $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{I}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$

q-Analog

A q-analog, also called a q-EXTENSION or q-GENERALIZATION, is a mathematical expression parameterized by a quantity q which generalizes a known expression and reduces to the known expression in the limit $q \to 1$. There are q-analogs of the FACTORIAL, BINOMIAL COEFFICIENT, DERIVATIVE, INTEGRAL, FIBONACCI NUMBERS, and so on. Koornwinder, Suslov, and Bustoz, have even managed some kind of q-Fourier analysis.

The q-analog of a mathematical object is generally called the "q-object", hence q-BINOMIAL COEFFICIENT, q-FACTORIAL, etc. There are generally several q-analogs if there is one, and there is sometimes even a multibasic analog with independent $q_1, q_2, \ldots$.

see also d-ANALOG, q-BETA FUNCTION, q-BINOMIAL COEFFICIENT, q-BINOMIAL THEOREM, q-COSINE, q-DERIVATIVE, q-FACTORIAL, q-GAMMA FUNCTION, q-SERIES, q-SINE, q-VANDERMONDE SUM

References

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q-Beta Function

A q-ANALOG of the BETA FUNCTION

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{q-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where $\Gamma(z)$ is a GAMMA FUNCTION, is given by

$$B_q(a,b) \equiv \int_0^1 t^{b-1}(qt;q)_{a-1}\,d(a,t) = \frac{\Gamma_q(b)\Gamma_q(a)}{\Gamma_q(a+b)},$$

where $\Gamma_q(a)$ is a q-GAMMA FUNCTION and $(a;q)_n$ is a q-SERIES coefficient (Andrews 1986, pp. 11–12).

see also q-FACTORIAL, q-GAMMA FUNCTION

References

Andrews, G. E. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* Providence, RI: Amer. Math. Soc., 1986.

q-Binomial Coefficient

A q-ANALOG for the BINOMIAL COEFFICIENT, also called the GAUSSIAN COEFFICIENT. It is given by

$$\binom{n}{m}_q \equiv \frac{(q)_n}{(q)_m(q)_{n-m}} = \prod_{i=0}^{j-1} \frac{1-q^{k-i}}{1-q^{i+1}}, \tag{1}$$

where

$$(q)_k \equiv \prod_{m=1}^{\infty} \frac{1-q^m}{1-q^{k+m}}. \tag{2}$$

For example, the first few q-binomial coefficients are

$$\binom{2}{1}_q = \frac{1-q^2}{1-q} = 1+q \tag{3}$$

$$\binom{3}{1}_q = \binom{3}{2}_q = \frac{1-q^3}{1-q} = 1+q+q^2 \tag{4}$$

$$\binom{4}{1}_q = \binom{4}{3}_q = \frac{1-q^4}{1-q} = 1+q+q^2+q^3 \tag{5}$$

$$\binom{4}{2}_q = \frac{(1-q^3)(1-q^4)}{(1-q)(1-q^2)} = (1+q)(1+q+q^2). \tag{6}$$

From the definition, it follows that

$$\binom{n}{1}_q = \binom{n}{n-1}_q = \sum_{i=0}^{n-1} q^n. \tag{7}$$

In the LIMIT $q \to 1$, the q-binomial coefficient collapses to the usual BINOMIAL COEFFICIENT.

see also CAUCHY BINOMIAL THEOREM, GAUSSIAN POLYNOMIAL

q-Binomial Theorem

The q-ANALOG of the BINOMIAL THEOREM

$$(1-z)^n = 1 - nz + \frac{n(n-1)}{1\cdot 2}z^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}z^3 + \ldots$$

is given by

$$\left(1-\frac{z}{q^n}\right)\left(1-\frac{z}{q^{n-1}}\right)\cdots\left(1-\frac{z}{q}\right)$$
$$= 1 - \frac{1-q^n}{1-q}\frac{z}{q^n} + \frac{1-q^n}{1-q}\frac{1-q^{n-1}}{1-q^2}\frac{z^2}{q^{n+(n-1)}} - \ldots \pm \frac{z^n}{q^{n(n+1)/2}}.$$

Written as a q-SERIES, the identity becomes

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty},$$

where

$$(a;q)_n = \prod_{m=0}^{\infty} \frac{(1-aq^m)}{(1-aq^{m+n})}$$

(Heine 1847, p. 303; Andrews 1986). The CAUCHY BINOMIAL THEOREM is a special case of this general theorem.

see also BINOMIAL SERIES, BINOMIAL THEOREM, CAUCHY BINOMIAL THEOREM, HEINE HYPERGEOMETRIC SERIES, RAMANUJAN PSI SUM

References

Andrews, G. E. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* Providence, RI: Amer. Math. Soc., p. 10, 1986.

Heine, E. "Untersuchungen über die Reihe $1+\frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}\cdot x+\frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}\cdot x^2+\ldots$" *J. reine angew. Math.* **34**, 285–328, 1847.

q-Cosine

The q-ANALOG of the COSINE function, as advocated by R. W. Gosper, is defined by

$$\cos_q(z,q) = \frac{\vartheta_2(z,p)}{\vartheta_2(0,p)},$$

where $\vartheta_2(z,p)$ is a THETA FUNCTION and p is defined via

$$(\ln p)(\ln q) = \pi^2.$$

This is a period 2π, EVEN FUNCTION of unit amplitude with double and triple angle formulas and addition formulas which are analogous to ordinary SINE and COSINE. For example,

$$\cos_q(2z,q) = \cos_q{}^2(z,q^2) - \sin_q{}^2(z,q^2),$$

where $\sin_q(z,a)$ is the q-SINE, and π_q is q-PI. The q-cosine also satisfies

$$\cos_q(\pi a) = \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+a)^2}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}}.$$

see also q-FACTORIAL, q-SINE

References

Gosper, R. W. "Experiments and Discoveries in q-Trigonometry." Unpublished manuscript.

q-Derivative

The q-ANALOG of the DERIVATIVE, defined by

$$\left(\frac{d}{dx}\right)_q f(x) = \frac{f(x)-f(qx)}{x-qx}.$$

For example,

$$\left(\frac{d}{dx}\right)_q \sin x = \frac{\sin x - \sin(qx)}{x-qx}$$

$$\left(\frac{d}{dx}\right)_q \ln x = \frac{\ln x - \ln(qx)}{x-qx} = \frac{\ln\left(\frac{1}{q}\right)}{(1-q)x}$$

$$\left(\frac{d}{dx}\right)_q x^2 = \frac{x^2-q^2x^2}{x-qx} = (1+q)x$$

$$\left(\frac{d}{dx}\right)_q x^3 = \frac{x^3-q^3x^3}{x-qx} = (1+q+q^2)x^2.$$

In the LIMIT $q \to 1$, the q-derivative reduces to the usual DERIVATIVE.

see also DERIVATIVE

q-Dimension

$$D_q \equiv \frac{1}{1-q}\lim_{\epsilon\to 0}\frac{\ln I(q,\epsilon)}{\ln\left(\frac{1}{\epsilon}\right)}, \qquad (1)$$

where

$$I(q,\epsilon) \equiv \sum_{i=1}^{N} {\mu_i}^q, \qquad (2)$$

ϵ is the box size, and μ_i is the NATURAL MEASURE. If $q_1 > q_2$, then

$$D_{q_1} \leq D_{q_2}. \qquad (3)$$

The CAPACITY DIMENSION (a.k.a. BOX COUNTING DIMENSION) is given by $q=0$,

$$D_0 = \frac{1}{1-0}\lim_{\epsilon\to 0}\frac{\ln\left(\sum_{i=1}^{N(\epsilon)} 1\right)}{-\ln\epsilon} = -\lim_{\epsilon\to 0}\frac{\ln[N(\epsilon)]}{\ln\epsilon}. \qquad (4)$$

If all μ_is are equal, then the CAPACITY DIMENSION is obtained for any q. The INFORMATION DIMENSION is defined by

$$D_1 = \lim_{q\to 1} D_q = \lim_{q\to 1}\frac{\lim_{\epsilon\to 0}\frac{\ln\left[\sum_{i=1}^{N(\epsilon)} {\mu_i}^q\right]}{-\ln\epsilon}}{1-q}$$

$$= \lim_{\epsilon\to 0}\lim_{q\to 1}\frac{\ln\left(\sum_{i=1}^{N(\epsilon)} {\mu_i}^q\right)}{\ln\epsilon(q-1)}. \qquad (5)$$

But

$$\lim_{q\to 1}\ln\left(\sum_{i=1}^{N(\epsilon)} {\mu_i}^q\right) = \ln\left(\sum_{i=1}^{N(\epsilon)} \mu_i\right) = \ln 1 = 0, \qquad (6)$$

so use L'HOSPITAL'S RULE

$$D_1 = \lim_{\epsilon\to 0}\left(\frac{1}{\ln\epsilon}\lim_{q\to 1}\frac{\sum q{\mu_i}^{q-1}}{\sum {\mu_i}^q}\right). \qquad (7)$$

Therefore,

$$D_1 = \lim_{\epsilon\to 0}\frac{\sum_{i=1}^{N(\epsilon)}\mu_i\ln\mu_i}{\ln\epsilon}. \qquad (8)$$

D_2 is called the CORRELATION DIMENSION. The q-dimensions satisfy

$$D_{q+1} \leq D_q. \qquad (9)$$

see also FRACTAL DIMENSION

Q.E.D.

An abbreviation for the Latin phrase "quod erat demonstrandum" ("that which was to be demonstrated"), a NOTATION which is often placed at the end of a mathematical proof to indicate its completion.

q-Extension

see q-ANALOG

q-Factorial

The q-ANALOG of the FACTORIAL (by analogy with the q-GAMMA FUNCTION). For a an integer, the q-factorial is defined by

$$\operatorname{faq}(a,q) = 1(1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{a-1}).$$

A reflection formula analogous to the GAMMA FUNCTION reflection formula is given by

$$\begin{aligned}\cos_q(\pi a) &= \sin_q[\pi(\tfrac{1}{2}-a)] \\ &= \frac{\pi_q q^{(a-1/2)(a+1/2)}}{\operatorname{faq}(a-\frac{1}{2},q^2)\operatorname{faq}(-(a+\frac{1}{2}),q^2)},\end{aligned}$$

where $\cos_q(z)$ is the q-COSINE, $\sin_q(z)$ is the q-SINE, and π_q is q-PI.

see also q-BETA FUNCTION, q-COSINE, q-GAMMA FUNCTION, q-PI, q-SINE

References

Gosper, R. W. "Experiments and Discoveries in q-Trigonometry." Unpublished manuscript.

Q-Function

Let

$$q = e^{-\pi K'/K} = e^{-i\pi\tau}, \tag{1}$$

then

$$Q_0 \equiv \prod_{n=1}^{\infty}(1-q^{2n}) \tag{2}$$

$$Q_1 \equiv \prod_{n=1}^{\infty}(1+q^{2n}) \tag{3}$$

$$Q_2 \equiv \prod_{n=1}^{\infty}(1+q^{2n-1}) \tag{4}$$

$$Q_3 \equiv \prod_{n=1}^{\infty}(1-q^{2n-1}). \tag{5}$$

The Q-functions are sometimes written using a lowercase q instead of a capital Q. The Q-functions also satisfy the identities

$$Q_0Q_1 = Q_0(q^2) \tag{6}$$

$$Q_0Q_3 = Q_0(q^{1/2}) \tag{7}$$

$$Q_2Q_3 = Q_3(q^2) \tag{8}$$

$$Q_1Q_2 = Q_1(q^{1/2}). \tag{9}$$

see also JACOBI IDENTITIES, q-SERIES

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Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, pp. 469–473 and 488–489, 1990.

q-Gamma Function

A q-ANALOG of the GAMMA FUNCTION defined by

$$\Gamma_q(x,q) \equiv \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \tag{1}$$

where $(x,q)_\infty$ is a q-SERIES. The q-gamma function satisfies

$$\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x) \tag{2}$$

(Andrews 1986).

A curious identity for the functional equation

$$\begin{aligned}f(a-b)f(a-c)f(a-d)f(a-e) - f(b)f(c)f(d)f(e) \\ = q^b f(a)f(a-b-c)f(a-b-d)f(a-b-e),\end{aligned} \tag{3}$$

where

$$b+c+d+e = 2a \tag{4}$$

is given by

$$f(\alpha) = \begin{cases}\sin(k\alpha) & \text{for } q=1 \\ \frac{1}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)} & \text{for } 0<q<1,\end{cases} \tag{5}$$

for any k.

see also q-BETA FUNCTION, q-FACTORIAL

References

Andrews, G. E. "W. Gosper's Proof that $\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$." Appendix A in *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* Providence, RI: Amer. Math. Soc., p. 11 and 109, 1986.

Wenchang, C. Problem 10226 and Solution. "A q-Trigonometric Identity." *Amer. Math. Monthly* **103**, 175–177, 1996.

q-Generalization

see q-ANALOG

q-Hypergeometric Series

see HEINE HYPERGEOMETRIC SERIES

Q-Matrix

see FIBONACCI Q-MATRIX

Q-Number

see HOFSTADTER'S Q-SEQUENCE

q-Pi

The q-ANALOG of PI π_q can be defined by taking $a = 0$ in the q-FACTORIAL

$$\mathrm{faq}(a,q) = 1(1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{a-1}),$$

giving

$$1 = \sin_q(\tfrac{1}{2}\pi) = \frac{\pi_q}{\mathrm{faq}^2(-\frac{1}{2},q^2)q^{1/4}},$$

where $\sin_q(z)$ is the q-SINE. Gosper has developed an iterative algorithm for computing π based on the algebraic RECURRENCE RELATION

$$\frac{4\pi_{q^4}}{q^4+1} = \frac{(q^2+1)^2{\pi_q}^2}{\pi_{q^2}} - \frac{(q^4+1){\pi_{q^2}}^2}{\pi_{q^4}}.$$

Q-Polynomial

see BLM/HO POLYNOMIAL

q-Product

see Q-FUNCTION

q-Series

A SERIES involving coefficients of the form

$$(a)_n \equiv (a;q)_n \equiv \prod_{k=0}^{\infty} \frac{(1-aq^k)}{(1-aq^{k+n})} \tag{1}$$

$$= \prod_{k=0}^{n-1}(1-aq^k) \tag{2}$$

(Andrews 1986). The symbols

$$[n] \equiv 1+q+q^2+\ldots+q^{n-1} \tag{3}$$

$$[n]! \equiv [n][n-1]\cdots[1] \tag{4}$$

are sometimes also used when discussing q-series.

There are a great many beautiful identities involving q-series, some of which follow directly by taking the q-ANALOG of standard combinatorial identities, e.g., the q-BINOMIAL THEOREM

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{5}$$

($|z| < 1$, $|q| < 1$; Andrews 1986, p. 10) and q-VANDERMONDE SUM

$${}_2\phi_1(a,q^{-n};c;q,q) = \frac{a^n(c/a,q)_n}{(c;q)_n}, \tag{6}$$

where ${}_2\phi_1(a,b;c;q,z)$ is a HEINE HYPERGEOMETRIC SERIES. Other q-series identities, e.g., the JACOBI IDENTITIES, ROGERS-RAMANUJAN IDENTITIES, and HEINE HYPERGEOMETRIC IDENTITY

$${}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1(c/b,a;az;q,b), \tag{7}$$

seem to arise out of the blue.

see also BORWEIN CONJECTURES, FINE'S EQUATION, GAUSSIAN COEFFICIENT, HEINE HYPERGEOMETRIC SERIES, JACKSON'S IDENTITY, JACOBI IDENTITIES, MOCK THETA FUNCTION, q-ANALOG, q-BINOMIAL THEOREM, q-COSINE, q-FACTORIAL, Q-FUNCTION, q-GAMMA FUNCTION, q-SINE, RAMANUJAN PSI SUM, RAMANUJAN THETA FUNCTIONS, ROGERS-RAMANUJAN IDENTITIES

References

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Berndt, B. C. "q-Series." Ch. 27 in *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 261–286, 1994.

Gasper, G. and Rahman, M. *Basic Hypergeometric Series.* Cambridge, England: Cambridge University Press, 1990.

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Q-Signature

see SIGNATURE (RECURRENCE RELATION)

q-Sine

The q-ANALOG of the SINE function, as advocated by R. W. Gosper, is defined by

$$\sin_q(z,q) = \frac{\vartheta_1(z,p)}{\vartheta_1(\frac{1}{2}\pi,p)},$$

where $\vartheta_1(z,p)$ is a THETA FUNCTION and p is defined via

$$(\ln p)(\ln q) = \pi^2.$$

This is a period 2π, ODD FUNCTION of unit amplitude with double and triple angle formulas and addition formulas which are analogous to ordinary SINE and COSINE. For example,

$$\sin_q(2z,q) = (q+1)\frac{\pi_q}{p_{q^2}}\cos_q(z,q^2)\sin_q(z,q^2),$$

where $\cos_q(z,a)$ is the q-COSINE, and π_q is q-PI.

see also q-COSINE, q-FACTORIAL

References

Gosper, R. W. "Experiments and Discoveries in q-Trigonometry." Unpublished manuscript.

q-Vandermonde Sum

$$ {}_2\phi_1(a, q^{-n}; c; q, q) = \frac{a^n (c/a, q)_n}{(c; q)_n}, $$

where ${}_2\phi_1(a, b; c; q, z)$ is a HEINE HYPERGEOMETRIC SERIES.

see also CHU-VANDERMONDE IDENTITY, HEINE HYPERGEOMETRIC SERIES

References
Andrews, G. E. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* Providence, RI: Amer. Math. Soc., pp. 15–16, 1986.

QR Decomposition

Given a MATRIX A, its QR-decomposition is of the form

$$ \mathsf{A} = \mathsf{QR}, $$

where R is an upper TRIANGULAR MATRIX and Q is an ORTHOGONAL MATRIX, i.e., one satisfying

$$ \mathsf{Q}^{\mathrm{T}}\mathsf{Q} = \mathsf{I}, $$

where I is the IDENTITY MATRIX. This matrix decomposition can be used to solve linear systems of equations.

see also CHOLESKY DECOMPOSITION, LU DECOMPOSITION, SINGULAR VALUE DECOMPOSITION

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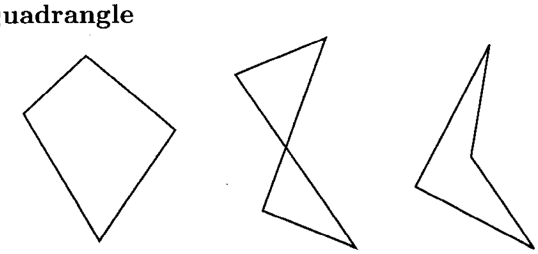
Stewart, G. W. "A Parallel Implementation of the QR Algorithm." *Parallel Comput.* **5**, 187–196, 1987. `ftp://thales.cs.umd.edu/pub/reports/piqra.ps`.

Quadrable

A plane figure for which QUADRATURE is possible is said to be quadrable.

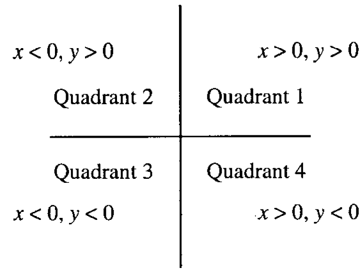
Quadrangle



A plane figure consisting of four points, each of which is joined to two other points by a LINE SEGMENT (where the line segments may intersect). A quadrangle may therefore be CONCAVE or CONVEX; if it is CONVEX, it is called a QUADRILATERAL.

see also COMPLETE QUADRANGLE, CYCLIC QUADRANGLE, QUADRILATERAL

Quadrant



One of the four regions of the PLANE defined by the four possible combinations of SIGNS $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$ for (x, y).

see also OCTANT, x-AXIS, y-AXIS

References
Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 73, 1996.

Quadratfrei

see SQUAREFREE

Quadratic Congruence

A CONGRUENCE of the form

$$ ax^2 + bx + c \equiv 0 \pmod{m}, $$

where a, b, and c are INTEGERS. A general quadratic congruence can be reduced to the congruence

$$ x^2 \equiv q \pmod{p} $$

and can be solved using EXCLUDENTS, although solution of the general polynomial congruence

$$ a_m x^m + \ldots + a_2 x^2 + a_1 x + a_0 \equiv 0 \pmod{n} $$

is intractable.

see also CONGRUENCE, EXCLUDENT, LINEAR CONGRUENCE

Quadratic Curve

The general 2-variable quadratic equation can be written

$$ ax^2 + 2bxy + cy^2 + 2dx + 2fy + g = 0. \qquad (1) $$

Define the following quantities:

$$ \Delta = \begin{vmatrix} a & b & d \\ b & c & f \\ d & f & g \end{vmatrix} \qquad (2) $$

$$ J = \begin{vmatrix} a & b \\ b & c \end{vmatrix} \qquad (3) $$

$$ I = a + c \qquad (4) $$

$$ K = \begin{vmatrix} a & d \\ d & g \end{vmatrix} + \begin{vmatrix} c & f \\ f & g \end{vmatrix}. \qquad (5) $$

Then the quadratics are classified into the types summarized in the following table (Beyer 1987). The real (nondegenerate) quadratics (the ELLIPSE, HYPERBOLA, and PARABOLA) correspond to the curves which can be created by the intersection of a PLANE with a (two-NAPPES) CONE, and are therefore known as CONIC SECTIONS.

Curve	Δ	J	Δ/I	K
coincident lines	0	0		0
ellipse (imaginary)	$\neq 0$	>0	>0	
ellipse (real)	$\neq 0$	>0	<0	
hyperbola	$\neq 0$	<0		
intersecting lines (imaginary)	0	>0		
intersecting lines (real)	0	<0		
parabola	$\neq 0$	0		
parallel lines (imaginary)	0	0		>0
parallel lines (real)	0	0		<0

It is always possible to eliminate the xy cross term by a suitable ROTATION of the axes. To see this, consider rotation by an arbitrary angle θ. The ROTATION MATRIX is

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \begin{bmatrix} x'\cos\theta + y'\sin\theta \\ -x'\sin\theta + y'\cos\theta \end{bmatrix}, \end{aligned} \tag{6}$$

so

$$x = x'\cos\theta + y'\sin\theta \tag{7}$$

$$y = -x'\sin\theta + y'\cos\theta \tag{8}$$

$$\begin{aligned} xy = &-x'^2\cos\theta\sin\theta + x'y'(\cos^2\theta - \sin^2\theta) \\ &+ y'^2\cos\theta\sin\theta \end{aligned} \tag{9}$$

$$x^2 = x'^2\cos^2\theta + 2x'y'\cos\theta\sin\theta + y'^2\sin^2\theta \tag{10}$$

$$y^2 = -x'^2\sin^2\theta - 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta. \tag{11}$$

Plugging these into (1) gives

$$\begin{aligned} &a(x'^2\cos^2\theta + 2x'y'\cos\theta + y'^2\sin^2\theta) \\ &\quad +2b(x'\cos\theta + y'\sin\theta)(-x'\sin\theta + y'\cos\theta) \\ &\quad +c(x'^2\sin^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\cos^2\theta) \\ &\quad +2d(x'\cos\theta + y'\sin\theta) \\ &\quad +2f(-x'\sin\theta + y'\cos\theta) + g = 0. \end{aligned} \tag{12}$$

Rewriting,

$$\begin{aligned} &a(x'^2\cos^2\theta + 2x'y'\cos\theta + y'^2\sin^2\theta) \\ &\quad +2b(-x^2\cos^2\theta\sin\theta - xy\sin^2\theta + xy\cos^2\theta + y^2\cos\theta\sin\theta) \\ &\quad +c(x'^2\sin^2\theta - 2x'y'\cos\theta\sin\theta + y'^2\cos^2\theta) \\ &\quad +2d(x'\cos\theta + y'\sin\theta) \\ &\quad +2f(-x'\sin\theta + y'\cos\theta) + g = 0. \end{aligned} \tag{13}$$

Grouping terms,

$$\begin{aligned} &x'^2(a\cos^2\theta + c\sin^2\theta - 2b\cos\theta\sin\theta) \\ &\quad +x'y'[2a\cos\theta\sin\theta - 2c\sin\theta\cos\theta + 2b(\cos^2\theta - \sin^2\theta)] \\ &\quad +y'^2(a\sin^2\theta + c\cos^2\theta + 2b\cos\theta\sin\theta) \\ &\quad +x'(2d\cos\theta - 2f\sin\theta) + y'(-2d\sin\theta + 2f\cos\theta) \\ &\quad +g = 0. \end{aligned} \tag{14}$$

Comparing the COEFFICIENTS with (1) gives an equation of the form

$$a'x'^2 + 2b'x'y' + c'y'^2 + 2d'x' + 2f'y' + g' = 0, \tag{15}$$

where the new COEFFICIENTS are

$$a' = a\cos^2\theta - 2b\cos\theta\sin\theta + c\sin^2\theta \tag{16}$$

$$b' = b(\cos^2\theta - \sin^2\theta) + (a-c)\sin\theta\cos\theta \tag{17}$$

$$c' = a\sin^2\theta + 2b\sin\theta\cos\theta + c\cos^2\theta \tag{18}$$

$$d' = d\cos\theta - f\sin\theta \tag{19}$$

$$f' = -d\sin\theta + f\cos\theta \tag{20}$$

$$g' = g. \tag{21}$$

The cross term $2b'x'y'$ can therefore be made to vanish by setting

$$\begin{aligned} b' &= b(\cos^2\theta - \sin^2\theta) - (c-a)\sin\theta\cos\theta \\ &= b\cos(2\theta) - \tfrac{1}{2}(c-a)\sin(2\theta) = 0. \end{aligned} \tag{22}$$

For b' to be zero, it must be true that

$$\cot(2\theta) = \frac{c-a}{2b} \equiv K. \tag{23}$$

The other components are then given with the aid of the identity

$$\cos[\cot^{-1}(x)] = \frac{x}{\sqrt{1+x^2}} \tag{24}$$

by defining

$$L \equiv \frac{K}{\sqrt{1+K^2}}, \tag{25}$$

so

$$\sin\theta = \sqrt{\frac{1-L}{2}} \tag{26}$$

$$\cos\theta = \sqrt{\frac{1+L}{2}}. \tag{27}$$

Rotating by an angle

$$\theta = \tfrac{1}{2}\cot^{-1}\left(\frac{c-a}{2b}\right) \tag{28}$$

therefore transforms (1) into

$$a'x'^2 + c'y'^2 + 2d'x' + 2f'y' + g' = 0. \tag{29}$$

COMPLETING THE SQUARE,

$$a'\left(x'^2+\frac{2d'}{a'}x\right)+c'\left(y'^2+\frac{2f'}{c'}y'\right)+g'=0 \quad (30)$$

$$a'\left(x'+\frac{d'}{a'}\right)^2+c'\left(y'+\frac{f'}{c'}\right)^2=-g'+\frac{d'^2}{a'}+\frac{f'^2}{c'}. \quad (31)$$

Defining $x''\equiv x'+d'/a'$, $y''\equiv y'+f'/c'$, and $g''\equiv -g'+d'^2/a'+f'^2/c'$ gives

$$a'x''^2+c'y''^2=g''. \quad (32)$$

If $g''\neq 0$, then divide both sides by g''. Defining $a''\equiv a'/g''$ and $c''\equiv c'/g''$ then gives

$$a''x''^2+c''y''^2=1. \quad (33)$$

Therefore, in an appropriate coordinate system, the general CONIC SECTION can be written (dropping the primes) as

$$\begin{cases} ax^2+cy^2=1 & a,c,g\neq 0 \\ ax^2+cy^2=0 & a,c\neq 0,\ g=0. \end{cases} \quad (34)$$

Consider an equation of the form $ax^2+2bxy+cy^2=1$ where $b\neq 0$. Re-express this using t_1 and t_2 in the form

$$ax^2+2bxy+cy^2=t_1x'^2+t_2y'^2. \quad (35)$$

Therefore, rotate the COORDINATE SYSTEM

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad (36)$$

so

$$\begin{aligned} ax^2+2bxy+cy^2 &= t_1x'^2+t_2y'^2 \\ &= t_1(x^2\cos^2\theta+2xy\cos\theta\sin\theta+y^2\sin^2\theta) \\ &\quad +t_2(x^2\sin^2\theta-2xy\sin\theta\cos\theta+y^2\cos^2\theta) \\ &= x^2(t_1\cos^2\theta+t_2\sin^2\theta)+2xy\cos\theta\sin\theta(t_1-t_2) \\ &\quad +y^2(t_1\sin^2\theta+t_2\cos^2\theta) \end{aligned} \quad (37)$$

and

$$a=t_1\cos^2\theta+t_2\sin^2\theta \quad (38)$$

$$b=(t_1-t_2)\cos\theta\sin\theta=\tfrac{1}{2}(t_1-t_2)\sin(2\theta) \quad (39)$$

$$c=t_1\sin^2\theta+t_2\cos^2\theta. \quad (40)$$

Therefore,

$$\begin{aligned} a+c &= (t_1\cos^2\theta+t_2\sin^2\theta)+(t_1\sin^2\theta+t_2\cos^2\theta) \\ &= t_1+t_2 \end{aligned} \quad (41)$$

$$\begin{aligned} a-c &= t_1\cos^2\theta+t_2\sin^2\theta-t_1\sin^2\theta+t_2\cos^2\theta \\ &= (t_1-t_2)(\cos^2\theta-\sin^2\theta)=(t_1-t_2)\cos(2\theta). \end{aligned} \quad (42)$$

From (41) and (42),

$$\frac{a-c}{b}=\frac{(t_1-t_2)\cos(2\theta)}{\frac{1}{2}(t_1-t_2)\sin(2\theta)}=2\cot(2\theta), \quad (43)$$

the same angle as before. But

$$\begin{aligned} \cos(2\theta) &= \cos\left[\cot^{-1}\left(\frac{a-c}{2b}\right)\right] \\ &= \cos\left[\tan^{-1}\left(\frac{2b}{a-c}\right)\right] \\ &= \frac{1}{\sqrt{1+\left(\frac{2b}{a-c}\right)^2}}, \end{aligned} \quad (44)$$

so

$$a-c=\frac{t_1-t_2}{\sqrt{1+\left(\frac{2b}{a-c}\right)^2}}. \quad (45)$$

Rewriting and copying (41),

$$\begin{aligned} t_1-t_2 &= (a-c)\sqrt{1+\left(\frac{2b}{a-c}\right)^2} \\ &= \sqrt{(a-c)^2+4b^2} \end{aligned} \quad (46)$$

$$t_1+t_2=a+c. \quad (47)$$

Adding (46) and (47) gives

$$t_1=\tfrac{1}{2}[a+c+\sqrt{(a-c)^2+4b^2}\,] \quad (48)$$

$$t_2=a+c-t_1=\tfrac{1}{2}[a+c-\sqrt{(a-c)^2+4b^2}\,]. \quad (49)$$

Note that these ROOTS can also be found from

$$(t-t_1)(t-t_2)=t^2-t(t_1+t_2)+t_1t_2=0 \quad (50)$$

$$\begin{aligned} t^2-&t(a+c)+\tfrac{1}{4}\{(a+c)^2-[(a-c)^2+4b^2]\} \\ &= t^2-t(a+c) \\ &\quad +\tfrac{1}{4}[a^2+2ac+c^2-a^2+2ac-c^2-4b^2] \\ &= t^2-t(a+c)+(ac-b^2)=(a-t)(c-t)-b^2 \\ &= \begin{vmatrix} a-t & b \\ b & c-t \end{vmatrix}=(a-t)(c-t)-b^2=0. \end{aligned} \quad (51)$$

The original problem is therefore equivalent to looking for a solution to

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=t\begin{bmatrix} x \\ y \end{bmatrix} \quad (52)$$

$$\begin{bmatrix} ax & bx \\ by & cy \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=t\begin{bmatrix} x^2 \\ y^2 \end{bmatrix}, \quad (53)$$

which gives the simultaneous equations

$$\begin{cases} ax^2+bxy=tx^2 \\ bxy+cy^2=ty^2. \end{cases} \quad (54)$$

Let $\mathbf{X}$ be any point (x, y) with old coordinates and (x', y') be its new coordinates. Then

$$ax^2 + 2bxy + cy^2 = t_+ x'^2 + t_- y'^2 = 1 \tag{55}$$

and

$$x' = \hat{\mathbf{X}}_+ \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{56}$$

$$y' = \hat{\mathbf{X}}_- \cdot \begin{bmatrix} x \\ y \end{bmatrix}. \tag{57}$$

If t_+ and t_- are both > 0, the curve is an ELLIPSE. If t_+ and t_- are both < 0, the curve is empty. If t_+ and t_- have opposite SIGNS, the curve is a HYPERBOLA. If either is 0, the curve is a PARABOLA.

To find the general form of a quadratic curve in POLAR COORDINATES (as given, for example, in Moulton 1970), plug $x = r\cos\theta$ and $y = r\sin\theta$ into (1) to obtain

$$ar^2\cos^2\theta + 2br^2\cos\theta\sin\theta + cr^2\sin^2\theta \\ +2dr\cos\theta + 2fr\sin\theta + g = 0 \tag{58}$$

$$(a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta) \\ +\frac{2}{r}(d\cos\theta + f\sin\theta) + \frac{g}{r^2} = 0. \tag{59}$$

Define $u \equiv 1/r$. For $g \neq 0$,we can divide through by $2g$,

$$\tfrac{1}{2}u^2 + \frac{1}{g}(d\cos\theta + f\sin\theta)u \\ +\frac{1}{2g}(a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta) = 0. \tag{60}$$

Applying the QUADRATIC FORMULA gives

$$u = -\frac{d}{g}\cos\theta - \frac{f}{g}\sin\theta \pm \sqrt{R}, \tag{61}$$

where

$$\begin{aligned} R &\equiv \frac{(d\cos\theta + f\sin\theta)^2}{g^2} \\ &\quad - 4\left(\frac{1}{2}\right)\left(\frac{1}{2g}\right)(a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta) \\ &= \frac{d^2}{g^2}\cos^2\theta + \frac{2df}{g^2}\cos\theta\sin\theta + \frac{f^2}{g^2}\sin^2\theta \\ &\quad - \frac{1}{g}(a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta). \end{aligned} \tag{62}$$

Using the trigonometric identities

$$\sin^2\theta = 1 - \cos^2\theta \tag{63}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta, \tag{64}$$

it follows that

$$\begin{aligned} R &= \left(\frac{d^2}{g^2} - \frac{a}{g} - \frac{f^2}{g^2} + \frac{c}{g}\right)\cos^2\theta \\ &\quad + \left(\frac{df}{g^2} - \frac{b}{g}\right)\sin(2\theta) + \left(\frac{f^2}{g^2} - \frac{c}{g}\right) \\ &= \tfrac{1}{2}[1 + \cos(2\theta)]\frac{d^2 - ag - f^2 + cg}{g^2} \\ &\quad + \sin(2\theta)\left(\frac{df - bg}{g^2}\right) + \frac{f^2 - cg}{g^2} \\ &= \frac{d^2 - ag - f^2 + cg}{2g^2}\cos(2\theta) + \frac{df - bg}{g^2}\sin(2\theta) \\ &\quad + \frac{d^2 - ag - f^2 + cg + 2f^2 - 2cg}{2g^2}. \end{aligned} \tag{65}$$

Defining

$$A \equiv -\frac{f}{g} \tag{66}$$

$$B \equiv -\frac{d}{g} \tag{67}$$

$$C \equiv \frac{df - bg}{g^2} \tag{68}$$

$$D \equiv \frac{d^2 - f^2 + cg - ag}{2g^2} \tag{69}$$

$$E \equiv \frac{d^2 + f^2 - ag - cg}{2g^2} \tag{70}$$

then gives the equation

$$u \equiv \frac{1}{r} = A\sin\theta + B\cos\theta \pm \sqrt{C\sin(2\theta) + D\cos(2\theta) + E} \tag{71}$$

(Moulton 1970). If $g = 0$, then (59) becomes instead

$$u \equiv \frac{1}{r} = -\frac{a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta}{2(d\cos\theta + f\sin\theta)}. \tag{72}$$

Therefore, the general form of a quadratic curve in polar coordinates is given by

$$u = \begin{cases} A\sin\theta + B\cos\theta \\ \quad \pm\sqrt{C\sin(2\theta) + D\cos(2\theta) + E} & \text{for } g \neq 0 \\ -\frac{a\cos^2\theta + 2b\cos\theta\sin\theta + c\sin^2\theta}{2(d\cos\theta + f\sin\theta)} & \text{for } g = 0. \end{cases} \tag{73}$$

see also CONIC SECTION, DISCRIMINANT (QUADRATIC CURVE), ELLIPTIC CURVE

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Quadratic Effect

see PRIME QUADRATIC EFFECT

Quadratic Equation

A quadratic equation is a second-order POLYNOMIAL

$$ax^2 + bx + c = 0, \tag{1}$$

with $a \neq 0$. The roots x can be found by COMPLETING THE SQUARE:

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{2}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \tag{3}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

Solving for x then gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{5}$$

This is the QUADRATIC FORMULA.

An alternate form is given by dividing (1) through by x^2:

$$a + \frac{b}{x} + \frac{c}{x^2} = 0 \tag{6}$$

$$c\left(\frac{1}{x^2} + \frac{b}{cx}\right) + a = 0 \tag{7}$$

$$c\left(\frac{1}{x} + \frac{b}{2c}\right)^2 = c\left(\frac{b}{2c}\right)^2 - a = \frac{b^2}{4c} - \frac{4ac}{4c} = \frac{b^2 - 4ac}{4c}. \tag{8}$$

Therefore,

$$\frac{1}{x} + \frac{b}{2c} = \pm\frac{\sqrt{b^2 - 4ac}}{2c} \tag{9}$$

$$\frac{1}{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c} \tag{10}$$

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}. \tag{11}$$

This form is helpful if $b^2 \gg 4ac$, in which case the usual form of the QUADRATIC FORMULA can give inaccurate numerical results for one of the ROOTS. This can be avoided by defining

$$q \equiv -\tfrac{1}{2}\left[b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}\right] \tag{12}$$

so that b and the term under the SQUARE ROOT sign always have the same sign. Now, if $b > 0$, then

$$q = -\tfrac{1}{2}(b + \sqrt{b^2 - 4ac}) \tag{13}$$

$$\begin{aligned}\frac{1}{q} &= \frac{-2}{b + \sqrt{b^2 - 4ac}}\,\frac{b - \sqrt{b^2 - 4ac}}{b - \sqrt{b^2 - 4ac}} = \frac{-2(b - \sqrt{b^2 - 4ac})}{b^2 - (b^2 - 4ac)}\\ &= \frac{-2(b - \sqrt{b^2 - 4ac})}{4ac} = \frac{-b + \sqrt{b^2 - 4ac}}{2ac},\end{aligned} \tag{14}$$

so

$$x_1 \equiv \frac{q}{a} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{15}$$

$$x_2 \equiv \frac{c}{q} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{16}$$

Similarly, if $b < 0$, then

$$q = -\tfrac{1}{2}(b - \sqrt{b^2 - 4ac}) = \tfrac{1}{2}(-b + \sqrt{b^2 - 4ac}) \tag{17}$$

$$\begin{aligned}\frac{1}{q} &= \frac{2}{-b + \sqrt{b^2 - 4ac}}\,\frac{b + \sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}} = \frac{2(b + \sqrt{b^2 - 4ac})}{-b^2 + (b^2 - 4ac)}\\ &= \frac{b + \sqrt{b^2 - 4ac}}{-2ac} = \frac{-b - \sqrt{b^2 - 4ac}}{2ac},\end{aligned} \tag{18}$$

so

$$x_1 \equiv \frac{q}{a} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{19}$$

$$x_2 \equiv \frac{c}{q} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{20}$$

Therefore, the ROOTS are always given by $x_1 = q/a$ and $x_2 = c/q$.

see also CARLYLE CIRCLE, CONIC SECTION, CUBIC EQUATION, QUARTIC EQUATION, QUINTIC EQUATION, SEXTIC EQUATION

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Quadratic Field

An ALGEBRAIC INTEGER of the form $a + b\sqrt{D}$ where D is SQUAREFREE forms a quadratic field and is denoted $\mathbb{Q}(\sqrt{D})$. If $D > 0$, the field is called a REAL QUADRATIC FIELD, and if $D < 0$, it is called an IMAGINARY QUADRATIC FIELD. The integers in $\mathbb{Q}(\sqrt{1})$ are simply called "the" INTEGERS. The integers in $\mathbb{Q}(\sqrt{-1})$ are called GAUSSIAN INTEGERS, and the integers in $\mathbb{Q}(\sqrt{-3})$ are called EISENSTEIN INTEGERS. The ALGEBRAIC INTEGERS in an arbitrary quadratic field do not necessarily have unique factorizations. For example, the fields $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-6})$ are not uniquely factorable, since

$$21 = 3 \cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) \qquad (1)$$

$$6 = -\sqrt{6}(\sqrt{-6}) = 2 \cdot 3, \qquad (2)$$

although the above factors are all primes within these fields. All other quadratic fields $\mathbb{Q}(\sqrt{D})$ with $|D| \leq 7$ *are* uniquely factorable.

Quadratic fields obey the identities

$$(a + b\sqrt{D}) \pm (c + d\sqrt{D}) = (a \pm c) + (b \pm d)\sqrt{D}, \quad (3)$$

$$(a + b\sqrt{D})(c + d\sqrt{D}) = (ac + bdD) + (ad + bc)\sqrt{D}, \quad (4)$$

and

$$\frac{a + b\sqrt{D}}{c + d\sqrt{D}} = \frac{ac - bdD}{c^2 - d^2D} + \frac{bc - ad}{c^2 - d^2D}\sqrt{D}. \qquad (5)$$

The INTEGERS in the real field $\mathbb{Q}(\sqrt{D})$ are of the form $r + s\rho$, where

$$\rho = \begin{cases} \sqrt{D} & \text{for } D \equiv 2 \text{ or } D \equiv 3 \pmod{4} \\ \frac{1}{2}(-1 + \sqrt{D}) & \text{for } D \equiv 1 \pmod{4}. \end{cases} \qquad (6)$$

There exist 22 quadratic fields in which there is a EUCLIDEAN ALGORITHM (Inkeri 1947).

see also ALGEBRAIC INTEGER, EISENSTEIN INTEGER, GAUSSIAN INTEGER, IMAGINARY QUADRATIC FIELD, INTEGER, NUMBER FIELD, REAL QUADRATIC FIELD

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Quadratic Form

A quadratic form involving n REAL variables $x_1, x_2, \ldots, x_n$ associated with the $n \times n$ MATRIX $\mathsf{A} = a_{ij}$ is given by

$$Q(x_1, x_2, \ldots, x_n) = a_{ij}x_ix_j, \qquad (1)$$

where EINSTEIN SUMMATION has been used. Letting $\mathbf{x}$ be a VECTOR made up of $x_1, \ldots, x_n$ and $\mathbf{x}^{\mathrm{T}}$ the TRANSPOSE, then

$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathsf{A}\mathbf{x}, \qquad (2)$$

equivalent to

$$Q(\mathbf{x}) = (\mathbf{x}, \mathsf{A}\mathbf{x}) \qquad (3)$$

in INNER PRODUCT notation. A BINARY QUADRATIC FORM has the form

$$Q(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2. \qquad (4)$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \alpha_{ij}x_ix_j \qquad (5)$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathsf{A}\mathbf{x}), \qquad (6)$$

where $\mathsf{A} = a_{ii}$ is a SYMMETRIC MATRIX given by

$$a_{ij} = \begin{cases} \alpha_{ii} & i = j \\ \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) & i \neq j. \end{cases} \qquad (7)$$

Any REAL quadratic form in n variables may be reduced to the diagonal form

$$Q(\mathbf{x}) = \lambda_1{x_1}^2 + \lambda_2{x_2}^2 + \ldots + \lambda_n{x_n}^2 \qquad (8)$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ by a suitable orthogonal point-transformation. Also, two real quadratic forms are equivalent under the group of linear transformations IFF they have the same RANK and SIGNATURE.

see also DISCONNECTED FORM, INDEFINITE QUADRATIC FORM, INNER PRODUCT, INTEGER-MATRIX FORM, POSITIVE DEFINITE QUADRATIC FORM, POSITIVE SEMIDEFINITE QUADRATIC FORM, RANK (QUADRATIC FORM), SIGNATURE (QUADRATIC FORM), SYLVESTER'S INERTIA LAW

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Quadratic Formula

The formula giving the ROOTS of a QUADRATIC EQUATION

$$ax^2 + bx + c = 0 \qquad (1)$$

as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

An alternate form is given by

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}. \qquad (3)$$

see also QUADRATIC EQUATION

Quadratic Integral

To compute integral of the form

$$\int \frac{dx}{a+bx+cx^2}, \tag{1}$$

COMPLETE THE SQUARE in the DENOMINATOR to obtain

$$\int \frac{dx}{a+bx+cx^2} = \frac{1}{c}\int \frac{dx}{\left(x+\frac{b}{2c}\right)^2 + \left(\frac{a}{c} - \frac{b^2}{4c^2}\right)}. \tag{2}$$

Let $u \equiv x + b/2c$. Then define

$$-A^2 \equiv \frac{a}{c} - \frac{b^2}{4c^2} = \frac{1}{4c^2}(4ac - b^2) \equiv \frac{1}{4c^2}q, \tag{3}$$

where

$$q \equiv 4ac - b^2 \tag{4}$$

is the NEGATIVE of the DISCRIMINANT. If $q < 0$, then

$$A = \frac{1}{2c}\sqrt{-q}. \tag{5}$$

Now use PARTIAL FRACTION DECOMPOSITION,

$$\frac{1}{c}\int \frac{du}{(u+A)(u-A)} = \frac{1}{c}\int \left(\frac{A_1}{u+A} + \frac{A_2}{u-A}\right) du \tag{6}$$

$$\begin{aligned}\left(\frac{A_1}{u+A} + \frac{A_2}{u-A}\right) &= \frac{A_1(u-A) + A_2(u+A)}{u^2 - A^2}\\ &= \frac{(A_1+A_2)u + A(A_2 - A_1)}{u^2 - A^2},\end{aligned} \tag{7}$$

so $A_2 + A_1 = 0 \Rightarrow A_2 = -A_1$ and $A(A_2 - A_1) = -2AA_1 = 1 \Rightarrow A_1 = -1/(2A)$. Plugging these in,

$$\begin{aligned}&\frac{1}{c}\int \left(-\frac{1}{2A}\frac{1}{u+A} + \frac{1}{2A}\frac{1}{u-A}\right) du\\ &= \frac{1}{2Ac}\left[-\ln(u+A) + \ln(u-A)\right]\\ &= \frac{1}{2Ac}\ln\left(\frac{u-A}{u+A}\right)\\ &= \frac{1}{2\left(\frac{1}{2c}\right)\sqrt{-q}\,c}\ln\left(\frac{x+\frac{b}{2c} - \frac{1}{2c}\sqrt{-q}}{x+\frac{b}{2c} + \frac{1}{2c}\sqrt{-q}}\right)\\ &= \frac{1}{\sqrt{-q}}\ln\left(\frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}\right)\end{aligned} \tag{8}$$

for $q < 0$. Note that this integral is also tabulated in Gradshteyn and Ryzhik (1979, equation 2.172), where it is given with a sign flipped.

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Quadratic Invariant

Given the BINARY QUADRATIC FORM

$$ax^2 + 2bxy + cy^2 \tag{1}$$

with DISCRIMINANT $b^2 - ac$, let

$$x = pX + qY \tag{2}$$

$$y = rX + sY. \tag{3}$$

Then

$$a(pX+qY)^2 + 2b(pX+qY)(rX+sY) + c(rX+sY)^2 = AX^2 + 2BXY + CY^2, \tag{4}$$

where

$$A = ap^2 + 2bpr + cr^2 \tag{5}$$

$$B = apq + b(ps+qr) + crs \tag{6}$$

$$C = aq^2 + 2bqs + cs^2, \tag{7}$$

so

$$\begin{aligned}B^2 - AC &= [a^2p^2q^2 + b^2(ps+qr)^2 + c^2r^2s^2\\ &\quad + 2abpq(ps+qr) + 2acpqrs + 2bcrs(ps+qr)]\\ &\quad - (ap^2 + 2bpr + cr^2)(aq^2 + 2bqs + cs^2)\\ &= a^2p^2q^2 + b^2p^2s^2 + 2b^2pqrs + b^2q^2r^2 + c^2r^2s^2\\ &\quad + 2abp^2qs + 2abpq^2r + 2acpqrs + 2bcprs^2 + 2bcqr^2s\\ &\quad - a^2p^2q^2 - 2abp^2qs - acp^2s^2 - 2abpq^2r - 4b^2pqrs\\ &\quad - 2bcprs^2 - acq^2r^2 - 2bcqr^2s - c^2r^2s^2\\ &= b^2p^2s^2 - 2b^2pqrs + b^2q^2r^2 + 2acpqrs - acp^2s^2\\ &\quad - acq^2r^2\\ &= p^2s^2(b^2 - ac) + q^2r^2(b^2 - ac) - 2pqrs(b^2 - ac)\\ &= (b^2 - ac)(p^2s^2 - 2pqrs + q^2r^2)\\ &= (ps - rq)^2(b^2 - ac).\end{aligned} \tag{8}$$

Surprisingly, this is the same discriminant as before, but multiplied by the factor $(ps - rq)^2$. The quantity $ps - rq$ is called the MODULUS.

see also ALGEBRAIC INVARIANT

Quadratic Irrational Number

An IRRATIONAL NUMBER of the form

$$\frac{P \pm \sqrt{D}}{Q},$$

where P and Q are INTEGERS and D is a SQUAREFREE INTEGER. Quadratic irrational numbers are sometimes also called QUADRATIC SURDS. In 1770, Lagrange proved that that any quadratic irrational has a CONTINUED FRACTION which is periodic after some point.

see also CONTINUED FRACTION, QUADRATIC SURD

Quadratic Map

A 1-D MAP often called "the" quadratic map is defined by

$$x_{n+1} = {x_n}^2 + c. \tag{1}$$

This is the real version of the complex map defining the MANDELBROT SET. The quadratic map is called attracting if the JACOBIAN $J < 1$, and repelling if $J > 1$. FIXED POINTS occur when

$$x^{(1)} = [x^{(1)}]^2 + c \tag{2}$$

$$(x^{(1)})^2 - x^{(1)} + c = 0 \tag{3}$$

$$x_\pm^{(1)} = \tfrac{1}{2}(1 \pm \sqrt{1-4c}). \tag{4}$$

Period two FIXED POINTS occur when

$$\begin{aligned} x_{n+2} &= {x_{n+1}}^2 + c = ({x_n}^2 + c)^2 + c \\ &= {x_n}^4 + 2c{x_n}^2 + (c^2 + c) = x_n \end{aligned} \tag{5}$$

$$x^4 + 2x^2 - x + (cx^2 + c) = (x^2 - x + c)(x^2 + x + 1 + c) = 0 \tag{6}$$

$$x_\pm^{(2)} = \tfrac{1}{2}[1 \pm \sqrt{1 - 4(1+c)}\,] = \tfrac{1}{2}(1 \pm \sqrt{-3-4c}). \tag{7}$$

Period three FIXED POINTS occur when

$$\begin{aligned} x^6 + x^5 + (3c+1)x^4 + (2c+1)x^3 + (c^2 + 3c + 1)x^2 \\ + (c+1)^2 x + (c^3 + 2c^2 + c + 1) = 0. \end{aligned} \tag{8}$$

The most general second-order 2-D MAP with an elliptic fixed point at the origin has the form

$$x' = x\cos\alpha - y\sin\alpha + a_{20}x^2 + a_{11}xy + a_{02}y^2 \tag{9}$$

$$y' = x\sin\alpha + y\cos\alpha + b_{20}x^2 + b_{11}xy + b_{02}y^2. \tag{10}$$

The map must have a DETERMINANT of 1 in order to be AREA preserving, reducing the number of independent parameters from seven to three. The map can then be put in a standard form by scaling and rotating to obtain

$$x' = x\cos\alpha - y\sin\alpha + x^2\sin\alpha \tag{11}$$

$$y' = x\sin\alpha + y\cos\alpha - x^2\cos\alpha. \tag{12}$$

The inverse map is

$$x = x'\cos\alpha + y'\sin\alpha \tag{13}$$

$$y = -x'\sin\alpha + y'\cos\alpha + (x'\cos\alpha + y'\sin\alpha)^2. \tag{14}$$

The FIXED POINTS are given by

$$ {x_i}^2\sin\alpha + 2x_i\cos\alpha - x_{i-1} - x_{i+1} = 0 \tag{15}$$

for $i = 0, \ldots, n-1$.

see also BOGDANOV MAP, HÉNON MAP, LOGISTIC MAP, LOZI MAP, MANDELBROT SET

Quadratic Mean

see ROOT-MEAN-SQUARE

Quadratic Reciprocity Law

see QUADRATIC RECIPROCITY THEOREM

Quadratic Reciprocity Relations

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \tag{1}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} \tag{2}$$

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)(-1)^{[(p-1)/2][(q-1)/2]}, \tag{3}$$

where $\left(\frac{p}{q}\right)$ is the LEGENDRE SYMBOL.

see also QUADRATIC RECIPROCITY THEOREM

Quadratic Reciprocity Theorem

Also called the AUREUM THEOREMA (GOLDEN THEOREM) by Gauss. If p and q are distinct ODD PRIMES, then the CONGRUENCES

$$\begin{aligned} x^2 &\equiv q \pmod{p} \\ x^2 &\equiv p \pmod{q} \end{aligned}$$

are both solvable or both unsolvable unless both p and q leave the remainder 3 when divided by 4 (in which case one of the CONGRUENCES is solvable and the other is not). Written symbolically,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4},$$

where

$$\left(\frac{p}{q}\right) \equiv \begin{cases} 1 & \text{for } x^2 \equiv p \pmod{q} \text{ solvable for } x \\ -1 & \text{for } x^2 \equiv p \pmod{q} \text{ not solvable for } x \end{cases}$$

is known as a LEGENDRE SYMBOL. Legendre was the first to publish a proof, but it was fallacious. Gauss was the first to publish a correct proof. The quadratic reciprocity theorem was Gauss's favorite theorem from NUMBER THEORY, and he devised many proofs of it over his lifetime.

see also JACOBI SYMBOL, KRONECKER SYMBOL, LEGENDRE SYMBOL, QUADRATIC RESIDUE, RECIPROCITY THEOREM

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Quadratic Recurrence

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

A quadratic recurrence is a RECURRENCE RELATION on a SEQUENCE of numbers $\{x_n\}$ expressing x_n as a second degree polynomial in x_k with $k < n$. For example,

$$x_n = x_{n-1}x_{n-2} \tag{1}$$

is a quadratic recurrence. Another simple example is

$$x_n = (x_{n-1})^2 \tag{2}$$

with $x_0 = 2$, which has solution $x_n = 2^{2^n}$. Another example is the number of "strongly" binary trees of height $\leq n$, given by

$$y_n = (y_{n-1})^2 + 1 \tag{3}$$

with $y_0 = 1$. This has solution

$$y_n = \left\lfloor c^{2^n} \right\rfloor, \tag{4}$$

where

$$c = \exp\left[\sum_{j=0}^{\infty} 2^{-j-1} \ln(1 + y_j{}^{-2})\right] = 1.502836801\ldots \tag{5}$$

and $\lfloor x \rfloor$ is the FLOOR FUNCTION (Aho and Sloane 1973). A third example is the closest strict underapproximation of the number 1,

$$S_n = \sum_{i=1}^{n} \frac{1}{z_i}, \tag{6}$$

where $1 < z_1 < \ldots < z_n$ are integers. The solution is given by the recurrence

$$z_n = (z_{n-1})^2 - z_{n-1} + 1, \tag{7}$$

with $z_1 = 2$. This has a closed solution as

$$z_n = \left\lfloor d^{2^n} + \tfrac{1}{2} \right\rfloor \tag{8}$$

where

$$d = \tfrac{1}{2}\sqrt{6} \exp\left\{\sum_{j=1}^{\infty} 2^{-j-1} \ln[1 + (2z_j - 1)^{-2}]\right\} = 1.2640847353\ldots \tag{9}$$

(Aho and Sloane 1973). A final example is the well-known recurrence

$$c_n = (c_{n-1})^2 - \mu \tag{10}$$

with $c_0 = 0$ used to generate the MANDELBROT SET.

see also MANDELBROT SET, RECURRENCE RELATION

References

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Quadratic Residue

If there is an INTEGER x such that

$$x^2 \equiv q \pmod{p}, \tag{1}$$

then q is said to be a quadratic residue of x mod p. If not, q is said to be a quadratic nonresidue of x mod p. For example, $4^2 \equiv 6 \pmod{10}$, so 6 is a quadratic residue (mod 10). The entire set of quadratic residues (mod 10) are given by 1, 4, 5, 6, and 9, since

$$\begin{array}{lll} 1^2 \equiv 1 \pmod{10} & 2^2 \equiv 4 \pmod{10} & 3^2 \equiv 9 \pmod{10} \\ 4^2 \equiv 6 \pmod{10} & 5^2 \equiv 5 \pmod{10} & 6^2 \equiv 6 \pmod{10} \\ 7^2 \equiv 9 \pmod{10} & 8^2 \equiv 4 \pmod{10} & 9^2 \equiv 1 \pmod{10} \end{array}$$

making the numbers 2, 3, 7, and 8 the quadratic nonresidues (mod 10).

A list of quadratic residues for $p \leq 29$ is given below (Sloane's A046071), with those numbers $< p$ not in the list being quadratic nonresidues of p.

p	Quadratic Residues
1	(none)
2	1
3	1
4	1
5	1, 4
6	1, 3, 4
7	1, 2, 4
8	1, 4
9	1, 4, 7
10	1, 4, 5, 6, 9
11	1, 3, 4, 5, 9
12	1, 4, 9
13	1, 3, 4, 9, 10, 12
14	1, 2, 4, 7, 8, 9, 11
15	1, 4, 6, 9, 10
16	1, 4, 9
17	1, 2, 4, 8, 9, 13, 15, 16
18	1, 4, 7, 9, 10, 13, 16
19	1, 4, 5, 6, 7, 9, 11, 16, 17
20	1, 4, 5, 9, 16

The UNITS in the integers (mod n), $\mathbb{Z}_n$, which are SQUARES are the quadratic residues.

Given an ODD PRIME p and an INTEGER a, then the LEGENDRE SYMBOL is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \\ -1 & \text{otherwise.} \end{cases} \tag{2}$$

If

$$r^{(p-1)/2} \equiv \pm 1 \pmod{p}, \tag{3}$$

then r is a quadratic residue (+) or nonresidue (−). This can be seen since if r is a quadratic residue of p, then there exists a square x^2 such that $r \equiv x^2 \pmod{p}$, so

$$r^{(p-1)/2} \equiv (x^2)^{(p-1)/2} \equiv x^{p-1} \pmod{p}, \tag{4}$$

and x^{p-1} is congruent to 1 (mod p) by FERMAT'S LITTLE THEOREM. x is given by

$$\begin{cases} q^{k+1} \pmod{p} \\ \quad \text{for } p = 4k+3 \\ q^{k+1} \pmod{p} \\ \quad \text{for } p = 8k+5 \text{ and } q^{2k+1} \equiv 1 \pmod{p} \\ (4q)^{k+1}\left(\frac{p+1}{2}\right) \pmod{p} \\ \quad \text{for } p = 8k+5 \text{ and } q^{2k+1} \equiv -1 \pmod{p}. \end{cases} \tag{5}$$

More generally, let q be a quadratic residue modulo an ODD PRIME p. Choose h such that the LEGENDRE SYMBOL $(h^2 - 4q/p) = -1$. Then defining

$$V_1 = h \tag{6}$$
$$V_2 = h^2 - 2q \tag{7}$$
$$V_i = hV_{i-1} - qV_{i-2} \qquad \text{for } i \geq 3, \tag{8}$$

gives

$$V_{2i} = V_i^{\,2} - 2q^i \tag{9}$$
$$V_{2i+1} = V_iV_{i+1} - hn^i, \tag{10}$$

and a solution to the quadratic CONGRUENCE is

$$x = V_{(p+1)/2}\left(\frac{p+1}{2}\right) \pmod{p}. \tag{11}$$

The following table gives the PRIMES which have a given number d as a quadratic residue.

d	Primes
−6	$24k+1, 5, 7, 11$
−5	$20k+1, 3, 7, 9$
−3	$6k+1$
−2	$8k+1, 3$
−1	$4k+1$
2	$8k \pm 1$
3	$12k \pm 1$
5	$10k \pm 1$
6	$24k \pm 1, 5$

Finding the CONTINUED FRACTION of a SQUARE ROOT $\sqrt{D}$ and using the relationship

$$Q_n = \frac{D - P_n^{\,2}}{Q_{n-1}} \tag{12}$$

for the nth CONVERGENT P_n/Q_n gives

$$P_n^{\,2} \equiv -Q_nQ_{n-1} \pmod{D}. \tag{13}$$

Therefore, $-Q_nQ_{n-1}$ is a quadratic residue of D. But since $Q_1 = 1$, $-Q_2$ is a quadratic residue, as must be $-Q_2Q_3$. But since $-Q_2$ is a quadratic residue, so is Q_3, and we see that $(-1)^{n-1}Q_n$ are all quadratic residues of D. This method is not guaranteed to produce all quadratic residues, but can often produce several small ones in the case of large D, enabling D to be factored.

The number of SQUARES $s(n)$ in $\mathbb{Z}_n$ is related to the number $q(n)$ of quadratic residues in $\mathbb{Z}_n$ by

$$q(p^n) = s(p^n) - s(p^{n-2}) \tag{14}$$

for $n \geq 3$ (Stangl 1996). Both q and s are MULTIPLICATIVE FUNCTIONS.

see also EULER'S CRITERION, MULTIPLICATIVE FUNCTION, QUADRATIC RECIPROCITY THEOREM, RIEMANN HYPOTHESIS

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Quadratic Sieve Factorization Method

A procedure used in conjunction with DIXON'S FACTORIZATION METHOD to factor large numbers. The rs are chosen as

$$\lfloor \sqrt{n} \rfloor + k, \tag{1}$$

where $k = 1, 2, \ldots$ and $\lfloor x \rfloor$ is the FLOOR FUNCTION. We are then looking for factors p such that

$$n \equiv r^2 \pmod{p}, \tag{2}$$

which means that only numbers with LEGENDRE SYMBOL $(n/p) = 1$ (less than $N = \pi(d)$ for trial divisor d) need be considered. The set of PRIMES for which this is true is known as the FACTOR BASE. Next, the CONGRUENCES

$$x^2 \equiv n \pmod{p} \tag{3}$$

must be solved for each p in the FACTOR BASE. Finally, a sieve is applied to find values of $f(r) = r^2 - n$

which can be factored completely using only the FACTOR BASE. GAUSSIAN ELIMINATION is then used as in DIXON'S FACTORIZATION METHOD in order to find a product of the $f(r)$s, yielding a PERFECT SQUARE.

The method requires about $\exp(\sqrt{\log n \log\log n})$ steps, improving on the CONTINUED FRACTION FACTORIZATION ALGORITHM by removing the 2 under the SQUARE ROOT (Pomerance 1996). The use of multiple POLYNOMIALS gives a better chance of factorization, requires a shorter sieve interval, and is well-suited to parallel processing.

see also PRIME FACTORIZATION ALGORITHMS, SMOOTH NUMBER

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Quadratic Surd

see QUADRATIC IRRATIONAL NUMBER

Quadratic Surface

There are 17 standard-form quadratic surfaces. The general quadratic is written

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0. \quad (1)$$

Define

$$e = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad (2)$$

$$E = \begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{bmatrix} \quad (3)$$

$$\rho_3 = \operatorname{rank} e \quad (4)$$

$$\rho_4 = \operatorname{rank} E \quad (5)$$

$$\Delta = \det E, \quad (6)$$

and k_1, k_2, as k_3 are the roots of

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0. \quad (7)$$

Also define

$$k \equiv \begin{cases} 1 & \text{if the signs of nonzero } k\text{s are the same} \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Surface	Equation	ρ_3	ρ_4	sgn (Δ)	k
coincident planes	$x^2 = 0$	1	1		
ellipsoid ($\Im$)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$	3	4	+	1
ellipsoid ($\Re$)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	3	4	−	1
elliptic cone ($\Im$)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	3	3		1
elliptic cone ($\Re$)	$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	3	3		0
elliptic cylinder ($\Im$)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	2	3		1
elliptic cylinder ($\Re$)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	2	3		1
elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	2	4	−	1
hyperbolic cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$	2	3		0
hyperbolic paraboloid	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$	2	4	+	0
hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	3	4	+	0
hyperboloid of two sheets	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	3	4	−	0
intersecting planes ($\Im$)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	2	2		1
intersecting planes ($\Re$)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	2	2		0
parabolic cylinder	$x^2 + 2rz = 0$	1	3		
parallel planes ($\Im$)	$x^2 = -a^2$	1	2		
parallel planes ($\Re$)	$x^2 = a^2$	1	2		

see also CUBIC SURFACE, ELLIPSOID, ELLIPTIC CONE, ELLIPTIC CYLINDER, ELLIPTIC PARABOLOID, HYPERBOLIC CYLINDER, HYPERBOLIC PARABOLOID, HYPERBOLOID, PLANE, QUARTIC SURFACE, SURFACE

References

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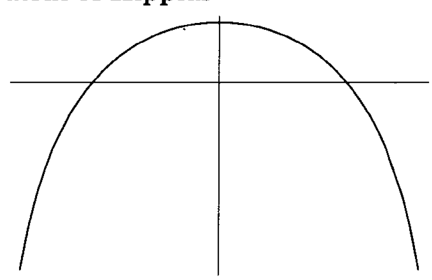
Quadratrix of Hippias



The quadratrix was discovered by Hippias of Elias in 430 BC, and later studied by Dinostratus in 350 BC (MacTutor Archive). It can be used for ANGLE TRISECTION or, more generally, division of an ANGLE into any integral number of equal parts, and CIRCLE SQUARING. In POLAR COORDINATES,

$$\pi\rho = 2r\theta \csc\theta,$$

so

$$r = \frac{\rho\pi \sin\theta}{\theta},$$

which is proportional to the COCHLEOID.

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 195 and 198, 1972.
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MacTutor History of Mathematics Archive. "Quadratrix of Hippias." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Quadratrix.html.

Quadrature

The word quadrature has (at least) three incompatible meanings. Integration by quadrature either means solving an INTEGRAL analytically (i.e., symbolically in terms of known functions), or solving of an integral numerically (e.g., GAUSSIAN QUADRATURE, QUADRATURE FORMULAS). The word quadrature is also used to mean SQUARING: the construction of a square using only COMPASS and STRAIGHTEDGE which has the same AREA as a given geometric figure. If quadrature is possible for a PLANE figure, it is said to be QUADRABLE.

For a function tabulated at given values x_i (so the ABSCISSAS cannot be chosen at will), write the function ϕ as a sum of ORTHONORMAL FUNCTIONS p_j satisfying

$$\int_a^b p_i(x)p_j(x)W(x)\,dx = \delta_{ij} \tag{1}$$

as

$$\phi(x) = \sum_{j=0}^{\infty} a_j p_j(x), \tag{2}$$

and plug into

$$\begin{aligned}\int_a^b \phi(x)W(x)\,dx &= \int_a^b \sum_{j=1}^{m} \frac{\pi(x)W(x)}{(x-x_j)\pi'(x_j)}\,dx\,f(x_j)\\ &\equiv \sum_{j=1}^{m} w_j f(x_j),\end{aligned} \tag{3}$$

giving

$$\int_a^b \sum_{j=0}^{\infty} a_j p_j(x)W(x)\,dx = \sum_{i=1}^{n} w_i \left[\sum_{j=0}^{\infty} a_j p_j(x_i)\right]. \tag{4}$$

But we wish this to hold for all degrees of approximation, so

$$a_j \int_a^b p_j(x)W(x)\,dx = a_j \sum_{i=1}^{n} w_i p_j(x_i) \tag{5}$$

$$\int_a^b p_j(x)W(x)\,dx = \sum_{i=1}^{n} w_i p_j(x_i). \tag{6}$$

Setting $i=0$ in (1) gives

$$\int_a^b p_0(x)p_j(x)W(x)\,dx = \delta_{0j}. \tag{7}$$

The zeroth order orthonormal function can always be taken as $p_0(x)=1$, so (7) becomes

$$\int_a^b p_j(x)W(x)\,dx = \delta_{0j} \tag{8}$$

$$= \sum_{i=1}^{n} w_i p_j(x_i), \tag{9}$$

where (6) has been used in the last step. We therefore have the MATRIX equation

$$\begin{bmatrix} p_0(x_1) & \cdots & p_0(x_n)\\ p_1(x_1) & \cdots & p_1(x_n)\\ \vdots & \ddots & \vdots\\ p_{n-1}(x_1) & \cdots & p_{n-1}(x_n)\end{bmatrix}\begin{bmatrix} w_1\\ w_2\\ \vdots\\ w_n\end{bmatrix} = \begin{bmatrix}1\\0\\ \vdots\\ 0\end{bmatrix}, \tag{10}$$

which can be inverted to solve for the w_is (Press *et al.* 1992).

see also CALCULUS, CHEBYSHEV-GAUSS QUADRATURE, CHEBYSHEV QUADRATURE, DERIVATIVE, FUNDAMENTAL THEOREM OF GAUSSIAN QUADRATURE, GAUSS-JACOBI MECHANICAL QUADRATURE, GAUSSIAN QUADRATURE, HERMITE-GAUSS QUADRATURE, HERMITE QUADRATURE, JACOBI-GAUSS QUADRATURE, JACOBI QUADRATURE, LAGUERRE-GAUSS QUADRATURE, LAGUERRE QUADRATURE, LEGENDRE-GAUSS QUADRATURE, LEGENDRE QUADRATURE, LOBATTO QUADRATURE, MECHANICAL QUADRATURE, MEHLER QUADRATURE, NEWTON-COTES FORMULAS, NUMERICAL INTEGRATION, RADAU QUADRATURE, RECURSIVE MONOTONE STABLE QUADRATURE

References
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Quadrature Formulas

see NEWTON-COTES FORMULAS

Quadric

An equation of the form

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1,$$

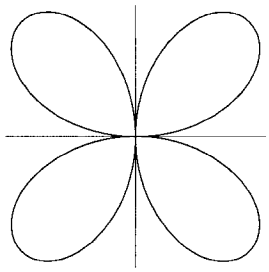
where θ is said to be the parameter of the quadric.

Quadricorn

A FLEXIBLE POLYHEDRON due to C. Schwabe (with the appearance of having four horns) which flexes from one totally flat configuration to another, passing through intermediate configurations of positive VOLUME.

see also FLEXIBLE POLYHEDRON

Quadrifolium



The ROSE with $n = 2$. It has polar equation

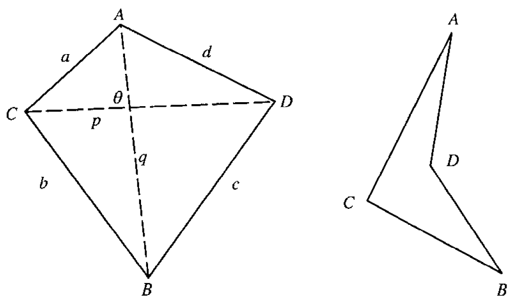
$$r = a\sin(2\theta),$$

and Cartesian form

$$(x^2 + y^2)^3 = 4a^2x^2y^2.$$

see also BIFOLIUM, FOLIUM, ROSE, TRIFOLIUM

Quadrilateral



A four-sided POLYGON sometimes (but not very often) also known as a TETRAGON. If not explicitly stated, all four VERTICES are generally taken to lie in a PLANE. If the points do not lie in a PLANE, the quadrilateral is called a SKEW QUADRILATERAL.

For a planar convex quadrilateral (left figure above), let the lengths of the sides be a, b, c, and d, the SEMIPERIMETER s, and the DIAGONALS p and q. The DIAGONALS are PERPENDICULAR IFF $a^2 + c^2 = b^2 + d^2$. An equation for the sum of the squares of side lengths is

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4x^2, \qquad (1)$$

where x is the length of the line joining the MIDPOINTS of the DIAGONALS. The AREA of a quadrilateral is given by

$$\begin{aligned} K &= \tfrac{1}{2}pq\sin\theta & (2)\\ &= \tfrac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta & (3)\\ &= \tfrac{1}{4}\sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2} & (4)\\ &= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2[\tfrac{1}{2}(A+B)]}, & (5) \end{aligned}$$

where (4) is known as BRETSCHNEIDER'S FORMULA (Beyer 1987).

A special type of quadrilateral is the CYCLIC QUADRILATERAL, for which a CIRCLE can be circumscribed so that it touches each VERTEX. For BICENTRIC quadrilaterals, the CIRCUMCIRCLE and INCIRCLE satisfy

$$2r^2(R^2 - s^2) = (R^2 - s^2)^2 - 4r^2s^2, \qquad (6)$$

where R is the CIRCUMRADIUS, r in the INRADIUS, and s is the separation of centers. A quadrilateral with two sides PARALLEL is called a TRAPEZOID.

There is a relationship between the six distances d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, and d_{34} between the four points of a quadrilateral (Weinberg 1972):

$$\begin{aligned} 0 = {}& d_{12}{}^4 d_{34}{}^2 + d_{13}{}^4 d_{24}{}^2 + d_{14}{}^4 d_{23}{}^2 + d_{23}{}^4 d_{14}{}^2\\ &+ d_{24}^4 d_{13}^2 + d_{34}^4 d_{12}^2\\ &+ d_{12}^2 d_{23}^2 d_{31}^2 + d_{12}^2 d_{24}^2 d_{41}^2 + d_{13}^2 d_{34}^2 d_{41}^2\\ &+ d_{23}^2 d_{34}^2 d_{42}^2 - d_{12}^2 d_{23}^2 d_{34}^2 - d_{13}^2 d_{32}^2 d_{24}^2\\ &- d_{12}^2 d_{24}^2 d_{43}^2 - d_{14}^2 d_{42}^2 d_{23}^2 - d_{13}^2 d_{34}^2 d_{42}^2\\ &- d_{14}^2 d_{43}^2 d_{32}^2 - d_{23}^2 d_{31}^2 d_{14}^2 - d_{21}^2 d_{13}^2 d_{34}^2\\ &- d_{24}^2 d_{41}^2 d_{13}^2 - d_{21}^2 d_{14}^2 d_{43}^2 - d_{31}^2 d_{12}^2 d_{24}^2\\ &- d_{32}^2 d_{21}^2 d_{14}^2. \qquad (7) \end{aligned}$$

see also BIMEDIAN, BRAHMAGUPTA'S FORMULA, BRETSCHNEIDER'S FORMULA, COMPLETE QUADRILATERAL, CYCLIC-INSCRIPTABLE QUADRILATERAL, CYCLIC QUADRILATERAL, DIAMOND, EIGHT-POINT CIRCLE THEOREM, EQUILIC QUADRILATERAL, FANO'S AXIOM, LÉON ANNE'S THEOREM, LOZENGE, ORTHOCENTRIC QUADRILATERAL, PARALLELOGRAM, PTOLEMY'S THEOREM, RATIONAL QUADRILATERAL, RHOMBUS, SKEW QUADRILATERAL, TRAPEZOID, VARIGNON'S THEOREM, VON AUBEL'S THEOREM, WITTENBAUER'S PARALLELOGRAM

References

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Routh, E. J. "Moment of Inertia of a Quadrilateral." *Quart. J. Pure Appl. Math.* **11**, 109–110, 1871.

Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: Wiley, p. 7, 1972.

Quadrillion

In the American system, 10^{15}.

see also LARGE NUMBER

Quadriplanar Coordinates

The analog of TRILINEAR COORDINATES for TETRAHEDRA.

see also TETRAHEDRON, TRILINEAR COORDINATES

References
Altshiller-Court, N. *Modern Pure Solid Geometry.* New York: Macmillan, 1935.
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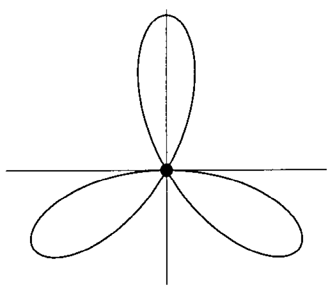
Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, pp. 193–196, 1961.

Quadruple

A group of four elements, also called a QUADRUPLET or TETRAD.

see also AMICABLE QUADRUPLE, DIOPHANTINE QUADRUPLE, MONAD, PAIR, PRIME QUADRUPLET, PYTHAGOREAN QUADRUPLE, QUADRUPLET, QUINTUPLET, TETRAD, TRIAD, TRIPLE, TWINS, VECTOR QUADRUPLE PRODUCT

Quadruple Point



A point where a curve intersects itself along four arcs. The above plot shows the quadruple point at the ORIGIN of the QUADRIFOLIUM $(x^2+y^2)^3-4x^2y^2=0$.

see also DOUBLE POINT, TRIPLE POINT

References
Walker, R. J. *Algebraic Curves.* New York: Springer-Verlag, pp. 57–58, 1978.

Quadruplet

see QUADRUPLE

Quadtree

A TREE having four branches at each node. Quadtrees are used in the construction of some multidimensional databases (e.g., cartography, computer graphics, and image processing). For a d-D tree, the expected number of comparisons over all pairs of integers for successful and unsuccessful searches are given analytically for $d=2$ and numerically for $d\geq 3$ by Finch.

References
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/qdt/qdt.html`.
Flajolet, P.; Gonnet, G.; Puech, C.; and Robson, J. M. "Analytic Variations on Quadtrees." *Algorithmica* **10**, 473–500, 1993.
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Quantic

An m-ary n-ic polynomial (i.e., a HOMOGENEOUS POLYNOMIAL with constant COEFFICIENTS of degree n in m independent variables).

see also ALGEBRAIC INVARIANT, FUNDAMENTAL SYSTEM, p-ADIC NUMBER, SYZYGIES PROBLEM

Quantifier

One of the operations EXISTS $\exists$ or FOR ALL $\forall$.

see also BOUND, EXISTS, FOR ALL, FREE

Quantization Efficiency

Quantization is a nonlinear process which generates additional frequency components (Thompson *et al.* 1986). This means that the signal is no longer band-limited, so the SAMPLING THEOREM no longer holds. If a signal is sampled at the NYQUIST FREQUENCY, information will be lost. Therefore, sampling faster than the NYQUIST FREQUENCY results in detection of more of the signal and a lower signal-to-noise ratio [SNR]. Let β be the OVERSAMPLING ratio and define

$$\eta_Q\equiv\frac{\mathrm{SNR_{quant}}}{\mathrm{SNR_{unquant}}}.$$

Then the following table gives values of η_Q for a number of parameters.

Quantization Levels	η_Q $(\beta=1)$	η_Q $(\beta=2)$
2	0.64	0.74
3	0.81	0.89
4	0.88	0.94

The Very Large Array of 27 radio telescopes in Socorro, New Mexico uses three-level quantization at $\beta=1$, so $\eta_Q=0.81$.

References
Thompson, A. R.; Moran, J. M.; and Swenson, G. W. Jr. Fig. 8.3 in *Interferometry and Synthesis in Radio Astronomy.* New York: Wiley, p. 220, 1986.

Quantum Chaos

The study of the implications of CHAOS for a system in the semiclassical (i.e., between classical and quantum mechanical) regime.

References
Ott, E. "Quantum Chaos." Ch. 10 in *Chaos in Dynamical Systems.* New York: Cambridge University Press, pp. 334–362, 1993.

Quarter

The UNIT FRACTION 1/4, also called one-fourth. It is the value of KOEBE'S CONSTANT.

see also HALF, QUARTILE